Grafting and cografting for species

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Species

Definition (Joyal):

A **species** is a functor $\mathbf{p} : \mathbf{Set}^{\times} \longrightarrow \mathbf{Vect}_{\mathbb{K}}$ from the category of finite sets to the category of vector spaces.

A species $\mathbf{p}$ associates to *any* finite set I a vector space $\mathbf{p}[I]$.

Definition:

A **morphism of species** $f : \mathbf{p} \longrightarrow \mathbf{q}$ is a natural transformation between the functors $\mathbf{p}$ and $\mathbf{q}$.

For each finite set I, a morphism of species $f : \mathbf{p} \longrightarrow \mathbf{q}$ gives a map

$$f[I] : \mathbf{p}[I] \longrightarrow \mathbf{q}[I].$$

Examples of species

Basic examples:

- Given a vector space V, $\mathbf{1}_V$ is the species defined by $\mathbf{1}_V[I] = V$ if $I = \emptyset$, $\mathbf{1}_V[I] = \{0\}$ otherwise.
- $\mathbf{L}$ is the species which to a set I associates the v.s. spanned by all linear orders one can endow I with:

$$\mathbf{L}[\{a,b\}] = \langle (a<b),(b<a) \rangle_{\mathbb{K}}.$$

Example

The tree species $\mathbf{t}$ associates to any finite I the v. s. spanned by all rooted trees structure on can endow I with.

$$\mathbf{t}[\{a,b\}] = \langle \overset{b}{\underset{a}{\mathsf{I}}}\ ,\ \overset{a}{\underset{b}{\mathsf{I}}}\ \rangle_{\mathbb{K}}, \qquad \mathbf{t}[\{a,b,c\}] = \langle \overset{c\ \ b}{\underset{a}{\mathsf{V}}}\ ,\ \overset{c\ \ a}{\underset{b}{\mathsf{V}}}\ ,\ \overset{a\ \ b}{\underset{c}{\mathsf{V}}}\ ,\ \overset{c}{\overset{b}{\underset{a}{\mathsf{I}}}} + \text{perm.} \rangle_{\mathbb{K}}$$

Why Species?

Why you should stop worrying and love species:

- Species were introduced to answer questions of **enumerative combinatorics** questions, as a categorification of generating series.
- Species are a machine to produce results: if two objects have similar properties...
  There is possibly a **species** lurking **behind them**!
- So it might be worth it to formulate interesting (universal) properties in the category of species.

Products for species

Definition:

If $\mathbf{p}$ and $\mathbf{q}$ are two species, the **Cauchy product** of $\mathbf{p}$ and $\mathbf{q}$, is defined by

$$(\mathbf{p} \cdot \mathbf{q})[I] := \bigoplus_{I = S \sqcup T} \mathbf{p}[S] \otimes \mathbf{q}[T].$$

Definition (Aguilar, Mahajan):

A **monoid** is a species $\mathbf{p}$ together with an associative product

$$\mu : \mathbf{p}.\mathbf{p} \longrightarrow \mathbf{p}.$$

So for any finite sets I, S and T with $I = S \sqcup T$ we have a map

$$\mu_{S,T} : \mathbf{p}[S] \otimes \mathbf{p}[T] \to \mathbf{p}[I]$$

associative in some sense.

Coproducts for species

The Cauchy product of $\mathbf{p}$ and $\mathbf{q}$ is defined by

$$(\mathbf{p} \cdot \mathbf{q})[I] := \bigoplus_{I = S \sqcup T} \mathbf{p}[S] \otimes \mathbf{q}[T].$$

Definition (Aguilar, Mahajan):

A **comonoid** is a species with a coassociative coproduct

$$\Delta : \mathbf{p} \longrightarrow \mathbf{p}.\mathbf{p}.$$

So for any finite sets I, S and T with $I = S \sqcup T$ we have a map

$$\Delta_{S,T} : \mathbf{p}[I] \to \mathbf{p}[S] \otimes \mathbf{p}[T]$$

coassociative in some sense.

Symmetric algebra

Definition:

For a species $\mathbf{p}$, its **symmetric algebra** $\mathcal{S}(\mathbf{p})$ is defined by

$$\mathcal{S}(\mathbf{p})[I] := \bigoplus_{\pi \vdash I} \mathbf{p}(\pi)$$

with $\pi = \{B_1, B_2, \ldots, B_k\}$ a partition of I and

$$\mathbf{p}(\pi) = \mathbf{p}[B_1] \odot \cdots \odot \mathbf{p}[B_k].$$

Example:

The symmetric algebra of the species of rooted trees is the species of rooted forests.

Symmetric algebra II

Example:

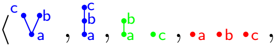
The symmetric algebra of the species of rooted trees is the species of rooted forests.

Example: $I = \{a, b, c\}$. Then

$$\pi = \{a, b, c\}, \{\{a, b\}, \{c\}\}, \{\{a, c\}, \{b\}\}, \{\{b, c\}, \{a\}\}, \{\{a\}, \{b\}, \{c\}\}.$$

Therefore:

$$\mathcal{S}(\mathbf{t})[I] = \mathbf{t}[I] \oplus (\mathbf{t}[a, b] \odot \mathbf{t}[c]) \oplus (\text{perm.}) \oplus (\mathbf{t}[a] \odot \mathbf{t}[b] \odot \mathbf{t}[c])$$

$\simeq \langle$  $, +\text{perm.}\rangle_{\mathbb{K}}$

Monoid structure of the symmetric algebra

Theorem (Aguiar, Mahajan):

$(S(\mathbf{p}), \mu)$ is a monoid for the μ build from the $\mu_{S,T}$ below.

- I, S and T finite sets with $I = S \sqcup T$;
- $\pi_S = \{B_1, \cdots, B_k\} \vdash S$, $\pi_T = \{C_1, \cdots, C_l\} \vdash T$,

Then $\pi_S \cup \pi_T \vdash I$ and define

$$\mu_{S,T}^{\pi_S,\pi_T} : \mathbf{p}(\pi_S) \otimes \mathbf{p}(\pi_T) \longrightarrow \mathbf{p}(\pi_S \cup \pi_T)$$
$$(x_1 \odot \cdots \odot x_k) \otimes (y_1 \odot \cdots \odot y_l) \longmapsto x_1 \odot \cdots x_k \odot y_1 \odot \cdots \odot y_l$$

$$\mu_{S,T} := \sum_{\pi_S \vdash S, \pi_T \vdash T} \mu_{S,T}^{\pi_S,\pi_T}.$$

Comonoid structure of the symmetric algebra

Theorem (Aguiar, Mahajan):

$(S(\mathbf{p}), \Delta)$ is a comonoid for $\Delta = (\Delta_I)_I$ described below

- I, S and T finite sets with $I = S \sqcup T$;
- $\pi_S = \{B_1, \cdots, B_k\} \vdash S$, $\pi_T = \{C_1, \cdots, C_l\} \vdash T$,

Then $\pi_S \cup \pi_T \vdash I$ and define

$$\Delta_{S,T}^{\pi_S,\pi_T} : \mathbf{p}(\pi_S \cup \pi_T) \longrightarrow \mathbf{p}(\pi_S) \otimes \mathbf{p}(\pi_T)$$

$$x_1 \odot \cdots \odot x_{k+l} \longrightarrow \bigodot_{i \text{ s.t. } x_i \in \mathbf{p}[B_j]} x_i \otimes \bigodot_{\alpha \text{ s.t. } x_\alpha \in \mathbf{p}[B_\beta]} x_\alpha .$$

$$\Delta_{S,T} := \sum_{\pi_S \vdash S, \pi_T \vdash T} \Delta_{S,T}^{\pi_S,\pi_T}, \quad \Delta_I := \sum_{S \sqcup T = I} \Delta_{S,T}.$$

Derivative for species

Definition:

Given a species $\mathbf{p}$, its **derivative** $\mathbf{p}'$ is defined by

$$\mathbf{p}'[I] := \mathbf{p}[I \cup \{*_I\}].$$

Given $f : \mathbf{p} \longrightarrow \mathbf{q}$, its **derivative** f' is defined by

$$f'[I] := f[I \cup \{*_I\}] : \mathbf{p}'[I] \longrightarrow \mathbf{q}'[I].$$

$$(\mathbf{p}.\mathbf{q})' = \mathbf{p}.\mathbf{q}' + \mathbf{p}'.\mathbf{q}.$$

Example:

The species $\mathbf{t}'$ is the species of non-empty trees.

Species with up and down operators

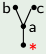
Definition (Guță, Maasen):

- A **species with up operator** is a species $\mathbf{p}$ with a morphism of species $u : \mathbf{p} \to \mathbf{p}'$.
- A **species with down operators** is a species $\mathbf{p}$ with a morphism of species $d : \mathbf{p}' \to \mathbf{p}$.

Example:

$B : \mathbf{t} \longrightarrow \mathbf{t}'$ is the up operator defined by adding a root decorated by $*_I$ below each trees in $\mathbf{t}[I]$:

$(B\mathbf{t})[\{a, b, c\}] = \langle$ 

, 

$, + \text{perm.}\rangle_{\mathbb{K}} \subsetneq \mathbf{t}[\{a, b, c\} \cup \{*\}].$

NB: the "inverse" map ᗺ is not a down op.: its image is not in $\mathbf{t}$.

Species with up and down (co)derivations

up: $u : \mathbf{p} \to \mathbf{p}'$, down: $d : \mathbf{p}' \to \mathbf{p}$.

Definition:

An up (resp. down) operator $u : \mathbf{p} \to \mathbf{p}'$ (resp. $d : \mathbf{p}' \to \mathbf{p}$) is **coderivation** (resp. **derivation**) if

$$
\begin{array}{ccc}
\mathbf{p} & \xrightarrow{\;u\;} & \mathbf{p}' \\
\Delta \downarrow & & \downarrow \Delta' \\
\mathbf{p}\cdot\mathbf{p} & \xrightarrow{\;u\cdot\mathrm{id}+\mathrm{id}\cdot u\;} & (\mathbf{p}\cdot\mathbf{p})'
\end{array}
\qquad \text{resp.:} \qquad
\begin{array}{ccc}
(\mathbf{p}\cdot\mathbf{p})' & \xrightarrow{\;d\cdot\mathrm{id}+\mathrm{id}\cdot d\;} & \mathbf{p}\cdot\mathbf{p} \\
\mu' \downarrow & & \downarrow \mu \\
\mathbf{p}' & \xrightarrow[\;d\;]{} & \mathbf{p}
\end{array}
$$

commutes.

Balanced up-down operators

For $[n] := \{1, \cdots, n\}$, $x \in \mathbf{p}[n]$ and $\sigma \in S_n$, write $\sigma.x := \mathbf{p}[\sigma](x)$.

Definition (Aguiar, Mahajan):

A species with up *and* down operators $(\mathbf{p}, u, d)$ is **balanced** if, $\forall n \in \mathbb{N}$

$$(1,2).u^2(x) = u^2(x) \quad \forall x \in \mathbf{p}[n] \tag{1}$$

$$d^2((1,2).x) = d^2(x) \quad \forall x \in \mathbf{p}[n+2] \tag{2}$$

$$d \circ u = \lambda_n \,\mathrm{Id} \quad \text{for some } \lambda_n \in \mathbb{K} \tag{3}$$

$$d((k+1,1).u(x)) = (k,1).u(d(x)) \quad \forall x \in \mathbf{p}[n],\ 1 \leq k < n. \tag{4}$$

These are difficult to produce!

Balanced up-down operators II

Why should we care?
There are functors $\mathcal{K} : \mathbf{Sp} \longrightarrow \mathbf{grVect}_{\mathbb{K}}$ that "realise" species.
Then

$$(\mathbf{p}, u, d) \text{ balanced } \implies (\mathcal{K}(\mathbf{p}), \mathcal{K}(u), \mathcal{K}(d)) \text{ a graded vector space}$$

with **creation-annihilation operators**.

(1) $\Rightarrow$ commutation of creation operator,
(2) $\Rightarrow$ commutation of annihilation operator,
(3)+(4) $\Rightarrow$ commutation of creation/annihilation operators.

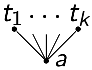
Graftings: for forests to species

For a set I

$$\mathcal{F}_I := \langle \text{forests decorated by } I \rangle_{\mathbb{K}}, \quad \mathcal{T}_I := \langle \text{trees decorated by } I \rangle_{\mathbb{K}}.$$

For $a \in I$, $B_+^a : \mathcal{F}_I \longrightarrow \mathcal{T}_I \setminus \emptyset$ defined by

$$B_+^a(t_1 \cdots t_k) =$$



Recall

$$\mathcal{F}_I \approx \mathcal{S}(\mathbf{t})[I], \qquad \mathcal{T}_I \setminus \emptyset \approx \mathbf{t}'[I].$$

Definition:

A **grafting map** for a species $\mathbf{p}$ is a map $\mathrm{B} : \mathcal{S}(\mathbf{p}) \longrightarrow \mathbf{p}'$.

Cograftings

A **grafting map** for a species $\mathbf{p}$ is a map $\mathrm{B} : S(\mathbf{p}) \longrightarrow \mathbf{p}'$.

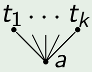
Definition:

A **cografting map** for a species $\mathbf{p}$ is a map $\reflectbox{B}\colon \mathbf{p}' \longrightarrow \mathcal{S}(\mathbf{p})$.

Example:

The inverse map $\reflectbox{B} := (B_+)^{-1}$ is now a cografting!

$$\reflectbox{B}\left(\quad \right) := t_1 \cdots t_k$$



Up operators from graftings and cograftings

$$\mathrm{B} : \mathcal{S}(\mathbf{p}) \longrightarrow \mathbf{p}'.$$

Definition:

Let $(\mathbf{p}, \mathrm{B})$ be a species with a grafting. Define

- the **set up operator** $u_{\mathrm{B}} : \mathcal{S}(\mathbf{p}) \longrightarrow \mathcal{S}(\mathbf{p})'$ by

$$u_{\mathrm{B}} : \mathcal{S}(\mathbf{p}) \xrightarrow{\mathrm{B}} \mathbf{p}' \hookrightarrow \mathcal{S}(\mathbf{p})'.$$

- the **algebraic up operator** $u^{\mathrm{B}} : \mathcal{S}(\mathbf{p}) \longrightarrow \mathcal{S}(\mathbf{p})'$ by

$$u^{\mathrm{B}} : \mathcal{S}(\mathbf{p}) \xrightarrow{\Delta} \mathcal{S}(\mathbf{p}) \cdot \mathcal{S}(\mathbf{p}) \xrightarrow{\mathrm{B} \cdot \mathrm{Id}} \mathbf{p}' \cdot \mathcal{S}(\mathbf{p}) = \mathcal{S}(\mathbf{p})'$$

Down operators from graftings and cograftings

$$\reflectbox{B} : \mathbf{p}' \longrightarrow \mathcal{S}(\mathbf{p}).$$

Definition:

Let $(\mathbf{p}, \reflectbox{B})$ be a species with a cografting. Define

- the **set down operator** $d_{\reflectbox{B}} : \mathcal{S}(\mathbf{p})' \longrightarrow \mathcal{S}(\mathbf{p})$ by

$$d_{\reflectbox{B}} : \mathcal{S}(\mathbf{p})' \twoheadrightarrow \mathbf{p}' \xrightarrow{\reflectbox{B}} \mathcal{S}(\mathbf{p}).$$

- the **algebraic down operator** $d^{\reflectbox{B}} : \mathcal{S}(\mathbf{p})' \longrightarrow \mathcal{S}(\mathbf{p})$ by

$$d^{\reflectbox{B}} : \mathcal{S}(\mathbf{p})' = \mathbf{p}'.\mathcal{S}(\mathbf{p}) \xrightarrow{\reflectbox{B} \cdot \mathsf{Id}} \mathcal{S}(\mathbf{p}).\mathcal{S}(\mathbf{p}) \xrightarrow{\mu} \mathcal{S}(\mathbf{p}).$$

Universal property

$$u_{\mathrm{B}} : \mathcal{S}(\mathbf{p}) \xrightarrow{\mathrm{B}} \mathbf{p}' \hookrightarrow \mathcal{S}(\mathbf{p})'.$$

Theorem (C., Paycha, Vargas):

Let $\mathrm{B} : \mathcal{S}(\mathbf{t}) \to \mathbf{t}'$ the grafting of forests. For every commutative monoid $(\mathbf{q}, \nu)$ with up operator $u : \mathbf{q} \to \mathbf{q}'$, there exists a unique map of monoids with up operators

$$\phi : (\mathcal{S}(\mathbf{t}), \mu, u_{\mathrm{B}}) \to (\mathbf{q}, \nu, u).$$

In particular, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{S}(\mathbf{t}) & \xrightarrow{\phi} & \mathbf{q} \\ \downarrow{\scriptstyle u_{\mathrm{B}}} & & \downarrow{\scriptstyle u} \\ \mathcal{S}(\mathbf{t})' & \xrightarrow{\phi'} & \mathbf{q}' \end{array}$$

(Co)Derivations from (co)graftings

Proposition (C., Paycha, Vargas):

Let $(\mathbf{p}, \mathrm{B})$ be a species with a grafting and $(\mathbf{q}, \text{ᗺ})$ a species with a cografting. Then $(\mathcal{S}(\mathbf{p}), \mu, u^{\mathrm{B}})$ is a comonoid with an up coderivation and $(\mathcal{S}(\mathbf{q}), \Delta, d_{\text{ᗺ}})$ is a monoid with a derivation:

$$
\begin{array}{ccc}
\mathbf{p} & \xrightarrow{u^{\mathrm{B}}} & \mathbf{p}' \\
\Big\downarrow \Delta & & \Big\downarrow \Delta' \\
\mathbf{p}\cdot\mathbf{p} & \xrightarrow{u^{\mathrm{B}}\cdot\mathrm{id}+\mathrm{id}\cdot u^{\mathrm{B}}} & (\mathbf{p}\cdot\mathbf{p})'
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
(\mathbf{p}\cdot\mathbf{p})' & \xrightarrow{d_{\text{ᗺ}}\cdot\mathrm{id}+\mathrm{id}\cdot d_{\text{ᗺ}}} & \mathbf{p}\cdot\mathbf{p} \\
\Big\downarrow \mu' & & \Big\downarrow \mu \\
\mathbf{p}' & \xrightarrow[d_{\text{ᗺ}}]{} & \mathbf{p}
\end{array}
$$

Proof: elegant or brute force.

Balanced up/down operators from co/graftings

Let $(\mathbf{p}, \mathrm{B}, \mathrm{Ꙏ})$ be a species with grafting and cografting. Then $(\mathcal{S}(\mathbf{p}), u_{\mathrm{B}}, d_{\mathrm{Ꙏ}})$ is balanced i.f.f.:

$$(1,2).\mathrm{B}^2(x) = \mathrm{B}^2(x) \quad \forall x \in \mathbf{p}[n]$$

$$\mathrm{Ꙏ}^2((1,2).x) = \mathrm{Ꙏ}^2(x) \quad \forall x \in \mathbf{p}[n+2]$$

$$\mathrm{Ꙏ} \circ \mathrm{B} = \lambda_n \,\mathrm{Id} \quad \text{for some } \lambda_n \in \mathbb{K}$$

$$\mathrm{Ꙏ}((k+1,1).\mathrm{B}(x)) = (k,1).\mathrm{B}(\mathrm{Ꙏ}(x)) \quad \forall x \in \mathbf{p}[n],\ 1 \leq k < n$$

This suggests a strategy to build (hopefully) non-trivial pairs of balanced up/down operators!

An example

Example:

Let $\mathbf{g}$ be the species of connected, non-oriented graphs without self-loops. $\mathcal{S}(\mathbf{g})$ is the species of non-connected, non-oriented graphs without self-loops. Define

$$\mathrm{B} : \mathcal{S}(\mathbf{g}) \longrightarrow \mathbf{g}', \quad \reflectbox{B} : \mathbf{g}' \longrightarrow \mathcal{S}(\mathbf{g}).$$

- B adds a new vertex linked to all former vertices,
- $\reflectbox{B}$ removes this vertex and all edges attached to it.

Then $(\mathcal{S}(\mathbf{g}), u_{\mathrm{B}}, d_{\reflectbox{B}})$ is balanced with $\lambda_n = 1$.

- First three relations (rather) trivial.
- $\reflectbox{B}((k+1,1).\mathrm{B}(x)) = (k,1).\mathrm{B}(\reflectbox{B}(x))$ requires some work.

Openings and conclusion

Openings:

- Connes-Kreimer comonoid for $\mathcal{S}(\mathbf{t})$ still to be investigated.
- General results about balanced up/down operators from graftings and cograftings?
- Other structures on species from graftings and cograftings?

Conclusion:

Openings and conclusion

Openings:

- Connes-Kreimer comonoid for $\mathcal{S}(\mathbf{t})$ still to be investigated.
- General results about balanced up/down operators from graftings and cograftings?
- Other structures on species from graftings and cograftings?

Conclusion:

Species are fun!

Thank you for your attention.