Some quantization questions related to the two-parameter quantum groups

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1. Some new isoclasses of one-parameter exotic small quantum groups arising from the two-parameter setting

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3. RLL realization of two parameter quantum affine algebra of type $D_n^{(1)}$

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Background

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Background

- In 1975, Kaplansky proposed ten conjectures about Hopf algebra in his book. These have been the focus of a great deal of researches.
- Kaplansky's tenth conjecture says that, for a given dimension there are only finitely many different iso-classes of Hopf algebras. But researchers have found plenty of counterexamples to this conjecture.
- Andruskiewitsch and Schneider had published a series of papers since 2000, they studied the classification of finite-dimensional pointed Hopf algebras over an algebraically closed field $\mathbb{K}$ with characteristic 0 in detail. They posed a principle of the so-called lifting method.

Open questions

- Using lifting method, they proved a classification of finite dimensional Hopf algebras, which requires that the coradical are abelian groups, with orders prime to 210.

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- Using lifting method, they proved a classification of finite dimensional Hopf algebras, which requires that the coradical are abelian groups, with orders prime to 210.
- We consider a class of pointed Hopf algebras whose orders are not prime to 210, and we have given a series of new representatives. This is independent of the classification theorem given by Andruskiewitsch and Schneider. Thus we can provide many new examples of finite dimensional pointed Hopf algebras.

Open questions

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Open questions

- The classification of pointed Hopf algebras with lower orders is an open question. From the perspective of two-parameter quantum groups, we obtain a series of new (exotic) iso-classes of pointed Hopf algebras by isomorphism theorem.
- For some cases that cannot be distinguished by isomorphism theorem, we calculate their dimensions, and the dimension distributions of simple Yetter-Drinfeld modules.
- And it does not rely on the lifting methods proposed by Andruskiewitsch and Schneider at all.

Our classification results

Results

When the orders of parameters are less than or equal to 8, we can get 209 new exotic isoclasses of one-parameter non-standard small quantum groups in all.

Indeed, for type A, we have 47 isoclasses.

For type B, we can get 31 isoclasses.

For type C, we have 62 isoclasses.

For type D, there are 36 isoclasses.

For type F_4, we have 31 isoclasses.

For type G_2, we have 47 isoclasses.

Among these isoclasses, there are 45 isoclasses of one-parameter standard small quantum groups.

Moreover, when the parameters have prime orders, we can also give the respective classifications.

Definition of two-parameter restricted quantum groups

- Let $C = (a_{ij})_{i,j\in I}$ be a Cartan matrix of finite type and $\mathfrak{g}$ be the associated semisimple Lie algebra over $\mathbb{Q}$. Let $\{d_i \mid i \in I\}$ be a set of relatively prime positive integers such that $d_i a_{ij} = d_j a_{ji}$, $i, j \in I$. Let $\langle \cdot, \cdot \rangle$ be the bilinear form , which is called the Euler form (or Ringel form), defined on the root lattice Q by

$$\langle i, j\rangle := \langle \alpha_i, \alpha_j\rangle = \begin{cases} d_i a_{ij}, & i < j, \\ d_i, & i = j, \\ 0, & i > j. \end{cases}$$

Definition of two-parameter restricted quantum groups

Definition(N. Hu, Y. Pei, Sci China Math, 2008)

The restricted two-parameter quantum group $\mathfrak{u}_{r,s}(\mathfrak{g})$ is a unital associative algebra over $\mathbb{Q}(r,s)$ generated by $e_i, f_i, \omega_i^{\pm}, \omega_i'^{\pm} (i \in I)$, subject to the following relations:

$(R1)$ $\omega_i^{\pm 1}, \omega_j'^{\pm 1}$ commute with each other,
$\omega_i\omega_i^{-1} = 1 = \omega_j'\omega_j'^{-1}$, $\omega_i^{\ell} = \omega_i'^{\ell} = 1$.

$$(R2)\quad \omega_i e_j \omega_i^{-1} = r^{\langle j,i\rangle} s^{-\langle i,j\rangle} e_i, \quad \omega_i' e_j \omega_i'^{-1} = r^{-\langle i,j\rangle} s^{\langle j,i\rangle} e_j,$$

$$(R3)\quad \omega_i f_j \omega_i^{-1} = r^{-\langle j,i\rangle} s^{\langle i,j\rangle} f_j, \quad \omega_i' f_j \omega_i'^{-1} = r^{\langle i,j\rangle} s^{-\langle j,i\rangle} f_j,$$

$$(R4)\quad [e_i, f_i] = \delta_{ij}\frac{\omega_i - \omega_i'}{r_i - s_i},$$

$$(R5)\quad \sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k}_{r_i s_i^{-1}} c_{ij}^{(k)}\, e_i^{1-a_{ij}-k} e_j e_i^k = 0, \quad i \neq j,$$

$$(R6)\quad \sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k}_{r_i s_i^{-1}} c_{ij}^{(k)}\, f_i^k f_j f_i^{1-a_{ij}-k} = 0, \quad i \neq j,$$

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（follow） $(R7)\quad e_i^\ell = f_i^\ell = 0,$
where $c_{ij}^{(k)} = (r_i s_i^{-1})^{\frac{k(k-1)}{2}} r^{k\langle j,i\rangle} s^{-k\langle i,j\rangle} (i \neq j)$, and for a symbol v, we set the notations：

$$(n)_v = \frac{v^n - 1}{v - 1}, \quad (n)_v! = (1)_v (2)_v \cdots (n)_v,$$

$$\binom{n}{k}_v = \frac{(n)_v!}{(k)_v!(n-k)_v!}, \quad n \geq k \geq 0.$$

The algebra $U_{r,s}(\mathfrak{g})$ has a Hopf structure with the comultiplication and the antipode given by

$$\begin{aligned} \Delta(e_i) &= e_i \otimes 1 + \omega_i \otimes e_i, & \Delta(f_i) &= 1 \otimes f_i + f_i \otimes \omega_i', \\ S(e_i) &= -\omega_i^{-1} e_i, & S(f_i) &= -f_i (\omega_i')^{-1}, \end{aligned}$$

and ω_i, ω_i' are group-like elements, for all $i \in I$.

Example given by Benkart et al.

Benkart, Pereira use isomorphism theorem of restricted quantum groups of type A established by N. Hu and X. Wang:

Theorem (N. Hu, X. Wang, J. Geom. Phys, 2010)

Assume that $rs^{-1}, r's'^{-1}$ are primitive ℓth roots of unity. Then $\varphi : \mathfrak{u}_{r,s}(\mathfrak{sl}_n) \cong \mathfrak{u}_{r',s'}(\mathfrak{sl}_n)$ as Hopf algebras if and only if either
$(1)(r', s') = (r, s), \varphi$ is a diagonal isomorphism:
$\varphi(\omega_i) = \widetilde{\omega}_i,\ \varphi(\omega_i') = \widetilde{\omega}_i',\ \varphi(e_i) = a_i\tilde{e}_i,\ \varphi(f_i) = a_i^{-1}\tilde{f}_i;$ or

$(2)(r', s') = (s, r),\ \varphi(\omega_i) = \widetilde{\omega}_i'^{-1},\ \varphi(\omega_i') = \widetilde{\omega}_i^{-1},$
$\varphi(e_i) = a_i\tilde{f}_i\widetilde{\omega}_i'^{-1},\ \varphi(f_i) = a_i^{-1}\widetilde{\omega}_i^{-1}\tilde{e}_i$;or

$(3)(r', s') = (s^{-1}, r^{-1}),\ \varphi(\omega_i) = \widetilde{\omega}_{n-i},\ \varphi(\omega_i') = \widetilde{\omega}_{n-i}',$
$\varphi(e_i) = a_{n-i}\tilde{e}_{n-i},\ \varphi(f_i) = (rs)^{-1}a_{n-i}^{-1}\tilde{f}_{n-i};$ or

$(4)(r', s') = (r^{-1}, s^{-1}),\ \varphi(\omega_i) = \widetilde{\omega}_{n-i}'^{-1},\ \varphi(\omega_i') = \widetilde{\omega}_{n-i}^{-1},$
$\varphi(e_i) = a_{n-i}\tilde{f}_{n-i}\widetilde{\omega}_{n-i}'^{-1},\ \varphi(f_i) = (rs)^{-1}a_{n-i}^{-1}\widetilde{\omega}_{n-i}^{-1}\tilde{e}_{n-i}(a_i \in \mathbb{k}^*).$

Example given by Benkart et al.

and the theorem of Hopf 2-cocycle twisting equivalence:

Theorem (G. Benkart, M. Pereira and S. Witherspoon, J. Algebra, 2010)

The following are equivalent:
(1) There is a cocycle σ from G that $H^{\sigma}_{r,s} \cong H_{r',s'}$ as graded Hopf algebras.
(2) $lcm(|r|, |s|) = lcm(|r'|, |s'|), rs^{-1} = r's'^{-1}$.
where $|r|$ denote the order of unity root r.

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Example (G. Benkart, M. Pereira and S. Witherspoon, J. Algebra, 2010)

Let q be a primitive 4th root of 1, and consider restricted quantum groups $\mathfrak{u}_{r,s}(\mathfrak{sl}_3)$. For $\mathfrak{u}_{r,s}(\mathfrak{sl}_3)$, such that $lcm(|r|,|s|) = 4$, we proceed with the following three steps:

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Firstly, we use isomorphism theorem of restricted quantum groups of type A , reducing them into three candidates:
Candidate 1: $\mathfrak{u}_{1,q}(\mathfrak{sl}_3)\cong\mathfrak{u}_{q^3,1}(\mathfrak{sl}_3)\cong\mathfrak{u}_{q,1}(\mathfrak{sl}_3)\cong\mathfrak{u}_{1,q^3}(\mathfrak{sl}_3)$;
Candidate 2: $\mathfrak{u}_{q,q^2}(\mathfrak{sl}_3)\cong\mathfrak{u}_{q^3,q^2}(\mathfrak{sl}_3)\cong\mathfrak{u}_{q^2,q}(\mathfrak{sl}_3)\cong\mathfrak{u}_{q^2,q^3}(\mathfrak{sl}_3)$;
Candidate 3: $\mathfrak{u}_{q,q^{-1}}(\mathfrak{sl}_3)\cong\mathfrak{u}_{q^{-1},q}(\mathfrak{sl}_3)$. (Notice that in this case, rs^{-1} is a primitive 2nd root of unity).

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Candidate 2: $\mathfrak{u}_{q,q^2}(\mathfrak{sl}_3)\cong\mathfrak{u}_{q^3,q^2}(\mathfrak{sl}_3)\cong\mathfrak{u}_{q^2,q}(\mathfrak{sl}_3)\cong\mathfrak{u}_{q^2,q^3}(\mathfrak{sl}_3)$;
Candidate 3: $\mathfrak{u}_{q,q^{-1}}(\mathfrak{sl}_3)\cong\mathfrak{u}_{q^{-1},q}(\mathfrak{sl}_3)$. (Notice that in this case, rs^{-1} is a primitive 2nd root of unity).
Secondly, the simple Yetter-Drinfeld modules dimension distributions of the Candidate 1, 2 are the same. So the above three candidates can actually be reduced to two small quantum groups with different sets of one parameter: $\mathfrak{u}_{1,q}(\mathfrak{sl}_3),\mathfrak{u}_{q,q^{-1}}(\mathfrak{sl}_3)$.

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Example (G. Benkart, M. Pereira and S. Witherspoon, J. Algebra, 2010)

（follow） Finally, calculate their dimension distribution of simple Yetter-Drinfeld modules, and find that their distributions are different:

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$$\dim(H_{1,q} \bullet_\beta g), g \in G(H_{1,q}), \beta \in \widehat{G(H_{1,q})}:$$

$$\{1^{16}, 3^{32}, 6^{32}, 8^{16}, 10^{32}, 12^{32}, 24^{32}, 26^{16}, 42^{32}, 64^{16}\};$$

$$\dim(H_{q,q^{-1}} \bullet_\beta g), g \in G(H_{q,q^{-1}}), \beta \in \widehat{G(H_{q,q^{-1}})}:$$

$$\{1^{16}, 3^{32}, 8^{16}, 16^{96}, 32^{96}\}.$$

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Since $\mathfrak{u}_{q,q^{-1}}(\mathfrak{sl}_3), \mathfrak{u}_{q,q^2}(\mathfrak{sl}_3)$ and $\mathfrak{u}_{1,q}(\mathfrak{sl}_3)$ have Drinfeld double structure , $\mathfrak{u}_{q,q^{-1}}(\mathfrak{sl}_3), \mathfrak{u}_{q,q^2}(\mathfrak{sl}_3), \mathfrak{u}_{1,q}(\mathfrak{sl}_3)$ are pairwise non-isomorphic.

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Since $\mathfrak{u}_{q,q^{-1}}(\mathfrak{sl}_3), \mathfrak{u}_{q,q^2}(\mathfrak{sl}_3)$ and $\mathfrak{u}_{1,q}(\mathfrak{sl}_3)$ have Drinfeld double structure , $\mathfrak{u}_{q,q^{-1}}(\mathfrak{sl}_3), \mathfrak{u}_{q,q^2}(\mathfrak{sl}_3), \mathfrak{u}_{1,q}(\mathfrak{sl}_3)$ are pairwise non-isomorphic.
That is to say, $\mathfrak{u}_{1,q}(\mathfrak{sl}_3), \mathfrak{u}_{q,q^2}(\mathfrak{sl}_3)$, apart from the one-parameter standard restricted quantum groups: $\mathfrak{u}_{q,q^{-1}}(\mathfrak{sl}_3)$.

Our Strategy

Step 1

Firstly, according to the isomorphism theorem of two-parameter restricted quantum groups, we can reduce $\mathfrak{u}_{r,s}(\mathfrak{g})$ to one-parameter restricted quantum groups, with some isomorphism classes representatives and candidates.

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Step 2

By calculating the dimension of $\mathfrak{u}_{r,s}(\mathfrak{g})$(In our paper, we consider the case $\mathfrak{g} = \mathfrak{sl}_3$), we can make further distinction.

Our Strategy

Step 3

Next, for type A_2, when $\mathfrak{u}_{r,s}(\mathfrak{g})$ has Drinfeld double structure (Our group have given sufficient condition), we can calculate their dimension distribution of simple modules (equivalently, simple Yetter Drinfeld modules) to get a new series of one-parameter restricted quantum groups representatives.

Isomorphism theorem of restricted quantum groups

Theorem: Type B (N. Hu, X. Wang, J. Geom. Phys, 2010)

Assume that $rs^{-1}, r's'^{-1}$ are primitive ℓth roots of unity with $\ell \neq 3, 4$, and ζ is a 2nd root of unity. Then $\varphi : \mathfrak{u}_{r,s}(\mathfrak{so}_{2n+1}) \cong \mathfrak{u}_{r',s'}(\mathfrak{so}_{2n+1})$ as Hopf algebras if and only if either

(1) $(r', s') = \zeta(r, s), \varphi$ is a diagonal isomorphism:

$\varphi(\omega_i) = \widetilde{\omega}_i,\ \varphi(\omega_i') = \widetilde{\omega}_i',\ \varphi(e_i) = a_i\tilde{e}_i,\ \varphi(f_i) = \zeta^{\delta_{i,n}} a_i^{-1}\tilde{f}_i$; or

(2) $(r', s') = \zeta(s, r),\ \varphi(\omega_i) = \widetilde{\omega}_i'^{-1},\ \varphi(\omega_i') = \widetilde{\omega}_i^{-1}, \varphi(e_i) = a_i\tilde{f}_i\widetilde{\omega}_i'^{-1},$

$\varphi(f_i) = \zeta^{\delta_{i,n}} a_i^{-1}\widetilde{\omega}_i^{-1}\tilde{e}_i, (a_i \in \mathbb{K}^*).$

Isomorphism theorem of restricted quantum groups

Theorem: Type C (R. Chen, Ph.D dissertation, 2008)

Assume that $n \geq 2$, and $rs^{-1}$and $r'(s')^{-1}$ are ℓth primitive roots of unity, Then $\varphi : \mathfrak{u}_{r,s}(\mathfrak{sp}_{2n}) \cong \mathfrak{u}_{r',s'}(\mathfrak{sp}_{2n})$ as Hopf algebras if and only if either $(r', s') = (r, s)$, or$(r', s') = (-r, -s)$.

Theorem: Type D (X. Bai, Ph.D dissertation, 2006)

Assume that $n \geq 5, rs^{-1}$and $r'(s')^{-1}$ are ℓth primitive roots ot unity, Then $\varphi : \mathfrak{u}_{r,s}(\mathfrak{so}_{2n}) \cong \mathfrak{u}_{r',s'}(\mathfrak{so}_{2n})$ as Hopf algebras if and only if either $(r', s') = (r, s)$, or $(r', s') = (s^{-1}, r^{-1})$.

Isomorphism theorem of restricted quantum groups

Theorem: Type F_4 (X. Chen, N. Hu and X. Wang, Acta Math. Sin.(Engl. Ser), 2023)

Assume that $rs^{-1}, r's'^{-1}$ are ℓth primitive roots of unity, and $l \neq 3, 4, \zeta$is 2nd root of unity. Then $\varphi : \mathfrak{u}_{r,s}(F_4) \cong \mathfrak{u}_{r',s'}(F_4)$ as Hopf algebras if and only if either:

(1) $(r', s') = \zeta(r, s), \varphi$ is diagonal isomorphism:

$\varphi(\omega_i) = \widetilde{\omega}_i,\ \varphi(\omega_i') = \widetilde{\omega}_i', \varphi(e_i) = a_i\widetilde{e}_i, \varphi(f_i) = \zeta^{\delta_{i,3}+\delta_{i,4}} a_i^{-1}\widetilde{f}_i;$ or

(2) $(r', s') = \zeta(s, r),\ \varphi(\omega_i) = \widetilde{\omega}_i'^{-1},\ \varphi(\omega_i') = \widetilde{\omega}_i^{-1}, \varphi(e_i) = a_i\widetilde{f}_i\widetilde{\omega}_i'^{-1},$

$\varphi(f_i) = \zeta^{\delta_{i,3}+\delta_{i,4}} a_i^{-1}\widetilde{\omega}_i^{-1}\widetilde{e}_i, (a_i \in \mathbb{K}^*).$

Isomorphism theorem of restricted quantum groups

Theorem: Type G_2 (N. Hu, X. Wang, Pacific J. Math, 2009)

Assume that $rs^{-1}, r's'^{-1}$ are ℓth primitive root of unity, and $l \neq 3, 4, \zeta$ is 3rd root of unity . Then $\varphi : \mathfrak{u}_{r,s}(G_2) \cong \mathfrak{u}_{r',s'}(G_2)$ as Hopf algebras if and only if either:

(1) $(r', s') = \zeta(r, s), \varphi$ is diagonal isomorphism:

$\varphi(\omega_i) = \widetilde{\omega}_i,\ \varphi(\omega_i') = \widetilde{\omega}_i',\ \varphi(e_i) = a_i\tilde{e}_i, \varphi(f_i) = \zeta^{\delta_{i,1}} a_i^{-1}\tilde{f}_i$; or

(2) $(r', s') = \zeta(s, r),\ \varphi(\omega_i) = \widetilde{\omega}_i'^{-1},\ \varphi(\omega_i') = \widetilde{\omega}_i^{-1}, \varphi(e_i) = a_i\tilde{f}_i\widetilde{\omega}_i'^{-1}$,

$\varphi(f_i) = \zeta^{\delta_{i,1}} a_i^{-1}\widetilde{\omega}_i^{-1}\tilde{e}_i, (a_i \in \mathbb{K}^*)$.

Classification results getting from isomorphism theorem

Theorem

Assume that $r^p = s^p = 1$, where p is an odd prime number, then the following table has given the complete classification of $\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$. In all, we have $\frac{p^2-1}{4}$ isoclasses.

Table: The isoclasses of type A restricted quantum groups when q is a primitive root of unity with odd prime order p

$$\mathfrak{u}_{q^k,q^{k+t}}(\mathfrak{sl}_n) \cong \mathfrak{u}_{q^{k+t},q^k}(\mathfrak{sl}_n) \cong \mathfrak{u}_{q^{p-k},q^{p-k-t}}(\mathfrak{sl}_n) \cong \mathfrak{u}_{q^{p-k-t},q^{p-k}}(\mathfrak{sl}_n);$$

$$(1 \leq t \leq \tfrac{p-1}{2}, 0 \leq k \leq p-1,\ \ t, k \in \mathbb{Z})$$

Classification results getting from isomorphism theorem

Theorem

Assume that $r^p = s^p = 1$, where p is an odd prime number, and $p > 3$, then $\mathfrak{u}_{r,s}(\mathfrak{so}_{2n+1}) \cong \mathfrak{u}_{r',s'}(\mathfrak{so}_{2n+1})$ if and only if $(r, s) = (r', s')$ or $(r, s) = (s', r')$. In all, we can get $\frac{p^2-p}{2}$ isoclasses.

Theorem

Assume that $r^p = s^p = 1$, where p is an odd prime number, and $p > 3$, then $\mathfrak{u}_{r,s}(\mathfrak{sp}_{2n}) \cong \mathfrak{u}_{r',s'}(\mathfrak{sp}_{2n})$ if and only if $(r, s) = (r', s')$. In this case, we can get $p^2 - p$ isoclasses.

Classification results getting from isomorphism theorem

Theorem

Assume that $r^p = s^p = 1$, where p is an odd prime number with $p > 3$, then the following table has given the complete classification of $\mathfrak{u}_{r,s}(\mathfrak{so}_{2n})$. In this case, we can get $\frac{p^2-1}{2}$ isoclasses.

Table: The isoclasses of type D restricted quantum groups when q is a primitive root of unity with odd prime order p

$\mathfrak{u}_{q^k,q^{k+t}}(\mathfrak{so}_{2n}) \cong \mathfrak{u}_{q^{p-k-t},q^{p-k}}(\mathfrak{so}_{2n});\ (0 < t, k \leq p-1,\ t, k \in \mathbb{Z})$

Classification results getting from isomorphism theorem

Theorem

Assume that $r^p = s^p = 1$, where p is an odd prime number, and $p > 3$, then $\mathfrak{u}_{r,s}(F_4) \cong \mathfrak{u}_{r',s'}(F_4)$ if and only if $(r, s) = (r', s')$ or $(r, s) = (s', r')$. As a result, we can get $\frac{p^2-p}{2}$ isoclasses.

Theorem

Assume that $r^p = s^p = 1$, where p is an odd prime number, and $p > 3$, then $\mathfrak{u}_{r,s}(G_2) \cong \mathfrak{u}_{r',s'}(G_2)$ if and only if $(r, s) = (r', s')$ or $(r, s) = (s', r')$. As a result, we can get $\frac{p^2-p}{2}$ isoclasses.

Classification results getting from isomorphism theorem

Theorem

Assume that $r^8 = s^8 = 1$, then the following table has given the complete classification of $\mathfrak{u}_{r,s}(G_2)$.

Table: The candidates of type G_2 restricted quantum groups when q is an 8th primitive root of unity

1	$\mathfrak{u}_{1,q}(G_2) \cong \mathfrak{u}_{q,1}(G_2)$;	9	$\mathfrak{u}_{1,q^3}(G_2) \cong \mathfrak{u}_{q^3,1}(G_2)$;
2	$\mathfrak{u}_{q,q^2}(G_2) \cong \mathfrak{u}_{q^2,q}(G_2)$;	10	$\mathfrak{u}_{q,q^4}(G_2) \cong \mathfrak{u}_{q^4,1}(G_2)$;
3	$\mathfrak{u}_{q^2,q^3}(G_2) \cong \mathfrak{u}_{q^3,q^2}(G_2)$;	11	$\mathfrak{u}_{q^2,q^5}(G_2) \cong \mathfrak{u}_{q^5,q^2}(G_2)$;
4	$\mathfrak{u}_{q^3,q^4}(G_2) \cong \mathfrak{u}_{q^4,q^3}(G_2)$;	12	$\mathfrak{u}_{q^3,q^6}(G_2) \cong \mathfrak{u}_{q^6,q^3}(G_2)$;
5	$\mathfrak{u}_{q^4,q^5}(G_2) \cong \mathfrak{u}_{q^5,q^4}(G_2)$;	13	$\mathfrak{u}_{q^4,q^7}(G_2) \cong \mathfrak{u}_{q^7,q^4}(G_2)$;
6	$\mathfrak{u}_{q^5,q^6}(G_2) \cong \mathfrak{u}_{q^6,q^5}(G_2)$;	14	$\mathfrak{u}_{q^5,1}(G_2) \cong \mathfrak{u}_{1,q^5}(G_2)$;
7	$\mathfrak{u}_{q^6,q^7}(G_2) \cong \mathfrak{u}_{q^7,q^6}(G_2)$;	15	$\mathfrak{u}_{q^6,q}(G_2) \cong \mathfrak{u}_{q,q^6}(G_2)$;
8	$\mathfrak{u}_{q^7,1}(G_2) \cong \mathfrak{u}_{1,q^7}(G_2)$;	16	$\mathfrak{u}_{q^7,q^2}(G_2) \cong \mathfrak{u}_{q^2,q^7}(G_2)$.

Dimension of $\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$

Theorem (Benkart, Witherspoon)

Assume r is a primitive dth root of unity, s is a primitive d'th root of unity, and ℓ is the least common multiple of d and d'. Then $\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$ is an algebra of dimension $\ell^{(n+2)(n-1)}$ with basis all

$$\mathcal{E}^{a_1}_{i_1,j_1}\cdots\mathcal{E}^{a_p}_{i_p,j_p}\omega_1^{b_1}\cdots\omega_{n-1}^{b_{n-1}}\omega_1^{b_1}\cdots\omega_{n-1}^{b_{n-1}}(\omega_1')^{b_1'}\cdots(\omega_{n-1}')^{b_{n-1}'}\mathcal{F}^{a_1'}_{i_1',j_1'}\cdots\mathcal{F}^{a_p'}_{i_p',j_p'},$$

where $(i_1,j_1)<\cdots<(i_p,j_p)$ and $(i_1',j_1')<\cdots<(i_p',j_p')$ lexicographically, and all powers range between 0 and $\ell-1$.

New examples from type A(q is a 4th root of unity)

Theorem

Assume that $r^4 = s^4 = 1$, then the following table has given the complete classification of $\mathfrak{u}_{r,s}(\mathfrak{sl}_3)$.

Table: The candidates of type A_2 restricted quantum groups when q is a 4th primitive root of unity

1	$\mathfrak{u}_{1,q}(\mathfrak{sl}_3) \cong \mathfrak{u}_{q,1}(\mathfrak{sl}_3) \cong \mathfrak{u}_{q^3,1}(\mathfrak{sl}_3) \cong \mathfrak{u}_{1,q^3}(\mathfrak{sl}_3)$;
2	$\mathfrak{u}_{q,q^2}(\mathfrak{sl}_3) \cong \mathfrak{u}_{q^2,q}(\mathfrak{sl}_3) \cong \mathfrak{u}_{q^3,q^2}(\mathfrak{sl}_3) \cong \mathfrak{u}_{q^2,q^3}(\mathfrak{sl}_3)$;
3°	$\mathfrak{u}_{1,q^2}(\mathfrak{sl}_3) \cong \mathfrak{u}_{q^2,1}(\mathfrak{sl}_3)$;
4	$\mathfrak{u}_{q,q^3}(\mathfrak{sl}_3) \cong \mathfrak{u}_{q^3,q}(\mathfrak{sl}_3)$.

New examples from type $A(q$ is a 4th root of unity)

Proof

Firstly, according to the isomorphism theorem, we know candidate 1 and 2 are nonisomorphic, and candidate 3 and 4 are nonisomorphic.

Secondly, candidate 3 are nonisomorphic with the others since its dimension is distinguished.

Lastly, we conclude that candidate 1, 2, 4 are pairwise nonisomorphic by calculating their dimension distribution of simple modules.

New examples from type $A(q$ is a 6th root of unity)

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2	$\mathfrak{u}_{q,q^2}(\mathfrak{sl}_n) \cong \mathfrak{u}_{q^2,q}(\mathfrak{sl}_n) \cong \mathfrak{u}_{q^4,q^5}(\mathfrak{sl}_n) \cong \mathfrak{u}_{q^5,q^4}(\mathfrak{sl}_n)$;
3	$\mathfrak{u}_{q^2,q^3}(\mathfrak{sl}_n) \cong \mathfrak{u}_{q^3,q^2}(\mathfrak{sl}_n) \cong \mathfrak{u}_{q^4,q^3}(\mathfrak{sl}_n) \cong \mathfrak{u}_{q^3,q^4}(\mathfrak{sl}_n)$;
4^*	$\mathfrak{u}_{1,q^2}(\mathfrak{sl}_n) \cong \mathfrak{u}_{q^2,1}(\mathfrak{sl}_n) \cong \mathfrak{u}_{1,q^4}(\mathfrak{sl}_n) \cong \mathfrak{u}_{q^4,1}(\mathfrak{sl}_n)$;
5^*	$\mathfrak{u}_{q,q^3}(\mathfrak{sl}_n) \cong \mathfrak{u}_{q^3,q}(\mathfrak{sl}_n) \cong \mathfrak{u}_{q^5,q^3}(\mathfrak{sl}_n) \cong \mathfrak{u}_{q^3,q^5}(\mathfrak{sl}_n)$;
6^*	$\mathfrak{u}_{q,q^5}(\mathfrak{sl}_n) \cong \mathfrak{u}_{q^5,q}(\mathfrak{sl}_n)$;
7^*	$\mathfrak{u}_{q^2,q^4}(\mathfrak{sl}_n) \cong \mathfrak{u}_{q^4,q^2}(\mathfrak{sl}_n)$;
8^{**}	$\mathfrak{u}_{1,q^3}(\mathfrak{sl}_n) \cong \mathfrak{u}_{q^3,1}(\mathfrak{sl}_n)$;
9^{**}	$\mathfrak{u}_{q^2,q^5}(\mathfrak{sl}_n) \cong \mathfrak{u}_{q^5,q^2}(\mathfrak{sl}_n) \cong \mathfrak{u}_{q,q^4}(\mathfrak{sl}_n) \cong \mathfrak{u}_{q^4,q}(\mathfrak{sl}_n)$.

New examples from type A_2(q is a 6th root of unity)

- When $n = 3$, by calculating the dimension of $\mathfrak{u}_{r,s}(\mathfrak{sl}_3)$, and the dimension of their dimension distribution (for those candidates who have Drinfeld double structure) :

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- When $n=3$, by calculating the dimension of $\mathfrak{u}_{r,s}(\mathfrak{sl}_3)$, and the dimension of their dimension distribution (for those candidates who have Drinfeld double structure) :

(1) $\dim(H_{1,q} \bullet_\beta g), g \in G(H_{1,q}), \beta \in \widehat{G(H_{1,q})}$:

$\{1^{36}, 3^{72}, 6^{72}, 8^{36}, 10^{72}, 15^{144}, 24^{72}, 25^{72}, 27^{108}, 48^{72}, 54^{72}, 56^{36},$

$87^{72}, 120^{72}, 124^{36}, 165^{72}, 216^{36}\},$

(2) $\dim(H_{q,q^5} \bullet_\beta g), g \in G(H_{q^3,q^5}), \beta \in \widehat{G(H_{q^3,q^5})}$:

$\{1^{36}, 3^{72}, 6^{72}, 7^{36}, 15^{72}, 27^{36}, 36^{324}, 72^{324}, 108^{324}\}.$

(3) $\dim(H_{q^2,q^5} \bullet_\beta g), g \in G(H_{q^2,q^5}), \beta \in \widehat{G(H_{q^2,q^5})}$:

$\{1^{36}, 3^{72}, 8^{36}, 36^{432}, 72^{432}, 216^{288}\}.$

New examples from type A_2(q is a 6th root of unity)

Table: The isoclasses of type A_2 restricted quantum groups when q is a 6th primitive root of unity

1	$\mathfrak{u}_{1,q}(\mathfrak{sl}_3) \cong \mathfrak{u}_{q,1}(\mathfrak{sl}_3) \cong \mathfrak{u}_{q^5,1}(\mathfrak{sl}_3) \cong \mathfrak{u}_{1,q^5}(\mathfrak{sl}_3)$;
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New examples from type A(q is an 8th root of unity)

Table: The candidates of type A restricted quantum groups getting from isomorphism theorem when q is an 8th primitive root of unity

1	$\mathfrak{u}_{1,q}(\mathfrak{sl}_n)\cong\mathfrak{u}_{q,1}(\mathfrak{sl}_n)\cong\mathfrak{u}_{q^7,1}(\mathfrak{sl}_n)\cong\mathfrak{u}_{1,q^7}(\mathfrak{sl}_n)$;
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12*	$\mathfrak{u}_{q^3,q^5}(\mathfrak{sl}_n)\cong\mathfrak{u}_{q^5,q^3}(\mathfrak{sl}_n)$;
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New examples from type A_2(q is an 8th root of unity)

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- The Candidates 1 — 7 have the same dimension distribution; Candidates 10 and 12 have the same dimension distribution; This can be reduced to the following candidates: $\mathfrak{u}_{1,q}(\mathfrak{sl}_3), \mathfrak{u}_{q,q^3}(\mathfrak{sl}_3), \mathfrak{u}_{q,q^5}(\mathfrak{sl}_3), \mathfrak{u}_{q,q^6}(\mathfrak{sl}_3), \mathfrak{u}_{q,q^7}(\mathfrak{sl}_3)$.

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- Finally, we calculate the dimension distribution respectively, three non-isomorphic quantum groups are obtained: $\mathfrak{u}_{1,q}(\mathfrak{sl}_3), \mathfrak{u}_{q,q^5}(\mathfrak{sl}_3), \mathfrak{u}_{q,q^{-1}}(\mathfrak{sl}_3)$.

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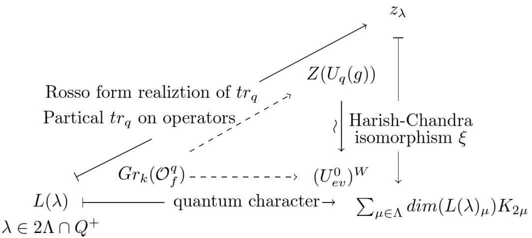
Table of Contents

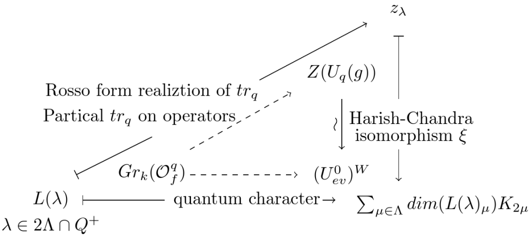
The centre of Drinfel'd-Jimbo quantum groups $U_q(\mathfrak{g})$

- To study the centre of $U_q(\mathfrak{g})$ (generic q and s.s. $\mathfrak{g}$), T. Tanisaki proved that the quantum Harish-Chandra homomorphism is an isomorphism.[T90, T92]
- In [T92] he used the quantum Killing form of $U_q(\mathfrak{g})$ (which is also called the Rosso form[Ro90]) to show the image of H-C, the Casimir element and a detailed proof of the existence of universal R-matrix.



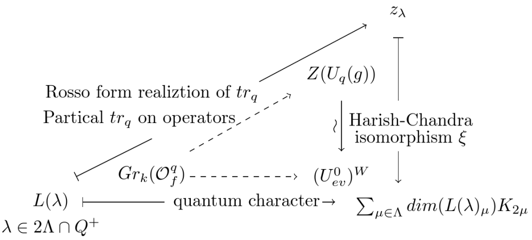
The centre of Drinfel'd-Jimbo quantum groups $U_q(\mathfrak{g})$

- Another way to construct central elements is given by N. Yu. Reshetikhin, et. al.[FRT89, Re89] and R. Zhang, et. al.[GZB91, ZGB91] Finding some operator Γ satisfying good centralizing properties, then its quantum partial trace is a central element. Γ is related to the R-matrix (e.g. $\Gamma = R^{21}R$).



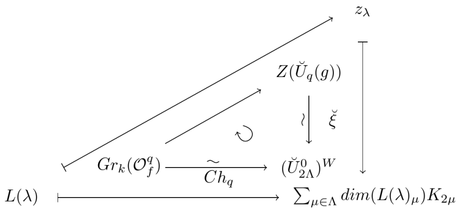
The centre of Drinfel'd-Jimbo quantum groups $U_q(\mathfrak{g})$

- A. Joseph and G. Letzter proved the H-C isomorphism in the same time[JZ92], they also pointed out the centre $Z(U_q)$ need not to be a polynomial algebra, $Gr_k(\mathcal{O}_f^q)$ and $Z(U_q)$ are not always isomorphic.
- Li-Xia-Zhang proved that $Z(U_q)$ is isomorphic to a polynomial algebra in A_1, B_n, C_n, D_{2k+2}, E_7, E_8, F_4, G_2 types, while in the remaining cases it is isomorphic to a quotient of polynomial algebra[LXZ21].



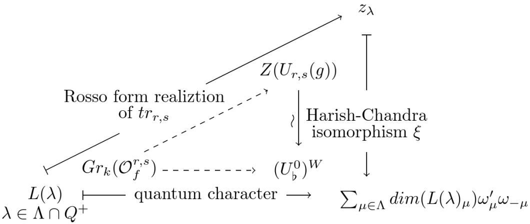
The centre of extended quantum groups $\breve{U}_q(\mathfrak{g})$

- To get a general description, P. I. Etingof considered them in the extended quantum group $\breve{U}_q(\mathfrak{g})$ (some new generator K_{ϖ_i}). He concluded that $Z(\breve{U}_q) \cong \mathbb{K}[c_{\varpi_1}, \cdots, c_{\varpi_n}]$ from [D86,Re89], where c_{ϖ_i} is the quantum partial trace of $\Gamma = R^{21}R$ on $L(\varpi_i)$. It was also generalized to the affine case[E95].
- Y. Dai [D23] gave a detailed proof of the theorem of $Z(\breve{U}_q)$ and a explicit formula of generator c_λ via an operator Γ from quasi-R matrix[ZGB91].



Main results in the two-parameter quantum group

- If $\mathfrak{g}$ is simple with even rank n, the Harish-Chandra homomorphism of the two-parameter quantum group $U_{r,s}(\mathfrak{g})$ is an isomorphism. That is, $\xi : Z(U_{r,s}) \cong (U_\flat^0)^W$.

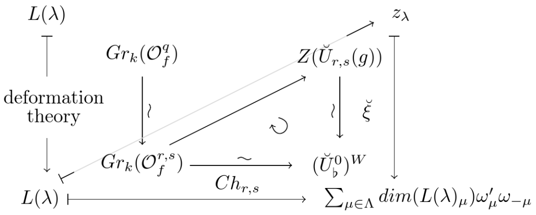


This covers the work of Benkart-Kang-Lee for $A_{2n}^{[BKL06]}$, Hu-Shi for $B_{2n}^{[HS14]}$ and Gan-Hu for $G_2^{[GH10]}$.

- If n is odd, then $\mathrm{Im}(\xi) \supseteq (U_\flat^0)^W \otimes \mathbb{K}[z_*, z_*^{-1}]$.
- We construct two kinds of central element of $U_{r,s}(\mathfrak{g})$, which are z_λ and c_Γ.

Main results

- If n is even, the extended two-parameter quantum group $\breve{U}_{r,s}(\mathfrak{g})$ has the same properties, $\breve{\xi} : Z(\breve{U}_{r,s}(\mathfrak{g})) \cong (\breve{U}_\flat^0)^W$, and the centre $Z(\breve{U}_{r,s}(\mathfrak{g}))$ is proved to be isomorphic to a polynomial algebra via the process



where $q^2 = rs^{-1}$ and $Z(\breve{U}_{r,s}(\mathfrak{g})) \cong \mathbb{K}[z_{\varpi_1}, \cdots, z_{\varpi_n}]$.

- If n is odd, then $Z(\breve{U}_{r,s}(\mathfrak{g})) \supseteq \mathbb{K}[z_{\varpi_1}, \cdots, z_{\varpi_n}] \otimes \mathbb{K}[z_*, z_*^{-1}]$.

Euler form

Definition (HP08)

The Euler form of $\mathfrak{g}$ is the bilinear form $\langle -, - \rangle$ on the root lattice Q satisfying

$$\langle i, j \rangle := \langle \alpha_i, \alpha_j \rangle = \begin{cases} d_i c_{ij} & i < j, \\ d_i & i = j, \\ 0 & i > j. \end{cases}$$

For type D, it is necessary to take the deformation $\langle n-1, n \rangle = -1$, $\langle n, n-1 \rangle = 1$.[BGH06, HP12]
It can be linearly extended to be defined on the weight lattice Λ such that

$$\langle \varpi_i, \varpi_j \rangle := \sum_{k,l=1}^{n} c^{ki} c^{lj} \langle k, l \rangle.$$

Two-parameter quantum groups

Now we let $a_{ij} = r^{\langle j,i\rangle} s^{-\langle i,j\rangle}$. Denote $A = (a_{ij})_{n\times n}$ the structure constant matrix of $U_{r,s}(\mathfrak{g})$, $R_{ij} = \langle j, i\rangle$ and $S_{ij} = -\langle i, j\rangle$. Then we have $a_{ij} = r^{R_{ij}} s^{S_{ij}}$ and $R = -S^T$.

Definition (BGH06,HP08)

Let $U = U_{r,s}(\mathfrak{g})$ be the unital associative algebra over $\mathbb{K}$ generated by elements $e_i, f_i, \omega_i^{\pm 1}, \omega_i'^{\pm 1}$ $(i = 1, \cdots, n)$ satisfying the following relation $(X_1) - (X_4)$:

(X1) For $1 \leqslant i,\ j \leqslant n$, we have

$$\omega_i^{\pm 1}\omega_j'^{\pm 1} = \omega_j'^{\pm 1}\omega_i^{\pm 1}, \qquad \omega_i\omega_i^{-1} = 1 = \omega_j'\omega_j'^{-1}.$$

(X2) For $1 \leqslant i \leqslant n, 1 \leqslant j < n$, we have

$$\omega_i e_j \omega_i^{-1} = a_{ij} e_j, \qquad \omega_i f_j \omega_i^{-1} = a_{ij}{}^{-1} f_j,$$
$$\omega_i' e_j \omega_i'^{-1} = a_{ji}{}^{-1} e_j, \qquad \omega_i' f_j \omega_i'^{-1} = a_{ji} f_j.$$

Two-parameter quantum groups

Definition (follow)

(X3) For $1 \leqslant i, j \leqslant n$, we have

$$[e_i, f_j] = \delta_{ij} \frac{\omega_i - \omega_i'}{r_i - s_i}.$$

(X4) For $i \neq j$, we have $(r, s)-$ Serre relations:

$$(\mathrm{ad}_l e_i)^{1-c_{ij}}(e_j) = 0, \qquad (\mathrm{ad}_r f_i)^{1-c_{ij}}(f_j) = 0,$$

where the definitions of left-adjoint action $\mathrm{ad}_l e_i$ and the right-adjoint action $\mathrm{ad}_r f_i$ are as follow: For any $x, y \in U_{r,s}(\mathfrak{g})$,

$$\mathrm{ad}_l x(y) = \sum_{(x)} x_{(1)} y S(x_{(2)}), \qquad \mathrm{ad}_r x(y) = \sum_{(x)} S(x_{(1)}) y x_{(2)}.$$

The extended two-parameter quantum group

Definition

The extended two-parameter quantum group $\breve{U}_{r,s}(\mathfrak{g})$ is the unital associative algebra generated by elements e_i, f_i, $\omega_{\varpi_i}^{\pm 1}$, ${\omega'_{\varpi_i}}^{\pm 1}$ $(i = 1, \cdots, n)$ over $\mathbb{K}$. The generators satisfy the following relations:
$(X1, \Lambda)$ For $1 \leqslant i,\ j \leqslant n$, we have

$$\omega_{\varpi_i}^{\pm 1} {\omega'_{\varpi_j}}^{\pm 1} = {\omega'_{\varpi_j}}^{\pm 1} \omega_{\varpi_i}^{\pm 1}, \qquad \omega_{\varpi_i} \omega_{\varpi_i}^{-1} = 1 = \omega'_{\varpi_j} {\omega'_{\varpi_j}}^{-1}.$$

$(X2, \Lambda)$ For $1 \leqslant i \leqslant n, 1 \leqslant j < n$, we have

$$\omega_{\varpi_i} e_j \omega_{\varpi_i}^{-1} = r^{\langle j, \varpi_i \rangle} s^{-\langle \varpi_i, j \rangle} e_j, \qquad \omega_{\varpi_i} f_j \omega_{\varpi_i}^{-1} = r^{-\langle j, \varpi_i \rangle} s^{\langle \varpi_i, j \rangle} f_j,$$
$$\omega'_{\varpi_i} e_j \omega'^{-1}_{\varpi_i} = r^{-\langle \varpi_i, j \rangle} s^{\langle j, \varpi_i \rangle} e_j, \qquad \omega'_{\varpi_i} f_j \omega'^{-1}_{\varpi_i} = r^{\langle \varpi_i, j \rangle} s^{-\langle j, \varpi_i \rangle} f_j,$$

The extended two-parameter quantum group

Definition (follow)

$(X3)$ For $1 \leqslant i, j \leqslant n$, we have

$$[e_i, f_j] = \delta_{ij} \frac{\omega_i - \omega_i'}{r_i - s_i},$$

where $\omega_i := \prod_{l=1}^{n} \omega_{\varpi_l}{}^{c_{li}}$, $\omega_j' := \prod_{k=1}^{n} \omega_{\varpi_k}'{}^{c_{kj}}$.

$(X4)$ For $i \neq j$, we have $(r, s)-$Serre relations:

$$(\mathrm{ad}_l e_i)^{1-c_{ij}}(e_j) = 0, \qquad (\mathrm{ad}_r f_i)^{1-c_{ij}}(f_j) = 0,$$

here we naturally extends the Hopf structure of $U_{r,s}(\mathfrak{g})$ to $\breve{U}_{r,s}(\mathfrak{g})$ where $\omega_{\varpi_i}, \omega_{\varpi_i}'$ are group-like elements.

Skew-Hopf pairing

Proposition (BGH06,HP08)

For $U_{r,s}(\mathfrak{g})$, there exists a unique skew-dual pairing $\langle\, -, -\,\rangle : \mathcal{B}' \times \mathcal{B} \to \mathbb{K}$ of the Hopf subalgebra $\mathcal{B}$ and $\mathcal{B}'$ satisfying

$$\langle f_i, e_j \rangle = \delta_{ij}\frac{1}{s_i - r_i}, \qquad 1 \leqslant i, j \leqslant n,$$

$$\langle \omega_i', \omega_j \rangle = a_{ji}, \qquad 1 \leqslant i, j \leqslant n,$$

$$\langle {\omega_i'}^{\pm 1}, \omega_j^{-1} \rangle = \langle {\omega_i'}^{\pm 1}, \omega_j \rangle^{-1} = \langle \omega_i', \omega_j \rangle^{\mp 1}, \qquad 1 \leqslant i, j \leqslant n,$$

and all other pairs of generators are 0. Further, we have

$$\langle S(a), S(b) \rangle = \langle a, b \rangle \,\forall\, a \in \mathcal{B}', b \in \mathcal{B}\ .$$

The Rosso form

Definition (BGH06)

The bilinear form $\langle - \, , \, - \rangle_U : U \times U \to \mathbb{K}$ defined by

$$\left\langle f_\alpha \omega'_\mu \omega_v e_\beta , f_\theta \omega'_\sigma \omega_\delta e_\gamma \right\rangle_U = \langle \omega'_\sigma , \omega_v \rangle \langle \omega'_\mu , \omega_\delta \rangle \langle f_\theta , e_\beta \rangle \left\langle S^2 \left(f_\alpha \right) , e_\gamma \right\rangle .$$

is called the Rosso form of $U_{r,s}(\mathfrak{g})$.

Theorem (BGH06,PHR10)

The Rosso form $\langle -, - \rangle_U$ on $U \times U$ is ad_l-invariant, that is,

$$\langle \mathrm{ad}_l(a) b, c \rangle_U = \langle b, \mathrm{ad}_l(S(a)) c \rangle_U, \quad \forall \ a, b, c \in U.$$

Theorem

The Rosso form $\langle -, - \rangle_U$ of $U_{r,s}(\mathfrak{g})$ is non-degenerate.

The category $\mathcal{O}_f^{r,s}$

Definition (HP12)

The category $\mathcal{O}_f^{r,s}$ consists of finite-dimensional $U_{r,s}(\mathfrak{g})$ −modules V (of type 1) satisfying the following conditions:
(1) V has a weight space decomposition $V = \bigoplus_{\lambda\in\Lambda} V_\lambda$,

$$V_\lambda = \{v \in V \,|\, \omega_\eta v = r^{\langle\lambda,\eta\rangle} s^{-\langle\eta,\lambda\rangle} v,\ \omega'_\eta v = r^{-\langle\eta,\lambda\rangle} s^{\langle\lambda,\eta\rangle} v, \quad \forall \eta \in Q\}$$

and dim(V_λ) is finite for all $\lambda \in \Lambda$;
(2) there exist a finite number of weights $\lambda_1, \cdots, \lambda_t \in \Lambda$ such that

$$\mathrm{Wt}(V) \subset \bigcup_{i=1}^{t} D(\lambda_i),$$

where $D(\lambda) := \{\mu \in \Lambda \,|\, \mu < \lambda\}$. The morphisms are usual $U_{r,s}(\mathfrak{g})$−module homomorphisms.

The Harish-Chandra homomorphism

Let $Z(U)$ be the centre of $U_{r,s}(\mathfrak{g})$. It follows that $Z(U) \subseteq U_0$. Now we define an algebra homomorphism $\gamma^{-\rho} : U^0 \to U^0$ as

$$\gamma^{-\rho}(\omega'_\eta \omega_\phi) = \varrho^{-\rho}(\omega'_\eta \omega_\phi)\omega'_\eta \omega_\phi.$$

Definition

Denote $\xi : Z(U) \to U^0$ as the restricted map $\gamma^{-\rho} \circ \pi|_{Z(U)}$,

$$\gamma^{-\rho}\pi : U_0 \to U^0 \to U^0,$$

where $\pi : U_0 \to U^0$ is the canonical projection. We call ξ the Harish - Chandra homomorphism of U.

Main theorem I

Theorem

When n is even, the Harish-Chandra homomorphism $\xi : Z(U) \to (U_\flat^0)^W$ is an isomorphism for all $U_{r,s}(\mathfrak{g})$; when n is odd, the Harish-Chandra image $\xi(Z(U)) \supseteq (U_\flat^0)^W \otimes \mathbb{K}[z_*, z_*^{-1}]$.

Our strategy:

- The morphism $\xi : Z(U) \to U^0$ is injective when n is even.
- The image $\mathrm{Im}(\xi) \subseteq (U_\flat^0)^W$ when n is even.
- To prove $(U_\flat^0)^W \subseteq \mathrm{Im}(\xi)$, we give an realization of quantum trace using the Rosso form. One set of the central elements can be obtained in this process.
- When n is odd, we construct the invertible central element z_* (a new generator of $Z(U)$).

- The proof of Step I and II relies on the non-degeneration of matrices $R-S$ and $R+S$.
- The matrices $R-S$ is equal to the symmetrized Cartan matrix, i.e.

$$R-S=DC=\frac{2}{\mathfrak{l}}\big((\alpha_i,\alpha_j)\big)_{n\times n}$$

where the matrix $D=\text{diag}(d_1,\cdots,d_n),\ d_i=(\alpha_i,\alpha_i)/\mathfrak{l}$. For type B_n and F_4 we have $\mathfrak{l}=1$, while $\mathfrak{l}=2$ for type A_n, C_n, D_n, E_6, E_7, E_8 and G_2.

Proposition

The matrix $R+S$ is invertible when n is even.

We give the direct results of the calculations for each type. For the case of type A_n, we have $\det(R+S) = \begin{cases} 1, & 2 \mid n \\ 0, & 2 \nmid n \end{cases}$ where

$$A = \begin{pmatrix} rs^{-1} & s & 1 & \cdots & 1 & 1 & 1 \\ r^{-1} & rs^{-1} & s & \cdots & 1 & 1 & 1 \\ 1 & r^{-1} & rs^{-1} & \cdots & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & rs^{-1} & s & 1 \\ 1 & 1 & 1 & \cdots & r^{-1} & rs^{-1} & s \\ 1 & 1 & 1 & \cdots & 1 & r^{-1} & rs^{-1} \end{pmatrix}, \quad R+S = \begin{pmatrix} & 1 & & & \\ -1 & & \ddots & & \\ & \ddots & & \ddots & \\ & & \ddots & & 1 \\ & & & -1 & & 1 \\ & & & & -1 & \end{pmatrix}.$$

For the case of type B_n, we have $\det\,(R+S)=\begin{cases}2^n, & 2\mid n\\ 0, & 2\nmid n\end{cases}$ where

$$A=\begin{pmatrix} r^2s^{-2} & s^2 & \cdots & 1 & 1 & 1\\ r^{-2} & r^2s^{-2} & \ddots & 1 & 1 & 1\\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots\\ 1 & 1 & \ddots & r^2s^{-2} & s^2 & 1\\ 1 & 1 & \cdots & r^{-2} & r^2s^{-2} & s^2\\ 1 & 1 & \cdots & 1 & r^{-2} & rs^{-1}\end{pmatrix},\ R+S=\begin{pmatrix} & 2 & & & & \\ -2 & & \ddots & & & \\ & \ddots & & \ddots & & \\ & & \ddots & & 2 & \\ & & & -2 & & 2\\ & & & & -2 & \end{pmatrix}.$$

For the case of type C_n, we have $\det\,(R+S)=\begin{cases}4, & 2\mid n\\ 0, & 2\nmid n\end{cases}$ where

$$A=\begin{pmatrix} rs^{-1} & s & \cdots & 1 & 1 & 1\\ r^{-1} & rs^{-1} & \ddots & 1 & 1 & 1\\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots\\ 1 & 1 & \ddots & rs^{-1} & s & 1\\ 1 & 1 & \cdots & r^{-1} & rs^{-1} & s^2\\ 1 & 1 & \cdots & 1 & r^{-2} & r^2s^{-2}\end{pmatrix},\ R+S=\begin{pmatrix} & 1 & & & & \\ -1 & & \ddots & & & \\ & \ddots & & \ddots & & \\ & & \ddots & & 1 & \\ & & & -1 & & 2\\ & & & & -2 & \end{pmatrix}.$$

For the case of type D_n, we have $\det\,(R+S)=\begin{cases}4, & 2\mid n\\ 0, & 2\nmid n\end{cases}$ where

$$A=\begin{pmatrix} rs^{-1} & s & \cdots & 1 & 1 & 1\\ r^{-1} & rs^{-1} & \ddots & 1 & 1 & 1\\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots\\ 1 & 1 & \ddots & rs^{-1} & s & s\\ 1 & 1 & \cdots & r^{-1} & rs^{-1} & rs\\ 1 & 1 & \cdots & r^{-1} & r^{-1}s^{-1} & rs^{-1}\end{pmatrix},\ R+S=\begin{pmatrix} & 1 & & & & \\ -1 & & \ddots & & & \\ & \ddots & & \ddots & & \\ & & \ddots & & 1 & 1\\ & & & -1 & & 2\\ & & & -1 & -2 & \end{pmatrix}.$$

For the case of exceptional type F_4, we have $\det\,(R+S)=4$ where

$$A=\begin{pmatrix} r^{-2}s^{-2} & s^2 & 1 & 1\\ r^{-2} & r^{-2}s^{-2} & s^2 & 1\\ 1 & r^{-2} & rs^{-1} & s\\ 1 & 1 & r^{-1} & rs^{-1}\end{pmatrix},\ R+S=\begin{pmatrix} & 2 & & \\ -2 & & 2 & \\ & -2 & & 1\\ & & -1 & \end{pmatrix}.$$

For the case of exceptional type G_2, we have $\det\,(R+S)=9$ where

$$A=\begin{pmatrix} rs^{-1} & s^3\\ r^{-3} & r^3s^{-3}\end{pmatrix},\ R+S=\begin{pmatrix} & 3\\ -3 & \end{pmatrix}.$$

For the family of exceptional type E, we first list the structural constant matrices $A=(a_{ij})_{n\times n}$ and matrices $R+S$ of type E_8.

$$A=\begin{pmatrix} rs^{-1} & 1 & s & 1 & 1 & 1 & 1 & 1 \\ 1 & rs^{-1} & 1 & s & 1 & 1 & 1 & 1 \\ r^{-1} & 1 & rs^{-1} & s & 1 & 1 & 1 & 1 \\ 1 & r^{-1} & r^{-1} & rs^{-1} & s & 1 & 1 & 1 \\ 1 & 1 & 1 & r^{-1} & rs^{-1} & s & 1 & 1 \\ 1 & 1 & 1 & 1 & r^{-1} & rs^{-1} & s & 1 \\ 1 & 1 & 1 & 1 & 1 & r^{-1} & rs^{-1} & s \\ 1 & 1 & 1 & 1 & 1 & 1 & r^{-1} & rs^{-1} \end{pmatrix},\ R+S=\begin{pmatrix} & & 1 & & & & & \\ & & & 1 & & & & \\ -1 & & & 1 & & & & \\ & -1 & -1 & & 1 & & & \\ & & & -1 & & 1 & & \\ & & & & -1 & & 1 & \\ & & & & & -1 & & 1 \\ & & & & & & -1 & \end{pmatrix}.$$

Hence, for the case of type E_8 and type E_6, we have $\det\,(R+S)=1$; for the case of type E_7, we have $\det\,(R+S)=0$.

The odd case

When n is odd, the zero-eigen subspace of the matrix $R+S$ has only 1 dimension. We list one non-zero solution vector for each type as follow.

$A_{2k+1}: \mathfrak{sl}_{2k+2}$	$v^* = (1,0,1,0,\cdots,1,0,1)^T$
$B_{2k+1}: \mathfrak{so}_{4k+3}$	$v^* = (1,0,1,0,\cdots,1,0,1)^T$
$C_{2k+1}: \mathfrak{sp}_{4k+2}$	$v^* = (2,0,2,0,\cdots,2,0,1)^T$
$D_{2k+1}: \mathfrak{so}_{4k+2}$	$v^* = (2,0,2,0,\cdots,2,1,-1)^T$
E_7	$v^* = (0,1,0,0,1,0,1)^T$

There is one extra central element $z_* := \prod_{i=1}^{n}(\omega_i\omega_i')^{v_i^*}$ in $U_{r,s}(\mathfrak{g})$. It is a fixed point of the Harish-Chandra homomorphism ξ, and

$$\xi(z_*) = z_* \notin (U_\flat^0)^W.$$

The realization from the Rosso form

In the proof we have shown one set of central elements.

Theorem

For $\lambda \in \Lambda^+$, there exists an element z_λ satisfying

$$\langle z_\lambda, - \rangle_U = \mathrm{tr}_{L(\lambda)}(- \circ \Theta)$$

where $\Theta \in \mathrm{End}_{\Bbbk}(L(\lambda))$,

$$\Theta : m \mapsto (rs^{-1})^{-\frac{2}{\ell}(\rho,\mu)} m. \quad \forall\, m \in L(\lambda)_\mu,\ \mu \in \Lambda.$$

and

$$z_\lambda = \sum_{\tau \in \Pi(L(\lambda))} \sum_{\mu \in Q^+, i,j} (rs^{-1})^{-\frac{2}{\ell}(\rho,\tau+\mu)} tr(v_j^\mu u_i^\mu \circ P_\tau)\ v_i^\mu \omega'_{\tau+\mu} \omega_{\tau+\mu}^{-1} u_j^\mu \ \in Z(U).$$

where $\{u_j^\mu\}_{j=1}^{d_\mu}$ is a basis of U_μ^+ and $\{v_i^\mu\}_{i=1}^{d_\mu}$ is the dual basis of $U_{-\mu}^-$ with respect to the restriction of $\langle - \,,\, - \rangle$ to $U_{-\mu}^- \times U_\mu^+$, and P_τ is the projector from $L(\lambda)$ to $L(\lambda)_\tau$.

Operators and quantum partial trace

It is worth to ask if there exists a realization from taking operators to partial quantum trace.

For $U_q(\mathfrak{g})$, R. Zhang, et. al. construct central elements by taking partial quantum trace on some operators on $U_q(\mathfrak{g}) \otimes \mathrm{End}(L(\lambda))$.[ZGB91] A similar properties holds in the two-parameter world.

Proposition

Let $\lambda \in \Lambda^+$ and $\zeta : U_{r,s}(\mathfrak{g}) \to End(L(\lambda))$ be the weight representation. If there is an operator $\Gamma \in U_{r,s}(\mathfrak{g}) \otimes \mathrm{End}(L(\lambda))$ such that

$$\Gamma \circ (id \otimes \zeta)\Delta(x) = (id \otimes \zeta)\Delta(x) \circ \Gamma, \quad \forall x \in U_{r,s}(\mathfrak{g}),$$

then the element $c_\Gamma := \mathrm{tr}_2(\Gamma(1 \otimes \Theta)) \in Z(U_{r,s}(\mathfrak{g}))$.

The motivation of the extension

- If the finite-dimensional weight module L of $U_{r,s}$ satisfies $\Pi(L) \subseteq Q$, one can naturally define a character of the finitely dimensional U-module L that

$$\mathrm{Ch}(L) = \sum_{\mu \in \Pi(L)} \dim L_\mu \, \omega'_\mu \omega_{-\mu} \in (U_\flat^0)^W.$$

However, the group-like elements in $U_{r,s}(\mathfrak{g})$ are not always enough to define this character map for all U-modules in $\mathcal{O}_f^{r,s}$.

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However, the group-like elements in $U_{r,s}(\mathfrak{g})$ are not always enough to define this character map for all U-modules in $\mathcal{O}_f^{r,s}$.

- Indeed, the same problem arises in the one-parameter quantum group $U_q(\mathfrak{g})$ which leads to a complex result[LXZ21].
- In order to obtain a uniform expression, it is necessary to extent $U_{r,s}(\mathfrak{g})$ with some group-like elements labeled by fundamental dominate weights $\{\varpi_i\}_{i=1}^n$.

The extended two-parameter quantum group

The extended two-parameter quantum group $\breve{U}_{r,s}(\mathfrak{g})$ has several extended structures which has the same portieres as $U_{r,s}(\mathfrak{g})$.

Proposition

There exists a unique skew-dual pairing $\langle -,- \rangle : \breve{\mathcal{B}}' \times \breve{\mathcal{B}} \to \mathbb{K}$ of the Hopf subalgebra $\breve{\mathcal{B}}$ and $\breve{\mathcal{B}}'$ satisfying

$$\langle f_i, e_j \rangle = \delta_{ij} \frac{1}{s_i - r_i}, \qquad 1 \leqslant i,j \leqslant n,$$

$$\langle \omega'_{\varpi_i}, \omega_{\varpi_j} \rangle = r^{\langle \varpi_i, \varpi_j \rangle} s^{-\langle \varpi_j, \varpi_i \rangle}, \qquad 1 \leqslant i,j \leqslant n,$$

$$\langle \omega_i'^{\pm 1}, \omega_j^{-1} \rangle = \langle \omega_i'^{\pm 1}, \omega_j \rangle^{-1} = \langle \omega_i', \omega_j \rangle^{\mp 1}, \qquad 1 \leqslant i,j \leqslant n,$$

and all other pairs of generators are 0.

The extended two-parameter quantum group

Lemma

The extended Rosso form $\langle -,-\rangle_{\breve{U}}$ is non-degenerate on $\breve{U}$.

Theorem

The Harish-Chandra homomorphism $\breve{\xi}: Z(\breve{U}_{r,s}(\mathfrak{g})) \to (\breve{U}^0_\flat)^W$ is an isomorphism of algebras when n is even, here $\breve{U}^0_\flat := \bigoplus_{\eta\in\Lambda} \mathbb{K}\omega'_\eta\omega_{-\eta}$. In particular, for each $\lambda \in \Lambda^+$,

$$\breve{\xi}(z_\lambda) = \sum_{\mu\in\Lambda} \dim(L(\lambda)_\mu)\, \omega'_\mu\omega_{-\mu}.$$

The character map

Now we are able to construct the character map to study the centre $Z(\breve{U}_{r,s}(\mathfrak{g}))$.

Theorem

Let $K(U_{r,s}) := Gr(\mathcal{O}_f^{r,s}) \otimes_{\mathbb{Z}} \mathbb{K}$, then the character map Ch : $K(U_{r,s}) \to (\breve{U}_\flat^0)^W$ is an isomorphism of algebras that

$$Ch([V]) = \sum_{\mu \in \Lambda} \dim(V_\mu)\, \omega'_\mu \omega_{-\mu}, \quad \forall\, V \in \mathcal{O}_f^{r,s}.$$

The deformed category $\mathcal{O}_f$

N. Hu and Y. Pei have studied the deformation theory of the representations of two-parameter quantum groups $U_{r,s}(\mathfrak{g})$. They define $\mathcal{O}_f^{r,s}$ as the category consisting of finite-dimensional weight modules (of type 1) of $U_{r,s}(\mathfrak{g})$ and prove that

Theorem (HP08)

Assume that $rs^{-1} = q^2$, there is an equivalence for braided tensor categories that

$$\mathcal{O}_f^{r,s} \simeq \mathcal{O}_f^{q,q^{-1}} \simeq \mathcal{O}_f^{q}.$$

where $\mathcal{O}_f^q$ is the category of the finite-dimensional weight modules (of type 1) of the quantum group $U_q(\mathfrak{g})$.

The category $\mathcal{O}_f^q$ is well-studied in many literatures.

Theorem (Etingof 95, Dai 23)

The $K(U_q) := Gr(\mathcal{O}_f^q) \otimes_{\mathbb{Z}} \mathbb{K}$, the Grothendieck ring (over $\mathbb{K}$) of the category $\mathcal{O}_f^q$ of the quantum group $U_q(\mathfrak{g})$, is a polynomial algebra. More precisely, let $\{\varpi_i\}_{i=1}^n$ be the set of fundamental weights of $\mathfrak{g}$, then

$$\begin{aligned} K(U_q) &= \mathbb{K}[[L(\varpi_1)], \cdots, [L(\varpi_n)]] \\ &\cong \mathbb{K}[z_{\varpi_1}, \cdots, z_{\varpi_n}] = Z(U_q). \end{aligned}$$

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It follows that

Theorem

The algebra $K(U_{r,s})$ is a polynomial algebra. When n is even, the centre of $\breve{U} = \breve{U}_{r,s}(\mathfrak{g})$ is a polynomial algebra $Z(\breve{U}) \cong (\breve{U}_\flat^0)^W \cong K(U_{r,s}) \cong K(U_q)$,

$$Z(\breve{U}) = \mathbb{K}[z_{\varpi_1}, \cdots, z_{\varpi_n}].$$

When n is odd, $Z(\breve{U}) \supseteq \mathbb{K}[z_{\varpi_1}, \cdots, z_{\varpi_n}] \otimes \mathbb{K}[z_*, z_*^{-1}]$.

Thank you