Conserved currents for the sine-Gordon model, their renormalizability and summability in pAQFT

Author: Fabrizio Zanello

Institute: University of Potsdam

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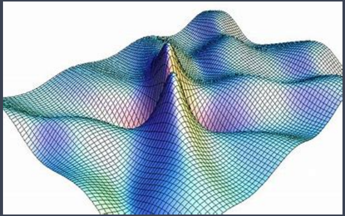
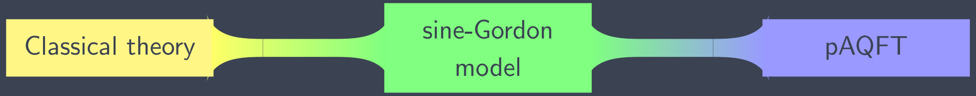


The sine-Gordon model in perspective I



Classical theory

- Spacetime: 2-D Minkowski space $\mathbb{M}_2 = (\mathbb{R}^2, \eta)$, with signature $(+,-)$. Light-cone coordinates (τ, ξ) are expressed in terms of cartesian coordinates $(t, \vec{x})$ by

$$\tau = \frac{1}{2}(\vec{x} + t), \qquad \xi = \frac{1}{2}(\vec{x} - t).$$

- Configurations: $\varphi \in \mathscr{E}(\mathbb{M}_2) := \Gamma^\infty(\mathbb{M}_2 \leftarrow E = \mathbb{M}_2 \times \mathbb{R}) = C^\infty(\mathbb{M}_2)$.
- Lagrangian: horizontal 2-form on $J^1 E$ with scalar density given by

$$L = L_0 + L_{\text{int}} = \frac{1}{2}\left[(\partial_t \varphi)^2 - (\partial_{\vec{x}}\varphi)^2\right] + \cos(a\varphi) \quad \longleftrightarrow \quad L = \varphi_\tau \varphi_\xi + \cos(a\varphi), \quad a > 0.$$

- Euler-Lagrange equation: also called sine-Gordon equation

$$-\Box\varphi - a\sin(a\varphi) = -\left(\partial_t^2 - \partial_{\vec{x}}^2\right)\varphi - a\sin(a\varphi) = 0 \quad \longleftrightarrow \quad \varphi_{\xi\tau} - a\sin(a\varphi) = 0.$$

Remark: Subscripts $_\tau$ and $_\xi$ indicate partial derivation.

Fact: From the theory of integrable systems, it is well-known that the sine-Gordon model admits an infinite number of on-shell conserved currents.

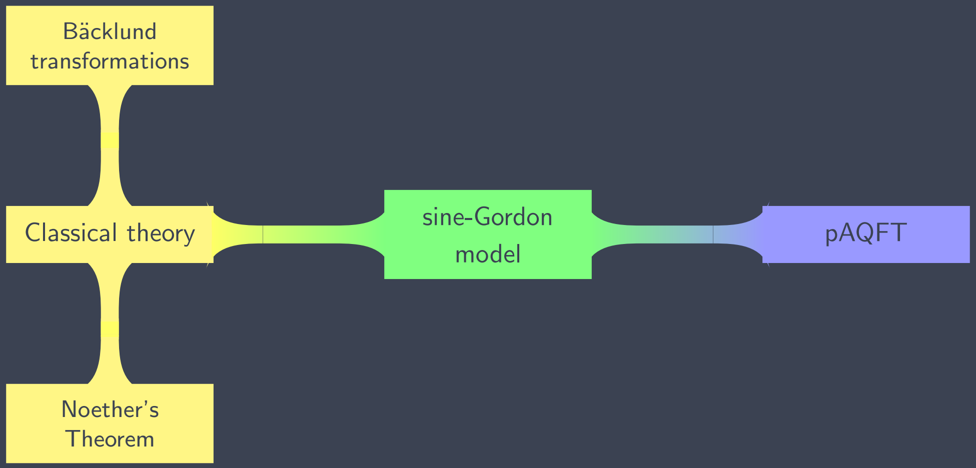
Definition

An on-shell conserved current is (in this setting) a horizontal 1-form $\rho = \rho_1 d\tau + \rho_2 d\xi$ on $J^k E$, for some $k \in \mathbb{N}$, such that

$$d\left((j^k\varphi)^*\rho\right) = 0, \tag{1}$$

whenever φ is a solution of the sine-Gordon equation. Equation (1) is called an on-shell conservation law.

The sine-Gordon model in perspective II



Bäcklund transformations

Definition

$\varphi' \in \mathscr{E}(\mathbb{M}_2)$ is obtained from a given $\varphi \in \mathscr{E}(\mathbb{M}_2)$ by a Bäcklund transformation B_α of parameter $\alpha \in \mathbb{R}$, in notation $\varphi' = B_\alpha \varphi$, if φ' satisfies the parametric system of first order PDEs:

$$\frac{1}{2}(\varphi' + \varphi)_\xi = \frac{1}{\alpha} \sin\left[\frac{a}{2}(\varphi' - \varphi)\right] \qquad (2) \qquad\qquad \frac{1}{2}(\varphi' - \varphi)_\tau = \alpha \sin\left[\frac{a}{2}(\varphi' + \varphi)\right]. \qquad (3)$$

Remark: Bäcklund transformations relate solutions of the sine-Gordon equation!

Definition

$\varphi' \in \mathscr{E}(\mathbb{M}_2)$ is obtained from a given $\varphi \in \mathscr{E}(\mathbb{M}_2)$ by an extended Bäcklund transformation $\hat{B}_\alpha$ of parameter $\alpha \in \mathbb{R}$, in notation $\varphi' = \hat{B}_\alpha \varphi$, if φ' satisfies (2).

The higher conserved currents

Extended Bäcklund transformations can be interpreted as "Lagrangian symmetries". The application of Noether's Theorem yields a family of on-shell conserved currents.

Proposition

The components of the on-shell conserved currents $s^N = s_1^N d\tau + s_2^N d\xi$, $N \in \mathbb{N}$, have the form:

$$s_2^N = 2\sum_{\mu=0}^{N}(-1)^{\mu}\left(\frac{a}{2}\right)^{2(\mu+1)} \sum_{\substack{n_0,\dots,n_{2(N-\mu)}\geq 0 \\ n_0+\cdots+n_{2(N-\mu)}=2(\mu+1) \\ 1\cdot n_1+\cdots+2(N-\mu)\cdot n_{2(N-\mu)}=2(N-\mu)}} \frac{A_1^{n_0}\cdots A_{2(N-\mu)+1}^{n_{2(N-\mu)}}}{n_0!\cdots n_{2(N-\mu)}!}.$$

The higher conserved currents

Proposition

$$s_1^N = -\left[2\sum_{\beta=1}^{N}(-1)^{\beta}\left(\frac{a}{2}\right)^{2\beta}\sum_{\substack{n_1,\ldots,n_{2N}\geq 0\\ n_1+\cdots+n_{2N}=2\beta\\ 1\cdot n_1+\cdots+2N\cdot n_{2N}=2N}}\frac{A_1^{n_1}\cdots A_{2N}^{n_{2N}}}{n_1!\cdots n_{2N}!}\right]\cos(a\varphi)$$

$$-\left[2\sum_{\beta=0}^{N-1}(-1)^{\beta+1}\left(\frac{a}{2}\right)^{2\beta+1}\sum_{\substack{n_1,\ldots,n_{2N}\geq 0\\ n_1+\cdots+n_{2N}=2\beta+1\\ 1\cdot n_1+\cdots+2N\cdot n_{2N}=2N}}\frac{A_1^{n_1}\cdots A_{2N}^{n_{2N}}}{n_1!\cdots n_{2N}!}\right]\sin(a\varphi),$$

where the coefficient of $\sin(a\varphi)$ is defined only for $N \geq 1$.

A notion of degree

$$\begin{cases} s_1^0 = -2\cos(a\varphi), \\ s_2^0 = \varphi_\xi^2, \end{cases} \qquad \begin{cases} s_1^1 = \varphi_\xi^2\cos(a\varphi) + \frac{2}{a}\varphi_{\xi\xi}\sin(a\varphi), \\ s_2^1 = \frac{1}{4}\varphi_\xi^4 + \frac{2}{a^2}\varphi_\xi\varphi_{\xi\xi\xi} + \frac{1}{a^2}\varphi_{\xi\xi}^2. \end{cases}$$

Definition

Consider $\varphi \in C^\infty(\mathbb{M}_2)$. Assign a degree to the k-th derivative w.r.t. ξ, by:

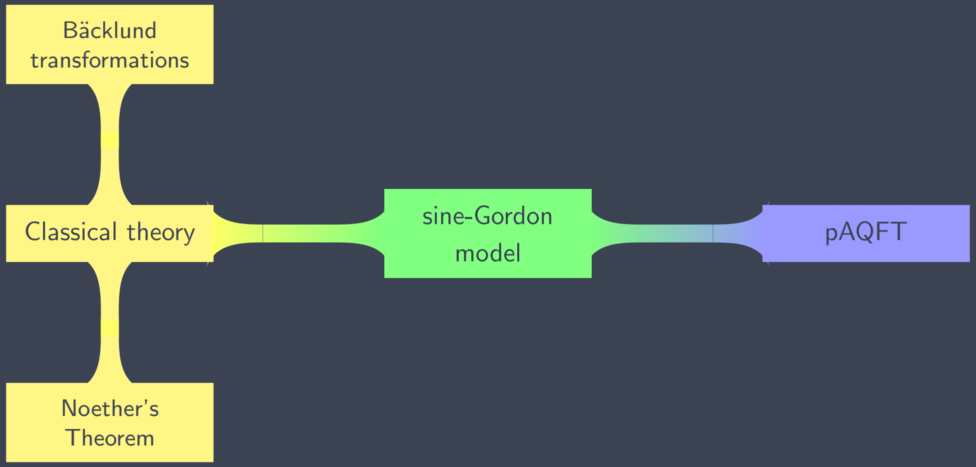
$$\deg(\varphi_{k\xi}) = k, \quad \forall k \in \mathbb{N}.$$

Extend to monomials in the derivatives of φ by additivity. A polynomial in the derivatives of φ is homogeneous of degree d if all its terms have degree d.

Proposition

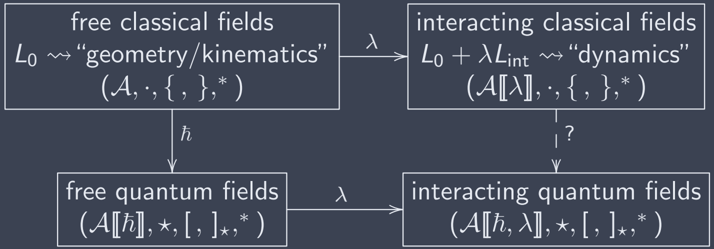
The components of s^N have homogeneous degrees $\deg(s_1^N) = 2N$ and $\deg(s_2^N) = 2(N+1)$.

The sine-Gordon model in perspective III



The general philosophy of pAQFT

A classical field theory is essentially described by its Lagrangian $L = L_0 + L_{\text{int}}$.



- $\hbar$-deformation $\longrightarrow$ (Formal) Deformation quantization.
- λ-deformation $\longrightarrow$ Perturbation.

(Formal) Deformation quantization

free classical fields
$\mathcal{A} = \{F : \mathscr{E}(\mathbb{M}_2) \to \mathbb{C},\ \mu\text{causal}\}$

$\downarrow \hbar$

free quantum fields
$\mathcal{A}[\![\hbar]\!] = \{\sum_{n=0}^{\infty} A_n \hbar^n \,|\, A_n \in \mathcal{A}\}$

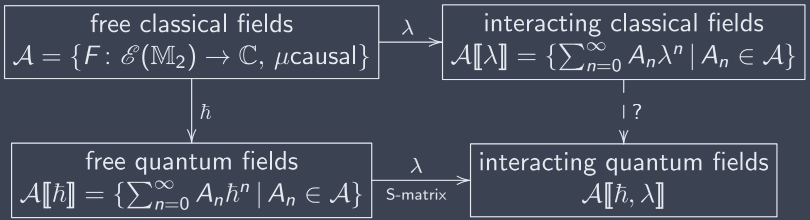
- Classical product: commutative
  $\forall F, G \in \mathcal{A} \to F \cdot G \in \mathcal{A}$
  $$(F \cdot G)[\varphi] = F[\varphi]G[\varphi], \qquad \varphi \in \mathscr{E}(\mathbb{M}_2).$$
- Star product: non-commutative
  $\forall F, G \in \mathcal{A} \to F \star G \in \mathcal{A}[\![\hbar]\!]$
  $$F \star G \xrightarrow[\hbar \to 0]{} F \cdot G.$$

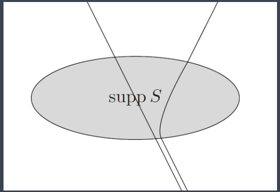
Microlocal analysis: imposing special requirements on the wavefront sets, define

$$(F \star G)[\varphi] = \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \left\langle F^{(n)}[\varphi], (W^{\otimes n}) * G^{(n)}[\varphi] \right\rangle, \qquad W \in \mathscr{D}'(\mathbb{M}_2).$$

Fact: Deformation of Poisson $*$-algebras $(\mathcal{A}[\![\hbar]\!], \star, [\ ,\ ]_\star, ^*) \xrightarrow[\hbar \to 0]{} (\mathcal{A}, \cdot, \{\ ,\ \}, ^*)$.

Perturbation





- S-matrix: $S(L_{\text{int}}) \in \mathcal{A}[\![\lambda]\!]((\hbar))$ encodes the notion of "Heisenberg interaction picture" $F(\varphi_{\text{ret}}) = (F)_{\text{ret}}(\varphi)$ in a perturbative way, by Bogoliubov formula:

$$F \to (F)_{\text{ret}} = \frac{\hbar}{i}\frac{d}{d\kappa}\left(S(\lambda L_{\text{int}})^{\star-1} \star S(\lambda L_{\text{int}} + \kappa F)\right)\Big|_{\kappa=0}.$$

Interaction picture: time-ordered products

The time-ordered products are multilinear maps $T_n\colon \mathcal{A}^{\otimes n} \to \mathcal{A}[\![\hbar]\!]$, that satisfy certain (physically motivated) axioms, and are used to define:

$$S(\lambda L_{\mathrm{int}}) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{i\lambda}{\hbar}\right)^n T_n\big(L_{\mathrm{int}}^{\otimes n}\big).$$

If L_{int} is a regular field, then $T_n\big(L_{\mathrm{int}}^{\otimes n}\big) = \underbrace{L_{\mathrm{int}} \star_{\Delta^F} \cdots \star_{\Delta^F} L_{\mathrm{int}}}_{n-\text{times}},$ where

$$(F \star_{\Delta^F} G)[\varphi] = \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \left\langle F^{(n)}[\varphi], (\Delta^F)^{\otimes n} * G^{(n)}[\varphi] \right\rangle .$$

Problem: What happens if L_{int} is not a regular field?

The renormalization problem

For a general interaction Lagrangian $L_{\text{int}} \in \mathcal{A}$, one could naively try to compute

$$\check{T}_n\big(L_{\text{int}}(x_1) \otimes \cdots \otimes L_{\text{int}}(x_n)\big) \text{ “ = ” } \underbrace{L_{\text{int}}(x_1) \star_{\Delta^F} \cdots \star_{\Delta^F} L_{\text{int}}(x_n)}_{n-\text{times}}$$

$$\downarrow$$

$$(\Delta^F)^{n_{12}}(x_1 - x_2)(\Delta^F)^{n_{13}}(x_1 - x_3) \cdots (\Delta^F)^{n_{23}}(x_2 - x_3) \cdots$$

These products are defined, by Hörmander's sufficient criterion, only on:

$$\check{\mathbb{M}}_2^n = \{\,(x_1, \ldots, x_n) \in \mathbb{M}_2^n \mid x_i \neq x_j,\ \forall 1 \leq i < j \leq n\,\}.$$

Fact: Renormalization is the inductive (on $n \geq 1$) construction of $T_n\big(L_{\text{int}}^{\otimes n}\big)$:

- by inductive hypotesis (and axioms), $\check{T}_n\big(L_{\text{int}}^{\otimes n}\big)$ is defined on $\mathbb{M}_2^n \setminus \Delta_n$;
- to complete the inductive step, $\check{T}_n$ is extended to the whole $\mathbb{M}_2^n$.

Scaling degree of distributions

Definition

The scaling degree of $t \in \mathscr{D}'(\mathbb{R}^d \setminus \{0\})$ in 0 is: $\quad \mathrm{sd}(t) = \inf\{\, r \in \mathbb{R} \mid \lim_{\rho \downarrow 0} \rho^r t(\rho x) = 0 \,\}$.

Theorem (Brunetti, Fredenhagen, Epstein, Glaser,...)

Let $t^0 \in \mathscr{D}'(\mathbb{R}^d \setminus \{0\})$. Then:

- If $\mathrm{sd}(t^0) < d \quad \Rightarrow \quad \exists!$ extension $t \in \mathscr{D}'(\mathbb{R}^d)$ s.t. $\mathrm{sd}(t) = \mathrm{sd}(t^0)$.
- If $d \leq \mathrm{sd}(t^0) < \infty \quad \Rightarrow \quad$ There are several extensions $t \in \mathscr{D}'(\mathbb{R}^d)$ s.t. $\mathrm{sd}(t) = \mathrm{sd}(t^0)$. Given a particular extension $\bar{t}$, the general extension t is of the form

$$t = \bar{t} + \sum_{|a| \leq \mathrm{sd}(t_0) - d} C_a \partial^a \delta, \qquad C_a \in \mathbb{C}.$$

Renormalizability of interacting fields

Unrenormalized retarded functionals are given by Bogoliubov formula, on $\mathbb{M}_2^{n+1} \setminus \Delta_{n+1}$:

$$(\check{F})_{\text{ret}} = \frac{\hbar}{i} \frac{d}{d\kappa} \left(\check{S}(\lambda L_{\text{int}})^{\star -1} \star \check{S}(\lambda L_{\text{int}} + \kappa F) \right) \Big|_{\kappa=0} = \sum_{n=0}^{\infty} \frac{\lambda^n}{n! \hbar^n} \check{R}_n(L_{\text{int}}^{\otimes n}, F).$$

The unrenormalized retarded products $\check{R}_n$ are then inductively extended to $\mathbb{M}_2^{n+1}$, $\forall n \geq 1$.

Definition

Consider $(\check{F})_{\text{ret}}$ as above. Let $N(L_{\text{int}}, F, \cdot) \colon \mathbb{N} \to \mathbb{N}$ be defined as: $N(L_{\text{int}}, F, 0) = 0$,

$$N(L_{\text{int}}, F, n) = \max \left\{ 0, \, \text{sd} \left(\check{R}_n(L_{\text{int}}^{\otimes n}, F) \right) - 2n - N(L_{\text{int}}, F, n-1) + 1 \right\}, \quad n \geq 1.$$

The unrenormalized retarded functional $(\check{F})_{\text{ret}}$ is:

(a) **renormalizable by power counting** if $N(L_{\text{int}}, F, \cdot)$ is bounded;

(b) **super-renormalizable by power counting** if the number of non-vanishing values of $N(L_{\text{int}}, F, \cdot)$ is finite.

The sine-Gordon model in pAQFT

- Interaction Lagrangian:

$$L_{\text{int}} = \cos(a\varphi) = \frac{1}{2}\left(e^{ia\varphi} + e^{-ia\varphi}\right) \quad \xrightarrow{\text{pAQFT}} \quad \frac{1}{2}\left(V_a + V_{-a}\right) \in \mathcal{A}.$$

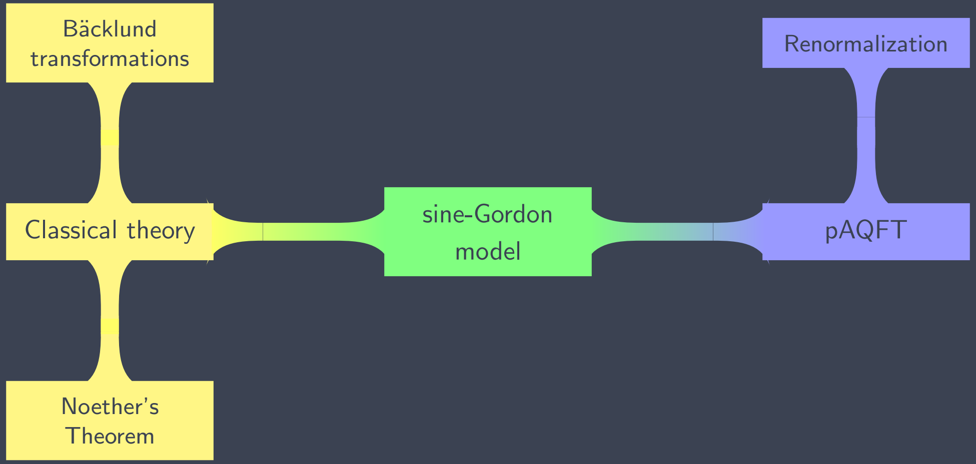
- Vertex operators: $V_a\colon \left(g \in C_c^\infty(\mathbb{M}_2),\ \varphi \in \mathscr{E}(\mathbb{M}_2)\right) \quad \mapsto \quad V_a(g)[\varphi] = \int_{\mathbb{M}_2} e^{ia\varphi} g.$
- S-matrix:

$$S(\lambda L_{\text{int}}) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{i\lambda}{2\hbar}\right)^n T_n(\underbrace{V_{\pm a} \otimes \cdots \otimes V_{\pm a}}_{n-\text{times}}) \in \mathcal{A}[\![\lambda]\!]((\hbar)).$$

- Observables: components of $s^N = (s_1^N)d\tau + (s_2^N)d\xi$ act in the following way

$$s_{1,2}^N\colon \left(g \in C_c^\infty(\mathbb{M}_2),\ \varphi \in \mathscr{E}(\mathbb{M}_2)\right) \mapsto s_{1,2}^N(g)[\varphi] = \int_{\mathbb{M}_2} s_{1,2}^N(\varphi)\, g.$$

The sine-Gordon model in perspective IV



First main result

We consider the unrenormalized retarded components

$$(\check{s}^N_{1,2})_{\mathrm{ret}} = \frac{\hbar}{i}\frac{d}{d\kappa}\left(\check{S}(\lambda L_{\mathrm{int}})^{\star -1} \star \check{S}(\lambda L_{\mathrm{int}} + \kappa s^N_{1,2})\right)\Big|_{\kappa=0} = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!\hbar^n}\check{R}_n\left(L_{\mathrm{int}}^{\otimes n}, s^N_{1,2}\right).$$

Theorem

The scaling degree of the unrenormalized retarded products above is uniformly bounded by the degree of the components. More specifically, for every $n \geq 1$ it holds:

$$\mathrm{sd}\left(\check{R}_n\left(L_{\mathrm{int}}^{\otimes n}, s^N_1\right)\right) = \deg(s^N_1) = 2N,$$
$$\mathrm{sd}\left(\check{R}_n\left(L_{\mathrm{int}}^{\otimes n}, s^N_2\right)\right) = \deg(s^N_2) = 2(N+1).$$

Super-renormalizability

Corollary

The unrenormalized retarded components $(\check{s}^N_{1,2})_{\mathrm{ret}}$ are super-renormalizable by power counting.

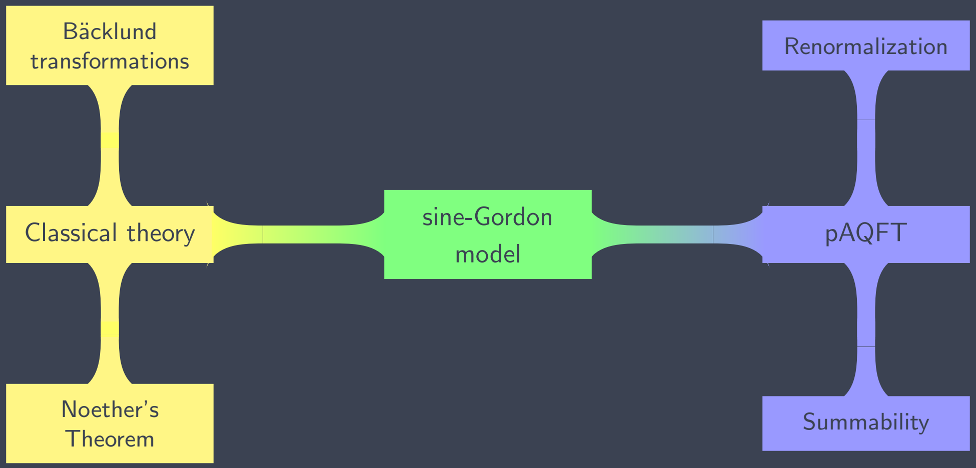
Idea: The uniform bound on the scaling degree of the retarded products implies:

$$\begin{aligned} N(L_{\mathrm{int}}, s_1^N, n) &= \mathrm{sd}\left(\check{R}_n(L_{\mathrm{int}}^{\otimes n}, s_1^N)\right) - 2n - N(L_{\mathrm{int}}, s_1^N, n-1) + 1 \\ &= 2N - 2n - N(L_{\mathrm{int}}, s_1^N, n-1) + 1 \leq 2N - 2n + 1, \end{aligned}$$

and

$$\begin{aligned} N(L_{\mathrm{int}}, s_2^N, n) &= \mathrm{sd}\left(\check{R}_n(L_{\mathrm{int}}^{\otimes n}, s_2^N)\right) - 2n - N(L_{\mathrm{int}}, s_2^N, n-1) + 1 \\ &= 2(N+1) - 2n - N(L_{\mathrm{int}}, s_2^N, n-1) + 1 \leq 2(N+1) - 2n + 1. \end{aligned}$$

The sine-Gordon model in perspective V



Gaussian states and summability of the S-matrix

Definition

Fix a configuration $\varphi \in \mathscr{E}(\mathbb{M}_2)$. The Gaussian state ω_φ is the evaluation map:

$$\begin{aligned} \omega_\varphi \colon \mathcal{A}[\![\hbar,\lambda]\!] &\to \mathbb{C}[\![\hbar,\lambda]\!] \\ F &\mapsto \omega_\varphi(F) = F[\varphi]. \end{aligned}$$

Theorem (Bahns, Rejzner)

Under proper technical conditions, there exists a constant $C = C(\gamma, f)$, $f \in \mathscr{E}(\mathbb{M}_2^n)$, such that for all n, the expectation value of the n-th order contribution to the S-matrix of sine-Gordon model in the state ω_φ satisfies the following inequality:

$$|\omega_\varphi\left(S_n(L_{\text{int}})(f)\right)| = |S_n(L_{\text{int}})(f)[\varphi]| \leq \frac{[\frac{n}{2}]C^n}{\left([\frac{n}{2}]!\right)^{1-\frac{1}{\gamma}}}$$

Second main result

Consider the renormalized retarded components $(s^N_{1,2})_{\text{ret}} = \sum_{n=0}^{\infty} \lambda^n \underbrace{\frac{1}{n!\hbar^n} R_n(L_{\text{int}}^{\otimes n}, s^N_{1,2})}_{\mathcal{R}_n(L_{\text{int}}, s^N_{1,2})}$.

Theorem

Under the same hypothesis as above, there exist two pairs of constants $\mathcal{K}^{s_1}_{\gamma,f,a\hbar,N}$, $\mathcal{C}^{s_1}_{\gamma,f}$ and $\mathcal{K}^{s_2}_{\gamma,f,a\hbar,N}$, $\mathcal{C}^{s_2}_{\gamma,f}$ such that for all $n \geq 1$, the expectation values $\omega_\varphi\left(\mathcal{R}_n(L_{\text{int}}, s^N_{1,2})(f)\right)$ satisfy the inequalities:

$$\left|\omega_\varphi\left(\mathcal{R}_n(L_{\text{int}}, s^N_1)(f)\right)\right| = \left|\mathcal{R}_n(L_{\text{int}}, s^N_1)(f)[\varphi]\right| \leq \mathcal{K}^{s_1}_{\gamma,f,a\hbar,N} \frac{(n+1)^2 n^{2N} \left(\mathcal{C}^{s_1}_{\gamma,g}\right)^n}{\left(\left[\frac{n}{2}\right]!\right)^{1-\frac{1}{\gamma}}},$$

$$\left|\omega_\varphi\left(\mathcal{R}_n(L_{\text{int}}, s^N_2)(f)\right)\right| = \left|\mathcal{R}_n(L_{\text{int}}, s^N_2)(f)[\varphi]\right| \leq \mathcal{K}^{s_2}_{\gamma,f,a\hbar,N} \frac{\left[\frac{n}{2}\right] n^{2N} \left(\mathcal{C}^{s_2}_{\gamma,f}\right)^n}{\left(\left[\frac{n}{2}\right]!\right)^{1-\frac{1}{\gamma}}},$$

Some future research directions

- Conservation and involutivity: Classically, the higher currents are conserved on-shell. Also, they are in involution w.r.t. the Peierl's bracket. In pAQFT, can renormalization be done in such a way to preserve conservation and involutivity?

  Idea: Adopt the point of view of Bahns and Wrochna in the analysis of the extensions of distributions satisfying a given set of PDEs.

- Symmetries: Is it possible to formulate the mechanism of production of the (classical) higher currents in a more general mathematical framework?

  Idea: Noether's Theorem for actions of Lie groupoids of symmetries, multisymplectic geometry . . .

"The art of doing mathematics consists in finding that special case which contains all the germs of generality."

David Hilbert