From Path Signatures to Algebraic Varieties

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Algebraic, analytic, geometric structures emerging from QFT

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Setup: Signatures

- Let $X : [0,1] \to \mathbb{R}^d$ be a piecewise differentiable path.
- Coordinate functions: $X_1, X_2, \dots, X_d : \mathbb{R} \to \mathbb{R}$
- Their differentials $\mathrm{d}X_i(t) = X_i'(t)\mathrm{d}t$ are the coordinates of the vector

$$\mathrm{d}X = (\mathrm{d}X_1, \mathrm{d}X_2, \dots, \mathrm{d}X_d)$$

- The kth signature of X is a **tensor** $\sigma^{(k)}(X)$ of order k and format $d \times d \times \cdots \times d$. It is the multivariate integral:

$$\sigma^{(k)}(X) \; = \; \int_\Delta \mathrm{d}X(t_1) \otimes \mathrm{d}X(t_2) \otimes \cdots \otimes \mathrm{d}X(t_k),$$

where $\Delta = \{(t_1, t_2, \dots, t_k) \in \mathbb{R}^k : 0 \leq t_1 \leq t_2 \leq \cdots \leq t_k \leq 1\}$.
- Its d^k entries $\sigma_{i_1 i_2 \dots i_k}$ are the *iterated integrals*

$$\sigma_{i_1 i_2 \cdots i_k} = \int_0^1 \int_0^{t_k} \cdots \int_0^{t_3} \int_0^{t_2} \mathrm{d}X_{i_1}(t_1)\, \mathrm{d}X_{i_2}(t_2) \cdots \mathrm{d}X_{i_k}(t_k).$$

Signatures

- $\sigma^{(k)}(X)$ has entries

$$\sigma_{i_1 i_2 \cdots i_k} = \int_0^1 \int_0^{t_k} \cdots \int_0^{t_3} \int_0^{t_2} \mathrm{d}X_{i_1}(t_1)\,\mathrm{d}X_{i_2}(t_2) \cdots \mathrm{d}X_{i_k}(t_k).$$

- Let's start with $k = 1$:
- Fundamental Theorem of Calculus:

$$\int_0^1 \mathrm{d}X_i(t) = X_i(1) - X_i(0)$$

- The first signature of the path X is

$$\sigma^{(1)}(X) = \int_0^1 \mathrm{d}X(t) = X(1) - X(0) \in \mathbb{R}^d$$

Signature Matrices

- Now let's consider $k = 2$. Then the second signature $S = \sigma^{(2)}(X)$ is the $d \times d$ matrix with entries

$$\sigma_{ij} = \int_0^1 \int_0^t \mathrm{d}X_i(s)\mathrm{d}X_j(t)$$

- Set $X(0) = 0$. Applying Fundamental Theorem of Calculus again:

$$\sigma_{ij} = \int_0^1 X_i(t)X_j'(t)\mathrm{d}t$$

- We obtain

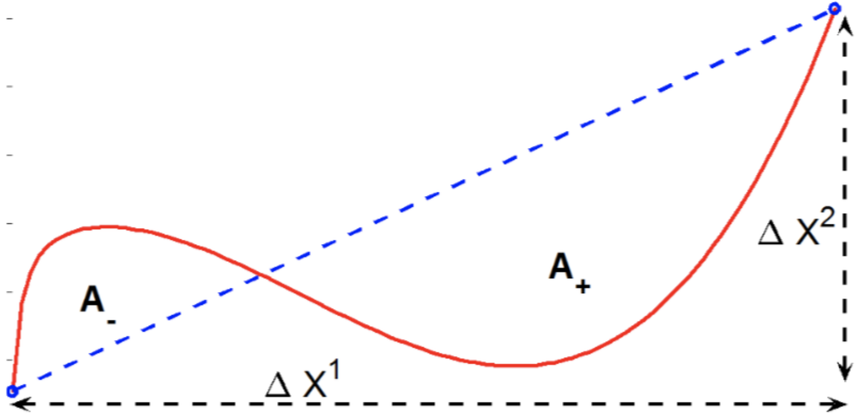
$$\sigma_{ij} + \sigma_{ji} = X_i(1) \cdot X_j(1)$$

- In matrix notation, $S + S^T = X(1) \cdot X(1)^T$
- In particular, the symmetric matrix $S + S^T$ has rank one!
- The skew-symmetric matrix $S - S^T$ measures *deviation from linearity*:

$$\sigma_{ij} - \sigma_{ji} = \int_0^1 (X_i(t)X_j'(t) - X_j(t)X_i'(t))\mathrm{d}t$$

Lévy Area

The entry $\frac{1}{2}(\sigma_{ij} - \sigma_{ji})$ of the skew-symmetric matrix $S - S^T$ is the area below the line minus the area above the line, known as a Lévy area:



Some History

- Introduced by Kuo Tsai Chen in the 1950s:
  K.-T. Chen: *Integration of paths, geometric invariants and a generalized Baker-Hausdorff formula*, Annals of Mathematics **65** (1957)
  K.-T. Chen: *Integration of paths – a faithful representation of paths by noncommutative formal power series*, Transactions AMS **89** (1958)
- The *signature* of a path X is the sequence of tensors

$$\sigma(X) = \big(1, \sigma^{(1)}(X), \sigma^{(2)}(X), \sigma^{(3)}(X), \ldots, \sigma^{(n)}(X), \ldots\big)$$

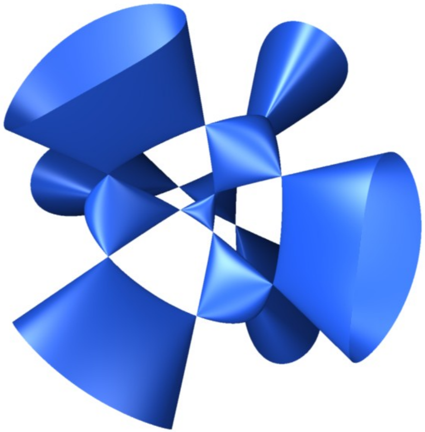
- *Essential question*: how much information does the signature reveal about the path X?
- Signature determines paths! (modulo starting point, parametrization and tree-like excursion)
  B. Hambly and T. Lyons: *Uniqueness for the signature of a path of bounded variation and the reduced path group*, Annals of Mathematics **171** (2010)

More History

- Signatures are central to the theory of rough paths, a revolutionary view on *Stochastic Analysis*.
- P. Friz and N. Victoir: *Multidimensional Stochastic Processes as Rough Paths*. Theory and Applications, Cambridge University Press, 2010.
  P. Friz and M. Hairer: *A Course on Rough Paths. With an introduction to regularity structures*, Universitext, Springer, Cham, 2014.
- They can be used to encode and model data!
  T. Lyons: *Rough paths, signatures and the modelling of functions on streams*, Proc. International Congress of Mathematicians 2014, Seoul
  I. Chevyrev and A. Kormilitzin: *A primer on the signature method in machine learning*, arXiv:1603.03788.

Recent developments

- C. Am., P. Friz and B. Sturmfels: *Varieties of Signature Tensors*, Forum of Mathematics, Sigma. Vol. 7, CUP (2019).
- M. Pfeffer, A. Seigal and B. Sturmfels: *Learning Paths from Signature Tensors*, SIMAX 40.2 (2019).
- F. Galuppi: *The Rough Veronese Variety*, Linear Algebra and Applications 583 (2019).
- L. Colmenajero , F. Galuppi and M. Michalek: *Toric geometry of path signature varieties*, Advances in Applied Mathematics 121 (2020).
- C. Am., D. Lee and C. Meroni: *Convex Hulls of Curves: Volumes and Signatures*, Geometric Science of Information (2023).
- C. Bellingeri and R. Penaguiao: *Discrete Signature Varieties*, arXiv:2303.13377
- C. Am., F. Galuppi, A. Ríos, P. Santarsiero and T. Seynnaeve: *Decomposing Tensor Spaces via Path Signatures*, arXiv:2308.11571



$$x^4 + y^4 + z^4 - x^2 - y^2 - z^2 - x^2y^2 - x^2z^2 - y^2z^2 + 1 = 0$$

Algebraic Varieties

- Solution set of a polynomial system of equations.
- $\mathcal{V} \subseteq K^n$ (affine) *algebraic variety* $\Rightarrow$ we can find a set of polynomials $\mathcal{F} \subseteq \mathbb{K}[s_1, \ldots, s_n]$ such that

$$\mathcal{V} = \{a \in \mathbb{K}^n | f(a) = 0 \text{ for all } f \in \mathcal{F}\}$$

- If polynomials are homogeneous $\Rightarrow$ work in *projective space* $\mathbb{P}^{n-1}$.
- The *ideal* V associated to a variety $\mathcal{V}$: set of all polynomials that vanish on $\mathcal{V}$.
- Key Fact: a polynomial map $\sigma : \mathbb{K}^m \to \mathbb{K}^n$ induces naturally an algebraic variety that contains the *image* of σ.
- We are interested in projective varieties in tensor space $\mathbb{P}^{d^k-1}$ that arise when X ranges over some nice families of paths.

Example of a Signature Variety

- Let $d = 2$ and consider quadratic paths in the plane $\mathbb{R}^2$:

$$X(t) = \left(x_{11}t + x_{12}t^2,\ x_{21}t + x_{22}t^2\right)^T = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \cdot \begin{pmatrix} t \\ t^2 \end{pmatrix}$$

- Their kth signature tensors depend *polynomially* of degree k on x_{ij}.
- $\sigma^{(1)}(X) = (\sigma_1, \sigma_2) = \left(x_{11} + x_{12}, x_{21} + x_{22}\right)$.
-

$$\begin{aligned} \sigma_{ij} &= \int_0^1 \int_0^t (x_{i1} + 2x_{i2}s)\mathrm{d}s\,(x_{j1} + 2x_{j2}t)\mathrm{d}t \\ &= \int_0^1 \left(x_{i1}t + x_{i2}t^2\right)\left(x_{j1} + 2x_{j2}t\right)\mathrm{d}t \\ &= \int_0^1 [x_{i1}x_{j1}t + (2x_{i1}x_{j2} + x_{i2}x_{j1})t^2 + 2x_{i2}x_{j2}t^3]\,\mathrm{d}t \\ &= \tfrac{1}{2}x_{i1}x_{j1} + \tfrac{2}{3}x_{i1}x_{j2} + \tfrac{1}{3}x_{i2}x_{j1} + \tfrac{1}{2}x_{i2}x_{j2}. \end{aligned}$$

- We can write $\sigma^{(2)}(X)$ as

$$\frac{1}{2}\begin{pmatrix} x_{11}+x_{12} \\ x_{21}+x_{22} \end{pmatrix}\left(x_{11}+x_{12},\ x_{21}+x_{22}\right) + \frac{1}{6}(x_{11}x_{22} - x_{12}x_{21})\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Example of a Signature Variety

- The variety of all such signature matrices is the solution set of the quadratic equation

$$(\sigma_{12} + \sigma_{21})^2 - 4\sigma_{11}\sigma_{22} \;\; = \;\; 0.$$

- This means that the image variety associated to signature matrices of polynomial paths of degree two in the plane is a *hypersurface* in $\mathbb{P}^3$.
- We will denote this surface by $\mathcal{P}_{2,2,2}$. Its prime ideal generated by the quadric above is $P_{2,2,2}$.
- We want to study and *understand* these varieties!
- Question: What is the resulting variety if we restrict to linear paths?
- *Answer*: Symmetric matrices of rank 1: the classical *Veronese* variety!

- The third signature $\sigma^{(3)}(X)$ is a $2 \times 2 \times 2$-tensor ($d = 2, k = 3$).

$$
\begin{aligned}
\sigma_{111} &= \tfrac{1}{6}(x_{11} + x_{12})^3 \\
\sigma_{112} &= \tfrac{1}{6}(x_{11}+x_{12})^2(x_{21}+x_{22}) + \tfrac{1}{60}(5x_{11} + 4x_{12})(x_{11}x_{22} - x_{12}x_{21}) \\
\sigma_{121} &= \tfrac{1}{6}(x_{11}+x_{12})^2(x_{21}+x_{22}) + \tfrac{1}{60}(2x_{12})(x_{11}x_{22} - x_{12}x_{21}) \\
\sigma_{211} &= \tfrac{1}{6}(x_{11}+x_{12})^2(x_{21}+x_{22}) - \tfrac{1}{60}(5x_{11} + 6x_{12})(x_{11}x_{22} - x_{12}x_{21}) \\
\sigma_{122} &= \tfrac{1}{6}(x_{11}+x_{12})(x_{21}+x_{22})^2 + \tfrac{1}{60}(5x_{21} + 6x_{22})(x_{11}x_{22} - x_{12}x_{21}) \\
\sigma_{212} &= \tfrac{1}{6}(x_{11}+x_{12})(x_{21}+x_{22})^2 - \tfrac{1}{60}(2x_{22})(x_{11}x_{22} - x_{12}x_{21}) \\
\sigma_{221} &= \tfrac{1}{6}(x_{11}+x_{12})(x_{21}+x_{22})^2 - \tfrac{1}{60}(5x_{21} + 4x_{22})(x_{11}x_{22} - x_{12}x_{21}) \\
\sigma_{222} &= \tfrac{1}{6}(x_{21} + x_{22})^3
\end{aligned}
$$

- Goal: find the polynomial relations among the eight entries of $\sigma^{(3)}(X)$
- Image signature variety is $\mathcal{P}_{2,3,2}$ and its prime ideal is $P_{2,3,2}$.
- An instance of the general $P_{d,k,m}$ of polynomial paths of degree m.

Universal Varieties

- Recall: $X : [0,1] \to \mathbb{R}^d$ be a piecewise differentiable path.
- The kth signature of X is a **tensor** $\sigma^{(k)}(X)$ of order k and format $d \times d \times \cdots \times d$.

$$\sigma_{i_1 i_2 \cdots i_k} = \int_0^1 \int_0^{t_k} \cdots \int_0^{t_3} \int_0^{t_2} \mathrm{d}X_{i_1}(t_1)\, \mathrm{d}X_{i_2}(t_2) \cdots \mathrm{d}X_{i_k}(t_k).$$

- In other words, the kth signature tensor of a path X in $\mathbb{R}^d$ is a point $\sigma^{(k)}(X)$ in the tensor space $(\mathbb{R}^d)^{\otimes k}$, and in projective space $\mathbb{P}^{d^k-1}$.
- Consider the set of signature tensors $\sigma^{(k)}(X)$ as X ranges over **all** smooth paths $X : [0,1] \to \mathbb{R}^d$. This is called the universal variety

$$\mathcal{U}_{d,k} \subset \mathbb{P}^{d^k-1}$$

- It is an algebraic variety! To truly understand them one needs the tools of *tensor algebras, Lie groups and free Lie algebras*...

Computing Universal Varieties

d	k	amb	dim	deg	gens
2	2	3	2	2	1
2	3	7	4	4	6
2	4	15	7	12	33
2	5	31	13	40	150
3	2	8	5	4	6
3	3	26	13	24	81
3	4	80	31	672	954
4	2	15	9	8	20
4	3	63	29	200	486

Invariants of the ideal $U_{d,k}$ that defines the universal variety $\mathcal{U}_{d,k}$

Example ($d = k = 2$)

The universal variety $\mathcal{U}_{2,2}$ of signature matrices consists of all 2×2 matrices whose symmetric part has rank ≤ 1: $(\sigma_{12} + \sigma_{21})^2 = 4\sigma_{11}\sigma_{22}$

We want to *understand* this table! Explain dimensions? degrees? generators?

Computing Dimension

$d \setminus k$	2	3	4	5	6	7	8	9
2	2	4	7	13	22	40	70	126
3	5	13	31	79	195	507	1317	3501
4	9	29	89	293	963	3303	11463	40583
5	14	54	204	828	3408	14568	63318	280318
6	20	90	405	1959	9694	49684	259474	1379194

The dimension of the universal variety $\mathcal{U}_{d,k}$ is much smaller than $d^k - 1$.

A word over the alphabet $\{1, 2, \ldots, d\}$ is Lyndon if it is strictly smaller in lexicographic order than all of its rotations, e.g. 11213.

Theorem

The dimension of the universal variety $\mathcal{U}_{d,k}$ equals the number of Lyndon words of length $\leq k$ over the alphabet $\{1, 2, \ldots, d\}$ (minus one).

$$\lambda_{d,n} = \dim(\mathrm{Lie}^n(\mathbb{R}^d)) = \sum_{k=1}^{n} \sum_{\ell \mid k} \frac{\mu(\ell)}{k} d^{k/\ell},$$ where μ is the Möbius function.

Shuffle Relations

- The *shuffle product* of two words of lengths r and s is the sum over all $\binom{r+s}{s}$ ways of interleaving the two words.
- Examples:

$$e_{12} \sqcup\!\sqcup\, e_{34} = e_{1234} + e_{1324} + e_{1342} + e_{3124} + e_{3142} + e_{3412},$$

$$e_{3} \sqcup\!\sqcup\, e_{134} = e_{3134} + 2e_{1334} + e_{1343}, \qquad e_{21} \sqcup\!\sqcup\, e_{21} = 2e_{2121} + 4e_{2211}$$

- These extend to the *shuffle linear forms* $e_{I \sqcup\!\sqcup J} := e_I \sqcup\!\sqcup\, e_J$, e.g. $e_{12 \sqcup\!\sqcup 34} = e_{12} \sqcup\!\sqcup\, e_{34}$.
- Further extension: replace e by σ so that we obtain polynomial functions on tensor entries, e.g. $\sigma_{21 \sqcup\!\sqcup 21} := 2\sigma_{2121} + 4\sigma_{2211}$

Theorem

For a smooth path X, the following shuffle relation holds:

$$\sigma_I(X)\sigma_J(X) = \sigma_{I \sqcup\!\sqcup J}(X) \quad \text{for all words } I, J$$

Example

$$\sigma_1^2 = 2\sigma_{11}, \quad \sigma_1\sigma_2 = \sigma_{12} + \sigma_{21}, \quad \sigma_2^2 = 2\sigma_{22}$$

$$\sigma_2\sigma_{21} = 2\sigma_{221} + \sigma_{212}, \quad \sigma_2\sigma_{22} = 3\sigma_{222}$$

Chen-Chow Theorem

- Step n signature map:

$$\sigma^{\leq n}(X) \;=\; \big(1,\, \sigma^{(1)}(X),\, \sigma^{(2)}(X),\, \sigma^{(3)}(X),\, \ldots,\, \sigma^{(n)}(X)\big)$$

- Key result attributed to Chow (1940) and Chen (1957):

Theorem (Chen-Chow)

Consider the image of the step n signature map applied to paths in $\mathbb{R}^d$:

$$\mathcal{G}^n(\mathbb{R}^d) \;=\; \left\{\, \sigma^{\leq n}(X) \;:\; X : [0,1] \to \mathbb{R}^d \text{ any smooth path} \,\right\}.$$

then it is an algebraic variety (known as the step-n free Lie group) in the space of truncated tensors defined by the shuffle relations

$$\sigma_I(P)\sigma_J(P) = \sigma_{I \sqcup\!\sqcup J}(P) \quad \text{for all words } I, J \text{ with } |I| + |J| \leq n.$$

Example ($d = k = 2$)

The universal variety $\mathcal{U}_{2,2}$ of signature matrices is defined by
$(\sigma_{12}+\sigma_{21})^2 = \sigma_{1\sqcup\!\sqcup 2}\sigma_{1\sqcup\!\sqcup 2} = \sigma_{1\sqcup\!\sqcup 2\sqcup\!\sqcup 1\sqcup\!\sqcup 2} = \sigma_{1\sqcup\!\sqcup 1\sqcup\!\sqcup 2\sqcup\!\sqcup 2} = \sigma_{1\sqcup\!\sqcup 1}\sigma_{2\sqcup\!\sqcup 2} = 4\sigma_{11}\sigma_{22}$

Polynomial Signature Varieties

- Look at nice family of paths whose signatures live inside $\mathcal{U}_{d,k}$: *polynomial paths*.
- The coordinates of $X : [0,1] \to \mathbb{R}^d$ are polynomials of degree $\leq m$.

$$X_i(t) \ = \ x_{i1}t + x_{i2}t^2 + x_{i3}t^3 + \cdots + x_{im}t^m.$$

- Each one is represented by a real $d \times m$ matrix $X = (x_{ij})$.
- The x_{ij} are homogeneous coordinates on the projective space $\mathbb{P}^{dm-1}$.
- We have the (rational) map

$$\sigma^{(k)} \ : \ \mathbb{P}^{dm-1} \dashrightarrow \mathbb{P}^{d^k-1}, \ X \mapsto \sigma^{(k)}(X).$$

- The closure of the image of this map is the *polynomial signature variety* $\mathcal{P}_{d,k,m}$.
- The homogeneous prime ideal $P_{d,k,m}$ of this variety in $\mathbb{R}[\sigma^{(k)}]$ is the *polynomial signature ideal*.

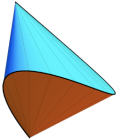
Example: $\mathcal{P}_{3,3,2}$

- The third signature variety $\mathcal{P}_{3,3,2}$ for quadratic paths in $\mathbb{R}^3$ lies in the universal variety $\mathcal{U}_{3,3}$ for $3 \times 3 \times 3$ tensors.

- Recall: $\mathcal{U}_{3,3}$ has dimension 13, degree 24, and cut out by 81 quadrics.
- $\mathcal{P}_{3,3,2}$ has dimension 5, degree 90, and cut out by 162 quadrics in $\mathbb{P}^{25}$!
- The linear span of $\mathcal{P}_{3,3,2}$ is a hyperplane $\mathbb{P}^{25}$. It is defined by

$$\sigma_{123} - \sigma_{132} - \sigma_{213} + \sigma_{231} + \sigma_{312} - \sigma_{321} \ = \ 0.$$

- This linear form is the signed volume of the convex hull of a path.



Piecewise Linear Signature Varieties

- Look at nice family of paths whose signatures live inside $\mathcal{U}_{d,k}$: *piecewise linear paths*.
- Paths $X : [0,1] \to \mathbb{R}^d$ that are piecewise linear with m pieces.
- Their steps are the vectors $X_1, \ldots, X_m \in \mathbb{R}^d$.

$$t \mapsto X_1 + \cdots + X_{i-1} + (mt - i + 1) \cdot X_i \quad \text{where} \quad \frac{i-1}{m} \leq t \leq \frac{i}{m}$$

- They are also represented by a real $d \times m$ matrix $X = (x_{ij})$.
- We again have a (rational) map

$$\sigma^{(k)} : \mathbb{P}^{dm-1} \dashrightarrow \mathbb{P}^{d^k-1}, \ X \mapsto \sigma^{(k)}(X).$$

- The closure of the image of this map is the *piecewise linear signature variety* $\mathcal{L}_{d,k,m}$.
- The homogeneous prime ideal $L_{d,k,m}$ of this variety in $\mathbb{R}[\sigma^{(k)}]$ is the *piecewise linear signature ideal*.

Piecewise-linear path parametrization

- By Chen (1954), the m-step signature of a piecewise linear path X is given by the *tensor product of tensor exponentials*:

$$\sigma^{\leq n}(X) \;=\; \exp(X_1) \otimes \exp(X_2) \otimes \cdots \otimes \exp(X_m) \quad \in \; T^n(\mathbb{R}^d).$$

- As a corollary, the kth signature tensor of X equals

$$\sigma^{(k)}(X) \;=\; \sum_{\tau} \prod_{\ell=1}^{m} \frac{1}{|\,\tau^{-1}(\ell)|\,!} \cdot X_{\tau(1)} \otimes X_{\tau(2)} \otimes X_{\tau(3)} \otimes \cdots \otimes X_{\tau(k)}$$

(sum is over weakly increasing $\tau : \{1,2,\ldots,k\} \to \{1,2,\ldots,m\}$).

- For example, for $k = 3$ we have

$$\sigma^{(3)}(X) = \frac{1}{6} \cdot \sum_{i=1}^{m} X_i^{\otimes 3} + \frac{1}{2} \cdot \sum_{1 \leq i < j \leq m} (X_i^{\otimes 2} \otimes X_j + X_i \otimes X_j^{\otimes 2}) \; + \sum_{1 \leq i < j < l \leq m} X_i \otimes X_j \otimes X_k.$$

Signature Matrices

Theorem (Am., Friz, Sturmfels 2019)

Let $k = 2$ and $m \leq d$. For $m \leq d$ we have the equality

$$\mathcal{P}_{d,2,m} = \mathcal{L}_{d,2,m}$$

- We denote $\mathcal{M}_{d,m} := \mathcal{P}_{d,2,m} = \mathcal{L}_{d,2,m}$.
- Any $d \times d$ matrix $S = \sigma^{(2)}(X)$ is uniquely the sum of a *symmetric* matrix and a *skew-symmetric* matrix:

$$S = P + Q \quad \text{where} \quad P = \frac{1}{2}(S + S^T)\,, Q = \frac{1}{2}(S - S^T)$$

- The $\binom{d+1}{2}$ entries p_{ij} of P and $\binom{d}{2}$ entries q_{ij} of Q serve as coordinates on the space $\mathbb{P}^{d^2-1}$ of matrices $S = (\sigma_{ij})$.

Determinantal Varieties

Theorem (Am., Friz, Sturmfels 2019)

For each d and m, the following subvarieties of $\mathbb{P}^{d^2-1}$ coincide:

1. *Signature matrices of piecewise linear paths with m segments.*
2. *Signature matrices of polynomial paths of degree m.*
3. *Matrices $S = P + Q$, with P symmetric and Q skew-symmetric, such that $\mathrm{rank}(P) \leq 1$ and $\mathrm{rank}([\,P\,Q\,]) \leq m$.*

For each fixed d, these varieties $\mathcal{M}_{d,m}$ form a nested family:

$$\mathcal{M}_{d,1} \subset \mathcal{M}_{d,2} \subset \mathcal{M}_{d,3} \subset \cdots \subset \mathcal{M}_{d,d} = \mathcal{M}_{d,d+1} = \cdots$$

For $m \leq d$, $\mathcal{M}_{d,m}$ is irreducible of dimension $md - \binom{m}{2} - 1$. For $m \geq d$, $\mathcal{M}_{d,m} = \mathcal{U}_{d,2}$ the universal variety.

Example ($d = 3, m = 2$)

The variety $\mathcal{M}_{3,2}$ has dimension 4 and degree 6 in $\mathbb{P}^8$. It is cut out by the 2×2 minors of the 3×3 symmetric matrix $P = (p_{ij})$ and the 3×3 minors of

$$[P\,Q] \quad = \quad \begin{bmatrix} p_{11} & p_{12} & p_{13} & 0 & q_{12} & q_{13} \\ p_{12} & p_{22} & p_{23} & -q_{12} & 0 & q_{23} \\ p_{13} & p_{23} & p_{33} & -q_{13} & -q_{23} & 0 \end{bmatrix}.$$

$$\sigma_{ii} = p_{ii}\,,\ \sigma_{ij} = p_{ij} + q_{ij} \quad \text{and} \quad \sigma_{ji} = p_{ij} - q_{ij} \quad \text{for} \quad 1 \leq i < j \leq d.$$

d	k	m	amb	dim	deg	gens
2	2	1	2	1	2	1
2	2	2	3	2	2	1
3	2	1	5	2	4	6
3	2	2	8	4	6	9
3	2	3	8	5	4	6
4	2	1	9	3	8	20
4	2	2	15	6	20	36
4	2	3	15	8	16	21
4	2	4	15	9	8	20
5	2	1	14	4	16	50
5	2	2	24	8	70	100
5	2	3	24	11	80	55
5	2	4	24	13	40	50, 5
5	2	5	24	14	16	50

Invariants of the ideal $M_{d,m}$ that defines the variety of signature matrices $\mathcal{M}_{d,m}$.

Chains of Inclusions

- If $m = 1$ then X is a linear path and $\mathcal{L}_{d,k,1} = \mathcal{P}_{d,k,1}$.
- This is the classical *Veronese variety* of symmetric tensors of rank 1.

Theorem (Am., Friz, Sturmfels 2019)

We have the following chains of inclusions between the kth Veronese variety and the kth universal variety:

$$\nu_k(\mathbb{P}^{d-1}) = \mathcal{L}_{d,k,1} \subset \mathcal{L}_{d,k,2} \subset \mathcal{L}_{d,k,3} \subset \cdots \subset \mathcal{L}_{d,k,M} = \mathcal{U}_{d,k} \subset \mathbb{P}^{d^k-1}$$
$$\nu_k(\mathbb{P}^{d-1}) = \mathcal{P}_{d,k,1} \subset \mathcal{P}_{d,k,2} \subset \mathcal{P}_{d,k,3} \subset \cdots \subset \mathcal{P}_{d,k,M'} = \mathcal{U}_{d,k} \subset \mathbb{P}^{d^k-1}$$

Here M and M' are constants that depend only on d and k.

Conjecture

$$M = M' = \left\lceil \frac{\lambda_{d,k}}{d} \right\rceil$$

For $m \geq M$, we have $\mathcal{P}_{d,k,m} = \mathcal{L}_{d,k,m} = \mathcal{U}_{d,k}$.

Piecewise linear vs. Polynomial

We have seen that $\mathcal{P}_{d,2,m} = \mathcal{L}_{d,2,m}$
The case $k = 2$ does not generalize!
For $k \geq 3$ we have $\mathcal{L}_{2,k,2} \neq \mathcal{P}_{2,k,2}$ in $\mathbb{P}^{2^k-1}$.

Example ($d = 2, k = 3, m = 3$)

The ideal $U_{2,3} = L_{2,3,3} = P_{2,3,3}$ is generated by six quadrics. Both $P_{2,3,2}$ and $L_{2,3,2}$ are generated by three quadrics modulo $U_{2,3}$. For $P_{2,3,2}$ these three generators can be written as

$$(2\beta_1 + \gamma_1)^2 - \mathbf{10}(\alpha_2\gamma_1 + 3\alpha_1\gamma_2),$$
$$(2\beta_1 + \gamma_1)(2\beta_2 + \gamma_2) + \mathbf{10}(\alpha_3\gamma_1 + \alpha_2\gamma_2),$$
$$(2\beta_2 + \gamma_2)^2 - \mathbf{10}(\alpha_3\gamma_2 + 3\alpha_4\gamma_1).$$

Corresponding generators of $L_{2,3,2}$: replace the coefficient **10** by **9**.

d	k	m	amb	dim	deg	gens
2	2	1	2	1	2	1
2	2	≥ 2	3	2	2	1
2	3	1	3	1	3	3
2	3	2	7	3	6	9
2	3	≥ 3	7	4	4	6
2	4	1	2	1	4	6
2	4	2	14	3	24	55
2	4	3	15	5	$192^{\mathcal{P}}, 64^{\mathcal{L}}$	$(33^{\mathcal{P}}, 34^{\mathcal{L}})$, $(0^{\mathcal{P}}, 3^{\mathcal{L}})$, ?
2	4	≥ 4	15	7	12	33
3	2	1	5	2	4	6
3	2	2	8	4	6	9
3	2	≥ 3	8	5	4	6
3	3	1	9	2	9	27
3	3	2	25	5	90	162
3	3	3	26	8	$756^{\mathcal{P}}, 396^{\mathcal{L}}$	$(83^{\mathcal{P}}, 91^{\mathcal{L}})$, ?

Invariants of the ideals $P_{d,k,m}$, $L_{d,k,m}$

Recall: Universal Varieties

d	k	amb	dim	deg	gens
2	2	3	2	2	1
2	3	7	4	4	6
2	4	15	7	12	33
2	5	31	13	40	150
3	2	8	5	4	6
3	3	26	13	24	81
3	4	80	31	672	954
4	2	15	9	8	20
4	3	63	29	200	486
5	2	24	14	16	50

Invariants of the ideal $U_{d,k}$ that defines the universal variety $\mathcal{U}_{d,k}$

Conclusion

We got a little taste of *Applied Algebraic Geometry* and *Nonlinear Algebra*. For related cool topics, check out the *SIAGA* and the *Algebraic Statistics* (MSP) journals:

谢谢*!*