On 2nd-stage quantization of quantum cluster algebras

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2-parameters quantization of $GL_r(2)$

We know from [N.Jing-M.Liu, 2014] that the 2-parameters quantum coordinate algebra $Fun_{\mathbb{C}}(GL_{r,s}(2))$ is generated by t_{ij}, $det_{r,s}^{\pm 1}$ with relations:

$$t_{11}t_{12} = r^{-1}t_{12}t_{11}, \quad t_{11}t_{21} = st_{21}t_{11}, \quad t_{21}t_{22} = r^{-1}t_{22}t_{21},$$

$$t_{12}t_{22} = st_{22}t_{12}, \quad t_{12}t_{21} = rst_{21}t_{12}, \quad t_{11}t_{22} - t_{22}t_{11} = (s-r)t_{21}t_{12},$$

$$det_{r,s}det_{r,s}^{-1} = det_{r,s}^{-1}det_{r,s} = 1, \quad det_{r,s}t_{ij} = (rs)^{i-j}t_{ij}(det_{r,s}),$$

$$det_{r,s} = t_{11}t_{22} - st_{21}t_{12} = t_{22}t_{11} - rt_{21}t_{12} = t_{11}t_{22} - r^{-1}t_{12}t_{21}.$$

How to do 2-parameters quantization of $SL_r(2)$?

If we consider this 2-parameters quantum algebra from $GL_{r,s}(2)$ to $SL_{r,s}(2)$, then we have $det_{r,s} = 1$.

Replacing it into the relation $det_{r,s} t_{ij} = (rs)^{i-j} t_{ij} (det_{r,s})$, we get $r = s^{-1}$, that is, 2-parameter quantum algebra $Fun(GL_{r,s}(2))$ is degenerated into one parameter quantum algebra $Fun(GL_r(2))$.

It means that this method of 2-parameters quantization $Fun(GL_{r,s}(2))$ of $Fun(GL_r(2))$ has no effect on the special quantum linear group $SL_r(2)$.

We will finish this task via the so-called 2nd-stage quantization of quantum cluster algebras.

Definition of quantum cluster algebras

- For $n \leqslant m \in \mathbb{N}$, denote T_n the **n-regular tree** with vertices $t \in T_n$. Let $\Lambda(t) = (\lambda_{ij})_{m \times m}$ be a skew-symmetric integer matrix.

- Let $\{e_i\}_{i=1}^{m}$ be the standard basis for $\mathbb{Z}^m$.
  Define a skew-symmetric bilinear form $\Lambda_t : \mathbb{Z}^m \times \mathbb{Z}^m \to \mathbb{Z}$ satisfying that

$$\Lambda_t(e, f) = \sum_{i,j=1}^{m} a_i b_j \Lambda_t(e_i, e_j) = \sum_{i,j=1}^{m} a_i b_j \lambda_{ij},$$

  where $e = \sum_{i=1}^{m} a_i e_i, f = \sum_{j=1}^{m} b_j e_j$.

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Definition of quantum cluster algebras

- Give a set of variables

$$\tilde{X}(t) = \{X_t^{e_1}, \cdots, X_t^{e_n}, X^{e_{n+1}}, \cdots, X^{e_m}\}$$

which is called the **extended cluster** at t, where $X_t^{e_i}, i \in [1, n]$ are called the **cluster variables** at t while $X^{e_i}, i \in [n+1, m]$ are called **frozen variables.**

- For the rational Laurent polynomial ring $\mathbb{Q}[q^{\pm\frac{1}{2}}]$, define a $\mathbb{Q}[q^{\pm\frac{1}{2}}]$-algebra $\mathcal{T}_t$ generated by $\tilde{X}(t)$ satisfying the following relations:

$$X_t^{e_i} X_t^{e_j} = q^{\frac{1}{2}\lambda_{ij}} X_t^{e_i+e_j}, \forall i, j \in [1, m]$$

We call $\mathcal{T}_t$ the **quantum torus** at t.
Denoted by $\mathscr{F}_q$ the skew-field of fractions of $\mathcal{T}_{t_0}$.

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Definition of quantum cluster algebras

- In general, $\forall e \in \mathbb{Z}^m$, let X_t^e denote the variable corresponding to e.
- Due to the bilinearity of Λ_t and e generated by $\{e_i | i \in [1, m]\}$, we obtain that

$$X_t^e X_t^f = q^{\frac{1}{2}\Lambda_t(e,f)} X_t^{e+f} \tag{1}$$

Definition of quantum cluster algebras

- Let

$$\tilde{B}(t) = \begin{pmatrix} B(t)_{n\times n} \\ B_1(t)_{(m-n)\times n} \end{pmatrix} = (b_{ij})_{m\times n}$$

be an integer matrix called the **extended exchange matrix** at t, such that $\exists$ diagonal matrix

$$D = \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{pmatrix}$$

$d_i \in \mathbb{Z}, \forall i \in [1, n]$ satisfying

$$\tilde{B}(t)^T \Lambda(t) = \begin{pmatrix} D & O \end{pmatrix}_{n\times m} \quad (2)$$

by this, $B(t)$ is a skew-symmetrizable matrix.

Then $B(t)$ is called the **exchange matrix** at t and $(\tilde{B}(t), \Lambda)$ is called a **compatible pair**.

Definition of quantum cluster algebras

Definition

[BZ] (a)Give a fixed $t_0 \in T_n$, denote $\Sigma(t_0) = (\tilde{X}(t_0), \tilde{B}(t_0), \Lambda(t_0))$ an initial quantum seed.
(b)Let $t \in T_n$ be an adjacent vertex of t_0, i.e. $t - t_0$ is an edge in T_n labeled $k \in [1, n]$. Let $b_k(t_0)$ be the k-th column of $\tilde{B}(t_0)$. Define the **mutation** μ_k at direction k satisfying that

$$X_t^{e_k} = \mu_k(X_{t_0}^{e_k}) = X_{t_0}^{-e_k+[b_k(t_0)]_+} + X_{t_0}^{-e_k+[-b_k(t_0)]_+}$$

where $[a]_+ = max\{a, 0\}$ for $a \in \mathbb{R}$. Then,

$$\tilde{X}(t) = (\tilde{X}(t_0) \backslash \left\{X_{t_0}^{e_k}\right\}) \cup \left\{X_t^{e_k}\right\}.$$

$$\tilde{B}(t) = \mu_k(\tilde{B}(t_0))$$

satisfying that

Definition of quantum cluster algebras

$$b_{ij}(t) = \begin{cases} -b_{ij}(t_0) & \textit{if} \quad i = k \textit{ or } \quad j = k \\ b_{ij}(t_0) + sgn(b_{ik}(t_0))[b_{ik}(t_0)b_{kj}(t_0)]_+ & \textit{otherwise} \end{cases}$$

And, $\Lambda(t) = (\lambda_{ij}(t))_{m\times m}$ where

$$\lambda_{ij}(t) = \begin{cases} -\lambda_{kj}(t_0) + \sum_{l=1}^{m} [b_{lk}(t_0)]_+ \lambda_{lj}(t_0) & \textit{if} \quad i = k \\ -\lambda_{ji}(t) & \textit{if} \quad j = k \\ \lambda_{ij}(t_0) & \textit{otherwise} \end{cases}$$

Also, write $\Lambda(t) = \mu_k(\Lambda(t_0))$.

Definition of quantum cluster algebras

[BZ] Given seeds $\Sigma(t) = (\tilde{X}(t), \tilde{B}(t), \Lambda(t))$ at $t \in T_n$, if $\Sigma(t)$ and $\Sigma(t')$ can do mutation to each other for any adjacent pair of vertices $t - t'$ in T_n, then the $\mathbb{Q}[q^{\pm\frac{1}{2}}]$-subalgebra of $\mathscr{F}_q$ generated by all variables in $\bigcup_{t\in T_n} \tilde{X}(t)$ is called the **quantum cluster algebra** $A_q(\Sigma)$ or simply A_q associated with Σ.

Here, the matrix $\Lambda(t)$ at t is called the **first deformation matrix** of A_q.

有意思的是，这个矩阵$\Lambda(t)$恰是量子丛代数A_q的对应的非量子丛代数$\mathcal{A}$在t-点的Poisson代数结构的Poisson矩阵.

而这给我们做二阶量子化提供了思路.

Compatible Poisson structures on A_q

Definition

- For a quantum cluster algebra A_q with Poisson structure $\{-,-\}$, a cluster $X = (X_1, \cdots, X_m)$ is said to be **log-canonical** if $\{X_i, X_j\} = \omega_{ij} X^{e_i+e_j}$, where $\omega_{ij} \in \mathbb{Q}[q^{\pm\frac{1}{2}}], \forall i, j \in [1, m]$.
- A Poisson structure $\{-,-\}$ on A_q is called **compatible** with A_q if all clusters in A_q are log-canonical with respect to $\{-,-\}$.
- $\Omega = (\omega_{ij})_{m\times m}$ is called the **Poisson matrix** of the extended cluster X.

- In the following, we always assume Poisson structures are nontrivial, that is, $\omega_{ij} \neq 0$ for some i, j.

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Mutation of Poisson matrices on A_q

Lemma 1

If a Poisson structure $\{-,-\}$ is compatible on A_q with $\{X_i, X_j\} = \omega_{ij} X^{e_i+e_j}, \forall i,j \in [1,m]$, then $\forall j \neq k$, where $j \in [1,m]$ while $k \in [1,n]$, we have

$$
\begin{aligned}
H &= \sum_{b_{tk}>0} \Big(\omega_{tj} q^{\frac{1}{2}} \sum_{h=1}^{[b_{tk}]_+} q^{\sum_{i=t}^{m}([b_{ik}]_+ - \delta_{ik})\lambda_{ji} - h\lambda_{jt}}\Big) - \omega_{kj} q^{\frac{1}{2}\lambda_{kj} + \sum_{i=k+1}^{m} \lambda_{ji}[b_{ik}]_+} \\
&= \sum_{b_{tk}<0} \Big(\omega_{tj} q^{\frac{1}{2}} \sum_{h=1}^{[-b_{tk}]_+} q^{\sum_{i=t}^{m}([-b_{ik}]_+ - \delta_{ik})\lambda_{ji} - h\lambda_{jt}}\Big) - \omega_{kj} q^{\frac{1}{2}\lambda_{kj} + \sum_{i=k+1}^{m} \lambda_{ji}[-b_{ik}]_+}
\end{aligned} \tag{3}
$$

Mutation of Poisson matrices on A_q

Lemma 1 (continue)

when X' is log-canonical with respect to $\{-,-\}$, we will have mutation of Ω at direction k

$$\omega'_{ij} = \begin{cases} q^{\frac{1}{2}(\lambda_{jk} - \sum_{t=1}^{m} [b_{tk}]_+ \lambda_{jt})} H & \text{if} \quad i = k \\ -\omega'_{ki} & \text{if} \quad j = k \\ \omega_{ij} & \text{otherwise} \end{cases}$$

where H denotes the left or right side of (3).

Compatible Poisson structures on A_q

The following is an equivalent condition for a poisson structure to be compatible with a quantum cluster algebra.

Lemma 3

If X is log-canonical with a nontrivial Poisson structure $\{-,-\}$ and $\{X_i, X_j\} = \omega_{ij} X^{e_i+e_j}$ for any $i, j \in [1, m]$, then $\mu_k(X) = X' = \{X'_i\}$ is log-canonical with $\{-,-\}$

if and only if the following conditions hold for any $j \in [1, m], k \in [1, n], k \neq j$:

- For any $u \in [1, m]$, if $b_{uk} \neq 0$, then $\frac{\omega_{uj}}{\omega_{kj}} = \frac{q^{\frac{1}{2}\lambda_{uj}} - q^{\frac{1}{2}\lambda_{ju}}}{q^{\frac{1}{2}\lambda_{kj}} - q^{\frac{1}{2}\lambda_{jk}}}$.
- For any $u, v \in [1, m]$, if $b_{uk} b_{vk} \neq 0$, then $\frac{\omega_{uj}}{\omega_{vj}} = \frac{q^{\frac{1}{2}\lambda_{uj}} - q^{\frac{1}{2}\lambda_{ju}}}{q^{\frac{1}{2}\lambda_{vj}} - q^{\frac{1}{2}\lambda_{jv}}}$.
- $\sum\limits_{\lambda_{tj}=0} \omega_{tj} b_{tk} = 0$.

2nd-stage quantization of A_q

Denote $[a]_q = \frac{q^a - q^{-a}}{q - q^{-1}} \in \mathbb{N}(q^{\pm 1})$ for $q \in \mathbb{C}$.

Let $(\tilde{X}(t), \tilde{B}(t), \Lambda(t))$ be a seed of a quantum cluster algebra A_q at $t \in \mathbb{T}_n$ and $\{-,-\}$ a compatible poisson bracket on A_q.

Let $\Omega(t)$ be the Poisson matrix of A_q associated to the seed at t.

Define an $m \times m$ skew-symmetric matrix $W(t) = (W_{ij})$ as

$$W_{ij} = \begin{cases} \frac{\omega_{ij}\lambda_{ij}}{[\lambda_{ij}]_{q^{\frac{1}{2}}}} & \lambda_{ij} \neq 0 \\ \omega_{ij} & \lambda_{ij} = 0 \end{cases} \tag{4}$$

The matrix $W(t)$ is called the **2nd-stage deformation matrix** of A_q at $t \in \mathbb{T}_n$.
注意，这里$W(t)$不是直接等于$\Omega(t)$，这与一阶量子化不完全同.

2nd stage quantization of A_q

Conversely, from a 2nd-stage deformation matrices $W(t) = (W_{ij})$ of A_q, we can obtain the Poisson matrices $\Omega(t)$ of a Poisson structure of A_q via the following:

$$\omega_{ij} = \begin{cases} \frac{W_{ij}[\lambda_{ij}]_{q^{\frac{1}{2}}}}{\lambda_{ij}} & \lambda_{ij} \neq 0 \\ W_{ij} & \lambda_{ij} = 0. \end{cases}$$

In fact, any one of $W(t), \Lambda(t), \Omega(t)$ can be determined by other two ones.

2nd-stage quantization of A_q

Definition

The triple $(\tilde{B}(t), \Lambda(t), \Omega(t))$ is called **compatible** if $(\tilde{B}(t), \Lambda(t))$ is a compatible pair for a quantum cluster algebra A_q and $\Omega(t)$ is a Poisson matrix for a Poisson structure compatible with A_q associated to $(\tilde{B}(t), \Lambda(t))$.

Theorem

Let $(\tilde{X}(t), \tilde{B}(t), \Lambda(t))$ be a seed of a quantum cluster algebra A_q at $t \in \mathbb{T}_n$ and $\{-,-\}$ a compatible Poisson structure on A_q. Then the 2nd-stage deformation matrix $W(t)$ satisfies that

$$\tilde{B}(t)^T W(t) = c(D\ O),$$

that is, $(\tilde{B}(t), W(t))$ is a compatible pair,
where $c \in \mathbb{Z}[q^{\pm\frac{1}{2}}]$ and D is the skew-symmetrizer of $\tilde{B}(t)$.

2nd-stage quantization of A_q

Given a compatible triple $(\tilde{B}(t), \Lambda(t), \Omega(t))$ assigned to vertex t, as usual we define the **cluster** at $t \in \mathbb{T}$ to be a set of variables

$$\tilde{Y}(t) = \left\{ Y_t^{e_1}, Y_t^{e_2}, \cdots, Y_t^{e_n}, Y^{e_{n+1}}, \cdots, Y^{e_m} \right\}.$$

where $e_i \in \mathbb{Z}^m$ are the standard basis.

2nd-stage quantization of A_q

For $p, q \in \mathbb{C}$, let $\mathcal{T}_t$ be the $\mathbb{Z}[p^{\pm\frac{1}{2}}, q^{\pm\frac{1}{2}}]$-algebra generated by $\tilde{Y}(t)$ satisfying the relation

$$Y_t^{e_i} Y_t^{e_j} = p^{\frac{1}{2}w_{ij}} q^{\frac{1}{2}\lambda_{ij}} Y_t^{e_i+e_j}, \forall i, j \in [1, m]. \tag{5}$$

We call $\mathcal{T}_t$ the **II-quantum torus** at t, or say, the (p, q)-**quantum torus.**

Denote by $\mathcal{F}_{p,q}$ the skew-field of fractions of $\mathcal{T}_t$. Thus, $\mathcal{T}_t$ is a subalgebra of $\mathcal{F}_{p,q}$.

We call $\Sigma_{II}(t) = (\tilde{Y}(t), \tilde{B}(t), \Lambda(t), \Omega(t))$ a **II-quantum seed** at t for a compatible triple $(\tilde{B}(t), \Lambda(t), \Omega(t))$.

2nd-stage quantization of A_q

Let $\Sigma_{II}(t)$ and $\Sigma_{II}(t')$ be two II-quantum seeds at t and t' respectively. Denote by b_i the i-column of $\tilde{B}(t)$.
Let t and t' be adjacent vertices by an edge labeled k in $\mathbb{T}_n$.

We say that $\Sigma_{II}(t')$ is obtained from $\Sigma_{II}(t)$ by a **mutation** in direction k if
$\Sigma_{II}(t') = \mu_k(\Sigma_{II}(t)) = (\mu_k(\tilde{Y}(t)), \mu_k(\tilde{B}(t)), \mu_k(\Lambda(t)), \mu_k(\Omega(t)))$,
where

$$\tilde{Y}(t') = \mu_k(\tilde{Y}(t)) = (\tilde{Y}(t) \setminus \{Y_t^{e_k}\}) \bigcup \{\mu_k(Y_t^{e_k})\},$$

$$Y_{t'}^{e_k} = \mu_k(Y_t^{e_k}) = Y_t^{-e_k+[b_k(t)]_+} + Y_t^{-e_k+[-b_k(t)]_+}$$

while the mutations of matrices $\tilde{B}(t)$, $\Lambda(t)$ and $\Omega(t)$ are the same as those we introduced before.

2nd-stage quantization of A_q

The 2nd-stage deformation matrix $W(t)$ is determined by $\Lambda(t)$ and $\Omega(t)$ in the II-quantum seed $\Sigma_{II}(t) = (\tilde{Y}(t), \tilde{B}(t), \Lambda(t), \Omega(t))$.

Theorem

- As the mutation of $\Sigma_{II}(t)$, $\mu_k(\Sigma_{II}(t)) = (\tilde{Y}', \tilde{B}', \Lambda', \Omega')$ is also a II-quantum seed.
- Assume $W(t) = (W_{ij}(t))$, $W(t') = (W_{ij}(t'))$ and $W(t') = \mu_k(W(t))$, then

$$W_{ij}(t') = \begin{cases} -W_{kj}(t) + \sum_{l=1}^{m} [b_{lk}(t)]_+ W_{lj}(t) & \text{if } i = k \neq j \\ -W_{ik}(t) + \sum_{l=1}^{m} [b_{lk}(t)]_+ W_{il}(t) & \text{if } j = k \neq i \\ W_{ij}(t) & \text{otherwise} \end{cases}$$

This means the formula of the mutation of $W(t)$.

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This means the formula of the mutation of $W(t)$.

2nd-stage quantization of A_q

Definition

Assign II-quantum seeds $\Sigma_{II}(t)$ to every vertex t in $\mathbb{T}_n$.

Denote by $A_{p,q} = A_{p,q}(\Sigma_{II})$ the $\mathbb{Z}[p^{\pm\frac{1}{2}}, q^{\pm\frac{1}{2}}]$-subalgebra of $\mathcal{F}_{p,q}$ generated by $\bigcup_{t\in\mathbb{T}_n} \tilde{Y}(t)$, which is defined as the **2nd-stage quantization** of A_q.

We call $A_{p,q}(\Sigma_{II})$ the **2nd-stage quantized cluster algebra** associated to Σ_{II} as a $\mathbb{Z}[p^{\pm\frac{1}{2}}, q^{\pm\frac{1}{2}}]$-subalgebra of $\mathcal{F}_{p,q}$.

2nd-stage quantization of A_q

Observation 1

Assume a quantum cluster algebra A_q with the exchange matrices $\tilde{B}(t)$. Then, we have the following one-by-one correspondence:

$\{$Compatible Poisson structures of $A_q\}$
. $\longleftrightarrow$ $\{$2nd-stage Quantizations of $A_q\}$
$=$ $\{$2nd stage quantized cluster algebras $A_{p,q}$ $\}$
via
. $\{$Poisson matrices of $A_q\}$ $\longleftrightarrow$ $\{$2nd-stage deformation matrices of $A_{p,q}\}$.

2nd-stage quantization of A_q

Observation 2

Therefore, we have the following two ways of quantization induced by a triple $(\tilde{B}, \Lambda, W)$:

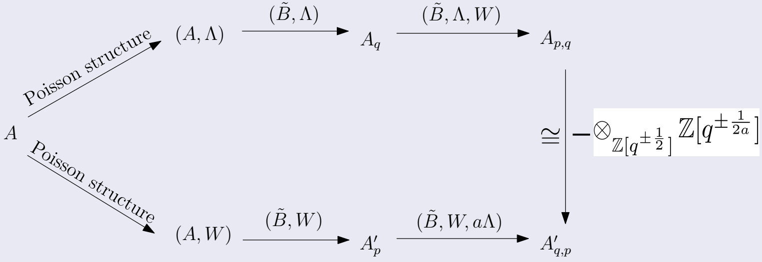

Figure: Two ways of quantization

In case A_q without coefficients

In the sequel, assume A_q is without coefficients.
In this case, the formula (2) becomes to that

$$B(t)^T \Lambda(t) = D_{n \times n} \tag{6}$$

Due to this, we obtain the invertibility of $B(t)$ and $\Lambda(t)$ for any t, which will be needed in our following proof. This is the reason we need the condition for A_q to be without coefficients.

Compatible Poisson structures on A_q without coefficients

Assume A_q is a quantum cluster algebra without coefficients and (X, B, Λ) is its initial seed.

Suppose B has decomposition $B = B_1 \bigsqcup B_2 \bigsqcup \cdots \bigsqcup B_s$, where B_i is indecomposable.

Then as a Poisson algebra, A_q has a decomposition $A_q = A_q(1) \bigoplus A_q(2) \bigoplus \cdots \bigoplus A_q(s)$.

Then, we have:

Theorem

Let A_q be a quantum cluster algebra without coefficients.

Then a Poisson structure $\{-,-\}$ on A_q is compatible with A_q if and only if it is piecewise standard on A_q.

2nd-stage quantization of A_q without coefficients

Therefore without loss of generality, in the following we can assume A_q is indecomposable. Then in a compatible triple,

$$\Omega = a \begin{pmatrix} 0 & [\lambda_{12}]_{q^{\frac{1}{2}}} & \cdots & [\lambda_{1n}]_{q^{\frac{1}{2}}} \\ [\lambda_{21}]_{q^{\frac{1}{2}}} & 0 & \cdots & [\lambda_{2n}]_{q^{\frac{1}{2}}} \\ \vdots & \vdots & \ddots & \vdots \\ [\lambda_{n1}]_{q^{\frac{1}{2}}} & [\lambda_{n2}]_{q^{\frac{1}{2}}} & \cdots & 0 \end{pmatrix}.$$

where a is an integer. Then $W_{ij} = a\lambda_{ij}$ for any $i, j \in [1, n]$. Therefore in this case,

$$Y_t^{e_i} Y_t^{e_j} = (p^a q)^{\frac{1}{2}\lambda_{ij}} Y_t^{e_i+e_j}, \forall i, j \in [1, n].$$

So, the 2nd-stage quantized cluster algebra $A_{p,q}$ is essentially a quantum cluster algebra, too.

2nd-stage quantization of A_q without coefficients

Hence, in general, for a (decomposable) quantum cluster algebra A_q, its 2nd-stage quantization $A_{p,q}$ can be decomposable into a direct sum of some quantum cluster algebras as $\mathbb{Z}[(p^a q)^{\pm\frac{1}{2}}]$-subalgebras.

So, this 2nd-stage quantization $A_{p,q}$ is essentially a sum of some **1st-stage** quantum cluster algebras.

In this case, we say **the 2nd-stage quantization $A_{p,q}$ to be trivial**.

Theorem

There is no non-trivial 2nd-stage quantization for a quantum cluster algebra without coefficients.

non-trivial 2nd-stage quantization

Example

The quantum coordinate algebra (or say, quantum matrix algebra) $Fun_{\mathbb{C}}(SL_q(2))$ is generated by a, b, c, d with relations:

$$ab = q^{-1}ba,\ ac = q^{-1}ca,\ db = qbd,$$
$$dc = qcd,\ bc = cb, ad - da = (q^{-1} - q)bc$$

and

$$ad - q^{-1}bc = 1,$$

where $0 \neq q \in \mathbb{C}$ is a parameter.

non-trivial 2nd-stage quantization

Example (continuing)

$Fun_{\mathbb{C}}(SL_q(2))$ has a quantum cluster structure:

Let $\mathbb{P} = \mathbb{C}[b, c]$. In the 1-regular tree T_1: $t_0 \bullet \text{———} \bullet t_1$, we assign the quantum seed $\Sigma(t_0) = (\widetilde{X}(t_0), \widetilde{B(t_0)}, \Lambda(t_0))$ on the vertex t_0, where $X(t_0) = \{a\}$, $X_{fr} = \{b, c\}$, $X_{t_0}^{e_1} = a$, $X_{t_0}^{e_2} = X^{e_2} = b$, $X_{t_0}^{e_3} = X^{e_3} = c$;

$$\Lambda(t_0) = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \ \widetilde{B}(t_0) = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

Then $Fun_{\mathbb{C}}(SL_q(2)) = A_q(\Sigma(t_0))$.

non-trivial 2nd-stage quantization

Example (continuing)

By definition, it can be calculated that the 2nd-stage deformation matrix $W(t_0)$ must be of the form

$$W(t_0) = \begin{pmatrix} 0 & -w_1 & -w_2 \\ w_1 & 0 & 0 \\ w_2 & 0 & 0 \end{pmatrix},$$

where $w_1 + w_2 \neq 0$. The 2nd-stage quantization induced by $(\widetilde{B}(t_0), \Lambda(t_0), W(t_0))$ is trivial if and only if $W(t_0) = h\Lambda(t_0)$ for some constant h, which means $w_1 = w_2$. Therefore, when $w_1 + w_2 \neq 0$ and $w_1 \neq w_2$, the obtained 2nd-stage quantization $A_{p,q}$ of $Fun_{\mathbb{C}}(SL_q(2))$ is non-trivial.

non-trivial 2nd-stage quantization

Example (continuing)

In this case, the relations of quantum tori are

$$X_*^{e_i} X_*^{e_j} = p^{\frac{1}{2}W_*(e_i,e_j)} q^{\frac{1}{2}\Lambda_*(e_i,e_j)} X^{e_i+e_j}, \quad \forall i,j = 1,2,3, * = t_0, t_1.$$

So the 2nd-stage quantized cluster algebra $A_{p,q}$ of $Fun_{\mathbb{C}}(SL_q(2))$ can be realized as the $\mathbb{C}[q^{\pm\frac{1}{2}}]$-algebra generated by a, b, c, d satisfying the relations as follows:

$$ab = r^{-1}ba,\ ac = s^{-1}ca,\ db = rbd,\ dc = scd,$$
$$bc = cb, ad - da = [(rs)^{-\frac{1}{2}} - (rs)^{\frac{1}{2}}]bc$$

and

$$ad - (rs)^{-\frac{1}{2}}bc = 1,$$

where $r = p^{w_1}q, s = p^{w_2}q$.

non-trivial 2nd-stage quantization

So, we can say that the 2nd-stage quantization $A_{p,q}(SL(2))$ provides a way to realize two-parameters quantization of the special quantum linear group $SL_q(2)$,

as a parallel supplement to the method of two parameters quantization of the general quantum linear group.

Other examples of non-trivial 2nd-stage quantizations

Example A

Let $\Sigma = (\tilde{X}, \tilde{B}, \Lambda)$ be an arbitrary quantum seed of a quantum cluster algebra A_q. Then there is a **cluster extension** $\Sigma' = (\tilde{X}', \tilde{B}', \Lambda')$ of Σ such that the cluster extension A'_q of A_q admits a non-trivial 2nd-stage quantization $A'_{p,q}$.

Example B from oriented Riemann surfaces

Let T be a triangulation of a surface with n marked points, where $n > 0$ is an odd number. Then,

A compatible pair $(\bar{B}_T, \bar{\Lambda}_T)$ can be constructed satisfying that $\mu_\gamma(\bar{\Lambda}_T) = \bar{\Lambda}_{T'}$ for any γ and T' is obtained from T by flipping at γ.

Following this, $(\bar{B}_T, \bar{\Lambda}_T)$ induces a quantum cluster algebra A_q admitting a non-trivial 2nd-stage quantization $A_{q,p}$.

Thanks for your attention!