Example: Spin Chains.

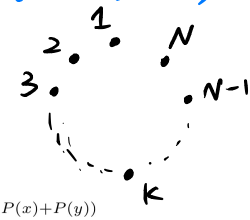
Consider $\mathbb{Z}_N := \mathbb{Z}/N\mathbb{Z}$.

Configuration space " path space.

$\mathbb{R}^N \cong \{\text{functions } \mathbb{Z}_N \to \mathbb{R}\}$

$\sigma \in \mathbb{R}^N$, $i \mapsto \sigma(i) \in \mathbb{R}$ = a config.

($\{\pm 1\}^N \leftrightarrow$ Ising model)



action / energy. $\leftarrow$ $S_N(\sigma) \overset{\text{def}}{=} \sum_{i=1}^{N} |\sigma(i+1)-\sigma(i)|^2 + \sum_{i=1}^{N} P(\sigma(i))$ — poly bounded below, ex. $P(x)=x^4$.

$$\rightsquigarrow \quad \mathrm{d}\mu_{P(\sigma)}(\sigma) \overset{\text{def}}{=} \frac{1}{\mathcal{Z}(N)} \mathrm{e}^{-S(\sigma)} \mathrm{d}^N\sigma, \quad \text{with} \quad \mathcal{Z}(N) = \int_{\mathbb{R}^N} \mathrm{e}^{-S_N(\sigma)} \mathrm{d}^N\sigma$$

(Gibbs measure; partition function)

Employ "splitting trick".

$$\rightsquigarrow \quad \exp(-S_N(\sigma)) = \prod_{i=1}^{N} K(\sigma(i+1), \sigma(i)) \quad \text{with} \quad K(x,y) = \mathrm{e}^{-|x-y|^2 - \frac{1}{2}(P(x)+P(y))}$$

$$\rightsquigarrow \quad \mathcal{Z}(N) = \int \prod_{i=1}^{N} [K(\sigma(i+1), \sigma(i))\mathrm{d}\sigma(i)] = \mathrm{tr}_{L^2(\mathbb{R})}(T^N), \qquad (TF)(x) = \int K(x,y)F(y)\mathrm{d}y,$$

"transfer operator"

$F \in L^2(\mathbb{R})$. "functional on config" above 1 site.
$\;\;\; \| $
$F(\sigma(i))$

But what's the kernel of T^N?

$$K_N(\sigma_{\text{out}}, \sigma_{\text{in}}) = \int K(\sigma_{\text{out}}, \sigma(N)) \cdots K(\sigma(2), \sigma_{\text{in}}) \prod_{i=2}^{N} \mathrm{d}\sigma(i) \quad (\#)$$

Path integral $\leftarrow$
$$\boxed{= \int_{\mathbb{R}^{N-1}} \mathrm{e}^{-S_N(\sigma|\sigma_{\text{in}},\sigma_{\text{out}})} \prod_{i=2}^{N} \mathrm{d}\sigma(i), \quad S_N(\sigma|\sigma_{\text{in}}, \sigma_{\text{out}}) \overset{\text{def}}{=} \sum_{i=1}^{N} |\sigma(i+1)-\sigma(i)|^2 + \sum_{i=2}^{N} P(\sigma(i)) + \frac{1}{2}(P(\sigma_{\text{in}}) + P(\sigma_{\text{out}})),}$$

Composition property $\Rightarrow$
$$\boxed{K_{N_2+N_1}(\sigma_{\text{out}}, \sigma_{\text{in}}) = \int K_{N_2}(\sigma_{\text{out}}, \sigma) K_{N_1}(\sigma, \sigma_{\text{in}}) \mathrm{d}\sigma.} \quad (S)$$

☆ Obvious because of (#). aka we know K_N is kernel of T^N.

? How about the other way round? define directly with

Goal. a 2-D, continuum, curved version of (S). w/ inward/outward pointing co-orient.

(Ω, g) = Riemannian surface w/ bdy $\partial\Omega = \Sigma_{in} \sqcup \Sigma_{out}$.

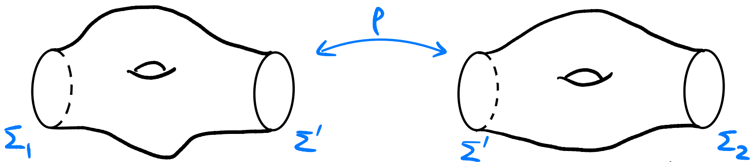
Config. space $= \mathcal{D}'(\Omega)$, config $\phi \in \mathcal{D}'(\Omega)$ $\longleftrightarrow$ $\sigma \in \mathbb{R}^N$.

$$\rightsquigarrow \quad \mathcal{A}_\Omega(\varphi_{\text{in}}, \varphi_{\text{out}}) = \int_{\{\phi|\partial\Omega = (\varphi_{\text{in}}, \varphi_{\text{out}})\}} \mathrm{e}^{-\boxed{\int_\Omega \frac{1}{2}(|\nabla\phi|_g^2 + m^2\phi^2) + P(\phi(x))\mathrm{d}V_\Omega(x)}} [\mathcal{L}\phi],$$

non-existent lebesgue measure

Let Ω_1, Ω_2 2 surfaces as above, $\partial\Omega_1 = \Sigma_1 \sqcup \Sigma'$, $\partial\Omega_2 = \Sigma' \sqcup \Sigma_2$,
$\Omega_1 \cup_\rho \Omega_2$ = surface glued along $\Sigma' \overset{\sim}{\underset{\rho}{\to}} \Sigma'$.

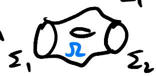
? $\mathcal{A}_{\Omega_1 \cup_\rho \Omega_2}(\varphi_2, \varphi_1) \overset{?}{=} \int \mathcal{A}_{\Omega_2}(\varphi_2, \varphi') \mathcal{A}_{\Omega_1}(\varphi', \varphi_1) [\mathcal{L}\varphi']$



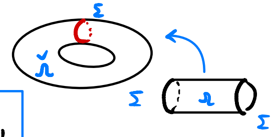
$\rightsquigarrow$ Segal's axioms.
rules that define abstractly a QFT (CFT)

(real)

① Circle $\Sigma \rightsquigarrow$ Hilbert Space $\mathcal{H}_\Sigma$.
$\Sigma_1 \sqcup \Sigma_2 \rightsquigarrow \mathcal{H}_{\Sigma_1} \otimes \mathcal{H}_{\Sigma_2}$.

② Cobordism  $\rightsquigarrow U_\Omega : \mathcal{H}_{\Sigma_1} \to \mathcal{H}_{\Sigma_2}$.

③ $Z_X = \mathrm{tr}(U_\Omega)$.



⑤ $U_{\Omega^*} = U_\Omega^\dagger$.
$\hookrightarrow$ co-orient. reversed

④ $\boxed{U_{\Omega_2 \cup_{\Sigma'} U_1} = U_{\Omega_2} \circ U_{\Omega_1}}$

Previously for $P(\phi)$, Pickrell '08
Related e.g. Liouville theory GKRV '21.
Perturbative approach: Kandel-Mnev-Wernli '21.

$\boxed{\text{Case } P = 0.}$ First let $M =$ closed surface.

$$\rightsquigarrow \quad e^{-\frac{1}{2}\int_M (|\nabla\phi|_g^2 + m^2\phi^2)\mathrm{d}V_M}[\mathcal{L}\phi] \overset{\text{heu}}{=} \text{“}\det\text{”}(\Delta + m^2)^{-\frac{1}{2}} \boxed{\mathrm{d}\mu^M_{\mathrm{GFF}}(\phi)}$$

a Gaussian Proba measure on $\mathcal{D}'(M)$.

Idea: $$\int_{\mathbb{R}^N} e^{-\frac{1}{2}\langle \mathbf{x}, C^{-1}\mathbf{x}\rangle} \mathrm{d}\mathcal{L}^N(\mathbf{x}) = \frac{(2\pi)^{N/2}}{(\det C^{-1})^{1/2}}$$

∞-dim'l generalization of determinant. (ζ-reg. det.).

$\boxed{\text{(massive) Gaussian Free Field (GFF).}}$ is a Prob. measure on $\mathcal{D}'(M)$. μ^M_{GFF}

s.t. $$\mathbb{E}[\phi(f)\phi(h)] = \langle f, \underbrace{(\Delta + m^2)^{-1}}_{\text{covariance operator}} h\rangle_{L^2(M)}, \quad \mathbb{E}[\phi(f)] \equiv 0,$$

$\forall$ test functions $f, h \in C^\infty(M)$. aka. $\phi \mapsto \langle \phi, f\rangle = \phi(f)$ defines a R.V. on sample space $\mathcal{D}'(M)$.

Existence: Bochner-Minlos. Uniqueness: Kolmogorov / Fourier transform.

(Many other equivalent constructions).

☆ $\|\Phi(f)\|_{L^2(\mu_{\mathrm{GFF}})} = \|f\|_{H^{-1}(M)}$ $\to$ Sobolev space.

$\Rightarrow H^{-1}(M) =$ Gaussian Hilbert Space.

$H^{-1}(M) \hookrightarrow L^2(\mu_{\mathrm{GFF}})$.
much bigger
contains $\Phi(f)^2$, $e^{i\Phi(f)}$, $e^{\Phi(f)}$, etc.

(for Gaussian R.V. $F \perp_{L^2} G \Leftrightarrow F, G$ indep).

$\Rightarrow \phi(f)$ makes sense as R.V. $\forall f \in H^{-1}$
& does not make sense as number!

Formally, take $f = \delta_x$, $h = \delta_y$,

$\Rightarrow \mathbb{E}[\Phi(x)\Phi(y)] \approx G(x,y)$ = Green's function.

(free) "propagator" in physics.

$\boxed{\text{How to define } \mathcal{A}^0_\Omega(\varphi_{in}, \varphi_{out}) \text{ for } P=0\,?}$

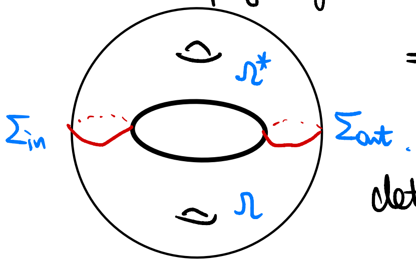
☆ $\mathcal{A}^0_\Omega$ is meant to be integrated against some measure on $\mathcal{D}'(\Sigma_{in})$ or $\mathcal{D}'(\Sigma_{out})$.

$\Rightarrow$ its value will depend on choice of these measures.

(different choice being mutually abs. cont.

$\Rightarrow$ value related by R-N densities).

Now, we apply Segal's rules ③ – ⑤.



$$\Rightarrow Z_{(\Omega^*\cup_{\Sigma_{out}}\Omega)^\vee} \overset{``|\Omega|^2"}{=} \mathrm{tr}\,(U_{\Omega^*}\circ U_\Omega)$$

$$\| \qquad\qquad\qquad \|$$

$$\det \times \int d\mu_{GFF}^{|\Omega|^2} \qquad \int |\mathcal{A}_\Omega(\varphi,\psi)|^2\, d\mu^?(\varphi) \otimes d\mu^?(\psi).$$

$\Rightarrow$ VERY simple minded choice: $\mathcal{A}_\Omega(\varphi,\psi) \equiv 1 \times \det$ ↳ a const.

if we let $d\mu^?(\varphi,\psi) \overset{\text{def}}{=} \underbrace{\tau_{\Sigma_{in}\cup\Sigma_{out}}(\mu_{GFF}^{|\Omega|^2})}$.

measure image under "trace" (restriction) $\phi \longmapsto (\phi|_{\Sigma_{in}}, \phi|_{\Sigma_{out}})$.

! this will NOT be the actual choice.

$\boxed{\text{What is the law of a random field under restriction onto a hypersurface } \Sigma \subseteq M\,?}$

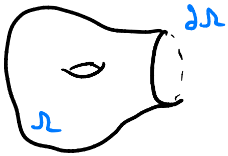
⇝ this will be a random dist. in $\mathcal{D}'(\Sigma)$.

① Is it Gaussian? What is the covariance?

Yes $\qquad \mathbb{E}[\Phi|_\Sigma(x)\,\Phi|_\Sigma(y)] = G|_{\Sigma\times\Sigma}(x,y)$!

$\Rightarrow$ i.e. the cov. op. must have int. kernel $= G|_{\Sigma\times\Sigma}$.

⇒ this Cov. op. is a "2-sided" version of what is called "Dirichlet-to-Neumann Operator".



$$DN : C^\infty(\partial\Omega) \longrightarrow C^\infty(\partial\Omega)$$

" 1-sided version.

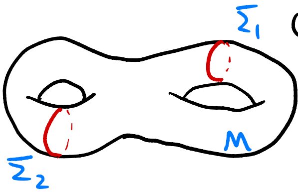
$$\varphi \longmapsto \partial_\nu \big(\underbrace{PI\,\varphi}_{\text{Harmonic extension.}}\big)\big|_{\partial\Omega}.$$

outward normal

Dirichlet data $\mapsto$ Neumann data.

$\boxed{\text{Moral.}}$ $\mathcal{A}_\Omega$ is related to induced Prob. measures under restriction to hypersurface.

What is behind Segal's composition axiom?



①
$$d\tau_{\Sigma_1\cup\Sigma_2}(\mu^M_{GFF})(\varphi_1,\varphi_2)$$
$$= d\tau_{\Sigma_1}(\mu^M_{GFF})(\varphi_1) \text{"}\otimes\text{"} d\mathbb{P}_{\tau_{\Sigma_2}\phi \,|\, \tau_{\Sigma_1}\phi=\varphi_1}(\varphi_2)$$
$$= d\tau_{\Sigma_2}(\mu^M_{GFF})(\varphi_2) \text{"}\otimes\text{"} d\mathbb{P}_{\tau_{\Sigma_1}\phi \,|\, \tau_{\Sigma_2}\phi=\varphi_2}(\varphi_1)$$

⇝ analogue of "Bayes formula".

② "Markov decomposition". for $\Sigma \subseteq M$.

$$\mu^M_{GFF} = PI\circ\tau_\Sigma(\mu^M_{GFF}) \otimes \mu^{M\setminus\Sigma,D}_{GFF} \Leftarrow W^{-1}(M) = \underset{\{f\in W^{-1} \,|\, \operatorname{supp} f\subseteq\Sigma\}}{\underset{\shortparallel}{W^{-1}_\Sigma(M)}} \overset{\perp}{\oplus} (\perp).$$

$$\phi = PI\circ\tau_\Sigma\phi + \underbrace{\phi^D_{M\setminus\Sigma}}$$

Gaussian random field w/ covariance $G_D(x,y)$ = Green's func. w/ Dirichlet condition on Σ.

$P \neq 0$. $\dim M = 2$. $\rightarrow$ define as R.V. in $L^1(\mu_{GFF})$.

$$\boxed{\int \boxed{e^{-\int_M \phi(x)^4 dx}} \, d\mu_{GFF}^M(\phi) < \infty}$$

Nelson's argument. '60s.

Recently, other approach from SPDE (regularity Str. / paracontrol)
especially effective in 3D. $\rightsquigarrow \Phi_3^4$ theory

$\boxed{\text{Problem}}$ ϕ = dist. low regularity, ϕ^2, ϕ^3, etc. not defined.

$\rightsquigarrow$ need renormalization.

Example. How to define $\frac{1}{x} \cdot 1_{(0,+\infty)} \in \mathcal{D}'(\mathbb{R})$?

Idea: $\exists$ distribution in $\mathcal{D}'(\mathbb{R} \setminus \{0\})$ agrees w/ $\frac{1}{x} \cdot 1_{(0,+\infty)}$.

Pick $\varphi \in C_c^\infty(\mathbb{R})$, $\varepsilon > 0$, I.B.P.

$$\Rightarrow \underbrace{\int_\varepsilon^\infty \varphi(x) \frac{dx}{x} = -\int_\varepsilon^\infty \varphi'(x) \log(x) dx}_{\text{agree with } \frac{1}{x} \cdot 1_{(0,+\infty)}} - \varphi(\varepsilon) \log(\varepsilon).$$

for $\varphi \in C_c^\infty(\mathbb{R} \setminus \{0\})$

$\frac{1}{x} \cdot 1_{(0,+\infty)} - \infty \cdot \delta_0$ = renormalized version.

In our case, ① first regularize $\phi_\varepsilon := K_\varepsilon \phi$, K_ε = Smoothing Op.

② replace $\phi_\varepsilon(x)^4 \rightsquigarrow :\phi_\varepsilon(x)^4: = \phi_\varepsilon(x)^4 - 6\mathbb{E}[\phi_\varepsilon(x)^2] \phi_\varepsilon(x)^2 + 3\mathbb{E}[\phi_\varepsilon(x)^2]^2$.

$\rightsquigarrow$ Corresponds to proj. of R.V. $\phi_\varepsilon(x)^4$ onto $\overline{\mathrm{Sym}^4(H^{-1}(M))}$ in $L^2(\mu_{GFF})$.

③ discover that $\int_M :\phi_\varepsilon(x)^4: dx \xrightarrow{\varepsilon \to 0}$ well-defined R.V. $\in L^2(\mu_{GFF})$.
$= S_M(\phi)$

☆ $\varepsilon, \varepsilon' > 0$, $G_{\varepsilon,\varepsilon'}(x,y) = \mathrm{Kernel}\left(K_{\varepsilon'}^* (\Delta + m^2)^{-1} K_\varepsilon\right)$
$= \mathbb{E}[\phi_\varepsilon(x) \phi_{\varepsilon'}(y)]$

$|G_{\varepsilon,\varepsilon'}(x,y)| \leq$ integrable uniformly in $\varepsilon, \varepsilon'$.

Example. $K_\varepsilon = e^{-\varepsilon(\Delta + m^2)}$. Remember $\dim M = 2$!

$\Rightarrow \left\{ \underbrace{\int_M :\phi_\varepsilon(x)^4: \, dx}_{= S_{M,\varepsilon}} \right\}$ Cauchy in $L^2(\mu_{GFF})$.

④ Hypercontractivity.

$X =$ deg $\le n$ poly. of Gaussian R.V.s

$$\Rightarrow \quad \mathbb{E}[|X|^p]^{\frac{1}{p}} \leqslant (p-1)^{\frac{n}{2}} \mathbb{E}[X^2]^{\frac{1}{2}}$$

$\Rightarrow$ Cauchy in L^p, $\forall \, 1 \le p < \infty$

⑤
$$\begin{aligned}
\mathbb{P}(\mathrm{e}^{-S_{M,\chi}} \geqslant \mathrm{e}^{b_2|\log(2\varepsilon)|^n+1}) &= \mathbb{P}(S_{M,\chi} \leqslant -b_2|\log(2\varepsilon)|^n - 1) \\
&\leqslant \mathbb{P}(|S_{M,\chi} - S_{M,\chi,\varepsilon}| \geqslant 1) \\
&\leqslant \|S_{M,\chi} - S_{M,\chi,\varepsilon}\|^p_{L^p(\mu_{\mathrm{GFF}})} \quad \leftarrow \text{Chebychev} \\
&\leqslant (p-1)^{\frac{np}{2}} C_1^p \varepsilon^{\frac{p}{2}} \|\chi\|^p_{L^4} \quad \leftarrow \text{③ \& ④, } \forall p! \\
&\lesssim \|\chi\|^p_{L^4} p^{\frac{n}{2}p} (C_1 \varepsilon^{\frac{1}{2}})^p, \\
&\lesssim \exp\left(-C_2 (\varepsilon^{\frac{1}{2}} \|\chi\|_{L^4})^{-\frac{1}{n}}\right). \quad \leftarrow \text{minimize over } 1 \le p < \infty!
\end{aligned}$$

(Goal: $e^{-S_M} \in L^1$)

⑥
$$\underbrace{\mathbb{E}[e^{-S_{M,\chi}}]}_{<\infty} = \int_0^\infty \mathbb{P}(\mathrm{e}^{-S_{M,\chi}} \geqslant t) \, \mathrm{d}t \Rightarrow 🏆$$

1 more step to show Segal for $P(\phi)$.

locality of the interaction S_M

Roughly speaking, $M = A \cup B$, then

$$\int_M :P(\phi(x)): \, dx = \int_A :P(\phi(x)): dx + \int_B :P(\phi(x)): dx.$$

Some technicality: ① K_ε needs to be local.

② $:X:$ needs to be local.

③ different K_ε define the same S_M.