# Basic prerequisites in differential geometry and operator theory in view of applications to quantum field theory

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# Introduction

These lecture notes are a written version of prerequisite lectures aimed at students attending summer schools in Villa de Leyva, Colombia (1999-2011) on geometric (and topological) methods for quantum field theory as well as parts of some postgraduate lectures delivered at the Université Blaise Pascal, in Clermont-Ferrand, France in 2008, the University of Los Andes, Bogotá in 2011 and 2015, the Lebanese University (Université libanaise) in Beyrouth, Lebanon in 2012. They are intended for graduate students in mathematics or physics who need some basic concepts in differential geometry, global analysis, operator algebras and pseudodifferential operators in view of understanding how these are used in quantum field theory.

Far from being complete, these notes offer a first guide for the layperson, suggesting further references for the interested reader. The list of references at the beginning of each section is also far from complete and is just meant to give the reader a first hint of the (often huge) literature on the subject. I have mostly chosen to refer to text books and survey type articles, in order to limit the number of references.
Due to lack of space, I unfortunately have had to leave out numerous examples that illustrate the sometimes rather abstract concepts presented here. The last chapter somewhat compensates for this lack of example by illustrating in Yang-Mills, Seiberg-Witten and string theory how the various concepts introduced in the previous chapter can come into play to investigate the structure of the configuration and moduli spaces.

For the sake of simplicity, I chose to introduce the concepts of manifold and vector bundle in the simplest infinite dimensional setting, namely the Hilbert setting, leaving aside the more subtle concepts needed in the Fréchet setting. A more general infinite dimensional setting is described in [KM].
The Hilbert setting offers various simplifications; we have the very useful implicit function theorem at hand and any closed subspace of a Hilbert space splits the Hilbert space as a direct sum of this subspace with its orthogonal complement. Also, a partition of unity can be defined on a Hilbert manifold, which is not always the case on Banach manifolds.
In fact the most appropriate setting for our needs is the I.L.H. setting, namely the inverse limit of Hilbert spaces [O] which we shall only briefly mention in the applications at the end of these notes.

These notes start at a leisurely pace but the material gets denser as one moves along in the chapters, relying on the fact that the reader who is acquainted with the first chapters is familiar enough with the geometric concepts to be able to use them rather loosely in the last chapters.
The present lecture notes are organised in ten chapters; the first four chapters are dedicated to prerequisites in differential geometry (including infinite dimensional Banach structures), chapters 5 to 8 are dedicated to operators and operator algebras of different types, including a few basic facts concerning pseudodifferential operators. Chapter 9 offers a brief incursion into index theory and Chapter 10 is dedicated to the geometry of configuration and moduli spaces one comes across in Yang-Mills, Seiberg-Witten and string theory.

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# 1 Manifolds, Lie algebras and Lie groups

## 1.1 Banach vector spaces

Useful references are [B], [Bre], [L], [Ru], [W].

Recall that a Banach vector space (we shall say Banach space for short) is a vector space equipped with a norm for which it is closed.

**Definition 1** *Let $E$ and $F$ be two Banach spaces and $U$ an open subset of $E$. A map $f : E \to F$ is differentiable at a point $x_0$ of $U$ provided there exists a continuous linear map $L : E \to F$ and a map $\phi : U \subset E \to F$ defined on a neighborhood $U$ of $0 \in E$ such that*

$$f(x_0 + y) = f(x_0) + L(y) + \phi(y)$$

*with $\lim_{y \to 0} \frac{\|\phi(y)\|_F}{\|y\|_E} = 0$, where $\| \cdot \|_E$ is the norm on $E$ and $\| \cdot \|_F$ the norm on $F$. Then $L$ is a uniquely defined map, called the differential at point $x_0$ and denoted by $D_{x_0} f$.*

The space $\mathcal{L}(E, F)$ of continuous linear maps from $E$ to $F$ is also a Banach space when equipped with the norm $|||L||| := \sup_{u \neq 0} \frac{\|Lu\|_F}{\|u\|_E}$.

**Definition 2** *Let $E$ and $F$ be two Banach spaces and $U$ an open subset of $E$. The map $f : U \subset E \to F$ is of class $C^1$ on $U$ provided $f$ is differentiable at any point of $U$ and the map:*

$$\begin{array}{rcl} Df : U & \to & \mathcal{L}(E, F) \\ x & \mapsto & D_x f \end{array}$$

*is continuous.*

Indentifying $\mathcal{L}\left(\mathcal{L}(E, \mathcal{L}\left(E, \cdots (E, F) \cdots, F\right) F\right)$ –where $E$ arises $k$ times– with the Banach space $\mathcal{L}^k(E, F)$ of $k$-linear maps from $E^k$ to $F$, we can define the notion of $C^k$ differentiability.

**Definition 3** *A differentiable map $f : U \subset E \to F$ is of class $C^k$ provided $Df$ is of class $C^{k-1}$.*
*Let $E$ and $F$ be two Banach spaces and $U$, $V$ two open subsets of $E$ and $F$, respectively. A differentiable map $f : U \to V$ is a diffeomorphism whenever it is one to one and onto and its inverse map is differentiable. It is a $C^k$ diffeomorphism whenever it is a diffeomorphism and both $f$ and its inverse $f^{-1}$ are of class $C^k$.*

The following well-known results in Banach spaces will be used later in these notes.

**Theorem 1 (Local inverse function theorem)** *Let $E$ and $F$ be Banach spaces, $U$ an open subset of $E$ and $f : U \to F$ a $C^k$ map for some $k \geq 1$. If for some point $x_0 \in U$ the map $D_{x_0} f : E \to F$ is an isomorphism then there exists a neighborhood $W$ of $x_0$ such that the restriction $f_{|W} : W \to f(W)$ of $f$ to $W$ is a $C^k$ diffeomorphism.*

**Theorem 2 (Hahn-Banach theorem)** *Let $F \subset E$ be a linear subspace of $E$ s.t. $\bar{F} \neq E$. Then there is a continuous linear form $L$ on $E$ such that $L(u) \neq 0 \quad \forall u \in F$.*

Applying this result to the vector space $F = \langle u_0 \rangle$ generated by some $u_0 \in E$ yields the following:

**Corollary 1** *Let* $u_0 \neq 0 \in E$*, where* $E$ *is a Banach space. Then there is a continuous linear form* $L$ *on* $E$ *such that* $L(u_0) \neq 0$.

Another fundamental result for the following is the

**Theorem 3 (Open mapping theorem)** *Let* $E$ *and* $F$ *be two Banach spaces. A continuous linear map* $L : E \to F$ *which is onto takes an open subset to an open subset. If it is continuous and one to one, it is a homeomorphism.*

**Corollary 2** *Let* $F$ *and* $G$ *be two closed linear subspaces in* $E$ *such that* $F \oplus G = E$*. Then the map:*

$$\begin{array}{rcl} F \times G & \to & E \\ (u, v) & \mapsto & u + v \end{array}$$

*is an isomorphism of Banach spaces.*

Restricting oneself to the Hilbert setting is convenient because of the existence of orthogonal complements for closed subspaces. This property can be formulated as follows.

**Definition 4** *A linear subspace* $F$ *of a Banach vector space* $E$ *splits the space* $E$ *if it is closed and if there exists a closed linear subspace* $G$ *of* $E$ *such that* $E = F \oplus G$.

In the finite dimensional setting, any subspace splits a vector space. In the Hilbert setting, any closed subspace of a Hilbert space splits the space; the orthogonal complement does the job and the above Corollary takes the following form.

**Corollary 3** *Let* $E$ *be a Hilbert vector space and* $F$ *a closed linear subspace of* $E$ *then the map:*

$$\begin{array}{rcl} F \times F^{\perp} & \to & E \\ (u, v) & \mapsto & u + v \end{array}$$

*is an isomorphism of Hilbert spaces.*

## 1.2 Manifolds, submanifolds

Useful references are [Hu], [KN],[L], [M], [Wu].

**Definition 5** *A manifold* $M$ *of class* $C^k$*,* $k \geq 0$ *(or* $C^k$*-manifold) modelled on a Banach space* $E$ *(called the model space) is a topological space equipped with a* $C^k$*-atlas i.e. a set of local charts* $\{(U_i, \phi_i), i \in I\}$ *satisfying the following requirements:*

*i) for any* $i \in I$ *the subset* $U_i$ *is open and* $M = \bigcup_{i \in I} U_i$*,*

*ii) for any* $i \in I$*, the map* $\phi_i : U_i \to \phi_i(U_i)$ *is a homomeorphism onto an open subset of* $E$*,*

*iii) for any* $i, j \in I$ *the maps*

$$\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$$

*are diffeomorphisms of class* $C^k$ *called transition maps.*

*It is of class $C^\infty$ if it is of class $C^k$ for all $k \geq 0$.*

An atlas is not unique and any $C^k$ (resp. $C^\infty$) atlas could do; one usually picks out the *maximal* atlas, i.e. one that contains all the others.

A (real) finite dimensional manifold of dimension $n$ is modelled on $E = I\!R^n$ and local charts provide *local coordinates*:

$$\begin{aligned} \phi_i : U_i &\rightarrow \phi_i(U_i) \\ x &\mapsto (x_1, \cdots, x_n) \in I\!R^n. \end{aligned}$$

Transition maps are elements of $GL(n, I\!R)$.

**Example 1**

- *The unit sphere in $R^{n+1}$ defined as:*

$$S^n = \{(x_0, \cdots, x_n) \in I\!R^{n+1}, \sum_{i=0}^{n} |x_i|^2 = 1\}$$

*is a smooth manifold of dimension $n$. Local charts are $(U_1, \phi_1)$ and $(U_2, \phi_2)$ where $U_1 = S^n - \{N\}$ and $U_2 - \{S\}$, $N = (0, \cdots, 0, 1)$ the south pole and $S = (0, \cdots, 0, -1)$ the south pole, $\phi_1(x_0, \cdots, x_n) = \left(\frac{x_0}{1-x_n}, \cdots, \frac{x_{n-1}}{1-x_n}\right)$,$\phi_2(x_0, \cdots, x_n) = \left(\frac{x_0}{1+x_n}, \cdots, \frac{x_{n-1}}{1+x_n}\right)$.*

- *The $n$-th dimensional torus $T^n = R^n / \sim$ where*

$$z_1 \sim z_2 \Leftrightarrow \exists\, n_i \in Z\!\!\!Z, i = 1 \cdots, n, \quad z_1 - z_2 = \sum_{i=1}^{n} n_i e_i$$

*where we have set $e_i = (0, \cdots, 0, 1, 0, \cdots, 0 \in I\!R^n$ with $1$ at the $i$-th place.*

- *The projective plane $P_n(R) = R^{n+1} - \{0\} / \sim = S^n / \simeq$ where $z_1 \sim z_2 \Leftrightarrow \exists \lambda \in I\!R, \quad z_1 = \lambda z_2$ and $z_1 \simeq z_2 \Leftrightarrow \exists \lambda \in \{-1, 1\}, \quad z_1 = \lambda z_2$. Local charts are given by $U_i = \{(x_0, x_1, \cdots, x_n) \in I\!R^{n+1}, x_i \neq 0\}$ and $\phi_i(x) = \left(\frac{x_0}{x_i}, \cdots, \frac{\hat{x_i}}{x_i}, \cdots, \frac{x_n}{x_i}\right)$ where the $\hat{}$ means we have deleted the variable.*

- *The Grassmann manifold $G^n_k(I\!R)$ of $k$-dimensional submanifolds of $I\!R^n$. Given $V \subset I\!R^n$, we can identify $I\!R^n = V \times V^\perp$. A neighborhood of $V \in G^n_k$ is mapped homeomorphically onto an open set in the vector space of linear maps $V \rightarrow V^\perp$. This makes $G^n_k$ a manifold of dimension $k(n-k)$. The case $k = 1$ yields back the projective space $P_n(I\!R)$.*

**Definition 6** *Let $F$ be a linear subspace of a Banach vector space $E$ that splits $E$. Given a $C^k$ manifold $M$ modelled on $E$, a subset $N$ of $M$ is a submanifold of $M$ modelled on $F$ provided there is a $C^k$-atlas $\{(U_i, \phi_i), i \in I\}$ on $M$ that induces an atlas on $N$, i.e. for any $i \in I$ there are open subsets $V_i, W_i$ of $E, F$ such that*

$$\phi_i(U_i) = V_i \oplus W_i$$

*and*

$$\phi_i(U_i \cap N) = V_i \oplus \{0\}.$$

In the case of an $n$-dimensional real manifold, the model space is $R^n$ and a subspace $F$ of dimension $k \leq n$ of $I\!R^n$ can be equipped with a basis which we complete to a basis of $I\!R^n$. In this basis local coordinates on $N$ will be of the form:

$$\begin{array}{rcl} (\phi_i)_{|_N} : U_i \cap N & \to & \phi_i(U_i) \\ x & \mapsto & (x_1, \cdots, x_k, 0, \cdots, 0) \in I\!R^n. \end{array}$$

In what follows, using local charts, we carry out to manifolds the notion of differentiability, $C^k$-regularity, and the notion of diffeomorphism, to maps between manifolds. Although the construction uses local charts, the concept thereby defined is shown to be independent of the choice of local chart. All the manifolds involved are Banach manifolds.

**Definition 7** *Let $M, N$ be two Banach manifolds of class $C^k, C^l$, $k, l \geq 1$ respectively and modelled on $E, F$ respectively. A map $f : M \to N$ is differentiable at a point $x_0 \in M$ provided there is a local chart $(U, \phi)$ of $M$ containing $x_0$, a local chart $(V, \psi)$ containing $f(x_0)$ such that the map*

$$\psi \circ f \circ \phi^{-1} : \phi(U) \subset E \to \psi(V) \subset F$$

*is differentiable at point $\phi(x_0)$.*

**Definition 8** *A tangent vector at a point $x$ of a $C^k$-Banach manifold $M$ $(k \geq 1)$ modelled on $E$ is an equivalence class $\xi$ of triples $(U, \phi, v)$ where $(U, \phi)$ is a local chart on $M$ containing $x$ and $v$ a vector in the Banach space $E$, the equivalence relation being defined by:*

$$(U, \phi, v) \sim (V, \psi, w) \Leftrightarrow w = D_{\phi(x)} \left(\psi \circ \phi^{-1}\right)(v).$$

*In other words, $v$ is the tangent vector $\xi$ read in the local chart $(U, \phi)$ whereas $w$ is the tangent vector $\xi$ read in the local chart $(V, \psi)$.*

In the finite dimensional setting, say in dimension $n$, given a local system of coordinates $(x_1, \cdots, x_n)$, a tangent vector reads $v := \sum_{i=1}^n v_i \frac{\partial}{\partial x_i}$.

The space $T_xM$ of tangent vectors at a point $x \in M$ can be equipped with a vector space structure induced from that of the model space $E$. Since transition maps are diffeomorphisms, the maps $D_{\phi(x)} \left(\psi \circ \phi^{-1}\right)$ are isomorphisms of Banach spaces and $T_xM$ can be equipped with a Banach structure induced from that on $E$. Thus $T_xM \simeq E$ is a Banach space.

**Proposition 1** *Let $M$ and $N$ be two Banach manifolds of class $C^k$, $C^l$ respectively, with $k, l \geq 1$. Let $f : M \to N$ be a differentiable map, then $D_xf : T_xM \to T_{f(x)}N$ is a linear map, called the differential of $f$ at point $x$ defined by:*

$$D_xf(\xi) = \eta \Leftrightarrow v = D_{\phi(x)} \left(\psi \circ f \circ \phi^{-1}\right) u$$

*whenever $u$ corresponds to $\xi$ read in a local chart $(U, \phi)$ containing $x$ and $v$ corresponds to $\eta$ read in a local chart $(V, \psi)$ containing $f(x)$.*

This definition is independent of the choice of local chart.

Given a manifold $M$ of class $C^k$ modelled on a Banach space $E$, $k \geq 1$, the set $TM := \bigcup_{x\in M} T_xM$ can be equipped with a $C^{k-1}$- manifold structure; a local chart at $(x, \xi)$, $\xi \in T_xM$ reads $(U \times E, \phi \otimes D\phi)$ where $(U, \phi)$ is a local chart on $M$ at point $x$.

**Definition 9** *Given manifolds $M, N$ of class $C^k, C^l$, $k, l \geq 1$, a map $f : M \to N$ is of class $C^j$, with $1 \leq j \leq inf\{k, l\}$ provided it is differentiable and $Df : TM \to TN$ is of class $C^{j-1}$.*

Submanifolds can be obtained via embeddings, a particular class of immersion. As we saw in the first section, a subspace of a Banach space does not automatically split the space $E$, so that we need to encode a "splitting" condition in the definition of immersion:

**Definition 10** *A differentiable map $f : M \to N$ from a $C^k$-manifold $M$, to a $C^l$-manifold $N$ with $k, l \geq 1$ is an immersion (resp. submersion) provided $D_xf$ is injective (resp. onto) and the range $R(D_xf)$ (resp. the kernel $Ker(D_xf)$ splits $T_{f(x)}N$ (resp. $T_xM$) for any $x \in M$.*

Here again, the Hilbert setting offers a simplification:

A differentiable map $f : M \to N$ from a Hilbert manifold $M$ to a Hilbert manifold $N$ is an immersion (resp. submersion) provided $D_xf$ is injective (resp. onto) and $R(D_xf)$ is closed for any $x \in M$. (Note that the kernel is always closed when the operator is closed).

An injective immersion is called an *embedding*. The following result which is a manifold version of the global inverse map theorem will be very useful in the slice theorem.

**Theorem 4 (Global inverse mapping theorem)** *An embedding $f : M \to N$ that is a homeomorphism onto its range yields a submanifold $f(M)$ of $N$ and $f(M) \simeq M$, namely $M$ is diffeomorphic to its range.*

## 1.3 Partitions of unity

Useful references are [KM], [L].

Partitions of unity provided means of gluing together local objects in order to build a globally defined one. It is therefore important to define conditions under which partitions of unity with a certain degree of regularity exist.

**Definition 11** *A partition of unity of class $C^k$ of a $C^k$-manifold $M$ is given by a locally finite covering $(u_i)_{i\in I}$ of $M$ and a family $\{\psi_i, i \in I\}$ of maps of class $C^k$:*

$$\psi_i : E \to \mathbb{R}$$

*such that:*

1. $0 \leq \psi_i, \quad \forall i \in I$
2. *The support of $\psi_i$ is contained in $U_i$*

*3.* $\sum_{i \in I} \psi_i(p) = 1$.

*Such a partition of unity is said to be subordinated to an atlas* $(W_i, \phi_i)_{i \in I}$ *on the manifold if* $\bar{U}_i \subset W_i$. *A partition of unity is smooth whenever it is of class* $C^k$ *for any* $k \in \mathbb{N}$.

Let us recall that a manifold is *paracompact* if from any cover of the manifold, one can extract a locally finite sub-cover, i.e. a subcover such that every point of the manifold admits a neighborhood which only intersects a finite number of the open subsets of this covering.
The following topological lemma will be useful in the sequel:

**Lemma 1** *1) Any paracompact manifold is normal, i.e. two disjoint closed subsets have dijoint neighborhoods.*

*2) (Urysohn's Lemma) Given two closed disjoints subsets* $A$ *and* $B$ *of a topological normal vector space* $E$, *there is a continuous map* $f : E \to [0, 1]$ *which vanishes on* $A$ *and is equal to* $1$ *on* $B$.

*3) Given a locally finite covering* $(U_i)$ *of a paracompact topological vector space, there is a locally finite subcovering* $(V_i)$ *such that* $\bar{V}_i \subset U_i$.

**Proposition 2** *Any paracompact topological manifold has a continuous partition of unity.*

**Proof 1** *Let* $U$ *be an open set on a manifold modelled on some sparable space* $E$ *and let* $x \in U$. *Let* $(U_i, \phi_i)$ *be an atlas and* $(U_{i_0}, \phi_{i_0})$ *be a local chart at point* $x_0$. $\phi_{i_0}$ *can be composed with a map that sends an open ball of* $E$ *onto* $E$ *in such a way that the resulting map (also denoted by* $\phi_0$*) satisfies* $\phi_{i_0}(U_{i_0}) = E$. *Since the manifold is Banach, it is metrisable, since it is moreover separable, it is paracompact ([L], chap. II, par. 3, Lemma 1). Thus one can extract from the above subcovering a locally finite one. The third part of the above lemma then yields locally finite subcoverings* $(V_i)$ *and* $(W_i)$ *such that* $\bar{W}_i \subset V_i \subset \bar{V}_i \subset U_i$. *Since every* $\bar{V}_i$ *is closed, given the way the* $\phi_i$ *were chosen, so are* $\phi_i(\bar{V}_i)$ *and* $\phi_i(\bar{W}_i)$ *closed subsets in* $E$. $E$ *being Banach and separable, it is paracompact and hence normal. The second part of the above lemma yields a continuous map* $\psi_i : E \to [0, 1]$ *which is* $1$ *on* $\bar{W}_i$ *and* $0$ *outside* $V_i$. *Composing it with* $\phi_i$ *yields continuous maps* $\Psi_i : M \to [0, 1]$ *which are* $1$ *on* $\bar{W}_i$ *and that vanish outside* $V_i$. *Setting* $\xi_i := \frac{\Psi_i}{\sum \Psi_i}$ *yields a partition of unity.*

However, smooth partitions of unity do not always exist on smooth Banach manifolds. They do on smooth Hilbert manifolds as a consequence of the following result which provides a smooth version of the Urysohn Lemma. It essentially relies on the smoothness of the squared norm on a Hilbert space.

**Lemma 2** *([L, Theorem 3.7]) Given two disjoint closed subets* $A$ *and* $B$ *of a separable Hilbert space, there is a smooth map* $\phi : H \to [0, 1]$ *which is* $1$ *on* $A$ *and which vanishes on* $B$.

**Proof 2** *(Idea of the proof) First of all, using the smoothness of the squared norm* $\|\cdot\|^2$ *on the Hilbert space* $H$, *given any open ball* $B(x, R)$ *in* $H$ *centered at point* $x$ *with radius* $R$, *one can build a smooth function* $\phi : H \to [0, 1]$ *which is positive on* $B(x, R)$ *and vanishes elsewhere. For this, one picks any smooth function* $\eta : \mathbb{R} \to [0, 1]$ *which is positive for* $t < R$ *and vanishes beyond* $R$, *and composes it with the squared norm* *to build $\phi := \eta \circ \| \cdot \|^2$. Using the separability and metrisability of $H$, one can build a countable set of open balls $\{U_i = B(x_i, r_i)\}$ (with $x_i \neq x_j$) which cover $A$ and do not meet $B$. One can then inductively construct a locally finite refinement $\{V_i \subset \bigcup_i U_i\}$, and correspondingly find smooth functions $\eta_i$ built as above, which are positive on $V_i$ and vanish outside $V_i$. The sum $\eta := \sum_i \eta_i$, which is finite at each point of $W$, defines a smooth function that is positive on $A$ and vanishes on $B$. Letting $\sigma$ be a smooth function positive on the complement $W^c$ of $W$ and that vanishes on $W$, the map $\phi := \frac{\eta}{\eta+\sigma}$ fulfills the requirements of the Lemma.*

**Proposition 3** *([L] Corollary 3.8) A paracompact manifold of class $C^p$ modelled on a separable Hilbert space admits a partition of unity of class $C^p$.*

**Proof 3** *(Idea of the Proof) The proof goes as in the construction of a continuous partition of unity. One transports an open covering (which by paracompactness is locally finite) of the manifold by local charts to the model space and applies the above lemma to the closure of the open subsets obtained this way. Carrying back the smooth functions thus obtained to the manifold via the local charts yields a smooth partition of unity on the manifold.*

## 1.4 Vector fields

Useful references are [Hu], [KN], [L], [M], [Wu].

Let $M$ be a $C^k$-manifold and let $j \leq k$. A *$C^j$-vector field* is a $C^j$-map $\xi : M \to TM$ such that $\xi(x) \in T_xM$ for all $x \in M$. If $M$ is smooth, a smooth vector field is one that is of class $C^j$ for any $j \geq 0$.
Let us denote by $\Xi(M)$ the vector space of smooth vector fields on $M$. If $M$ is $n$-dimensional, given a local system of coordinates $(x_1, \cdots, x_n)$ around a point $x$, a vector field $\xi \in \Xi(M)$ reads $\xi(x) := \sum_{i=1}^n \xi_i(x) \frac{d}{dx^i}$.

**Definition 12** *The integral curve of a vector field $\xi$ on a manifold $M$ is a curve $c : I \to M$ ($I$ an open interval in $\mathbb{R}$) with tangent vector $\xi(c(t))$ at point $c(t)$ i.e. such that:*

$$\frac{d}{dt}c(t) = \xi(c(t)) \quad \forall t \in I.$$

From the theory of classical differential equations in Banach spaces we know, that given some initial condition $c(0) = x$ (we assume $0 \in I$), for some $x \in M$, there exists a neighborhood $I$ of 0 and an integral curve uniquely defined on $I$. The union of the domains of all integral curves with a given initial condition $x$ is an open interval which we denote by $I_x$ with end points $t_x^- \leq t_x^+$ (which could be $+\infty$ or $-\infty$).
These integral curves are smooth w.r. to initial conditions namely, given an integral curve $c_x$ starting at point $x$, there is an open neighborhood $U_x$ of $x$ and a neighborhood $J_x$ of 0 such that for for $y \in U$, the integral curve $c_y$ starting at point $y$ is defined on $J_x$. Furthermore the map:

$$\begin{array}{rcl} J_x \times U_x & \to & M \\ (t, y) & \mapsto & c_y(t) \end{array}$$

is smooth.

For some given vector field $\xi$, let $\mathcal{D}(\xi)$ denote the subset in $I\!R \times M$ consisting of all points $(t, x)$ such that $t_x^- < t < t_x^+$. The *flow of* $\xi$ is a map:

$$\phi : \mathcal{D}(\xi) \to M$$

such that the map $\phi_x(t) := \phi(t, x)$ defined on $I_x$ is a morphism and an integral curve for $\xi$ with initial condition $x$. In particular it satisfies the differential equation:

$$\frac{d\phi_x}{dt} = \xi \circ \phi_x.$$

The flow $\phi_x$ is *complete* if it can be extended to $I_x = I\!R$. Fixing the starting point $x \in M$ and setting $\phi_t := \phi_x(t)$ for any $t \in I\!R$ yields a *one parameter semi-group*:

$$\phi_t \circ \phi_s = \phi_{t+s} \quad \forall t, s \in I\!R.$$

As a consequence of the above property it follows that:

$$\phi_t^{-1} = \phi_{-t}$$

so that $\phi_t$ so that $\{\phi_t, t \in I\!R\}$ defines a *one parameter group of diffeomorphisms.*

Given a diffeomorphism $\phi$ on $M$ and a vector field $\xi$, we call the vector field defined by:

$$\phi_* \xi(\phi(x)) = D_x\phi(\xi(x)),$$

the *push forward* of $\xi$ and

$$\phi^* \xi := \left(\phi^{-1}\right)_* \xi$$

the *pull-back* of $\xi$.

**Definition 13** *The Lie bracket of two vector fields $\xi, \tilde{\xi}$ on a smooth manifold $M$ is given by the variation of $\tilde{\xi}$ along an integral curve $\phi_t$ of $\xi$:*

$$[\xi, \tilde{\xi}] := \frac{d}{dt}_{|t=0} (\phi_t^* \tilde{\xi}) = \frac{d}{dt}_{|t=0} \left( (\phi_{-t})_* \tilde{\xi} \right).$$

Given a smooth map $\phi$ from a manifold $N$ to a manifold $M$ and $\xi, \tilde{\xi} \in \Xi(M)$, we have:

$$[\phi_* \xi, \phi_* \tilde{\xi}] = \phi_* [\xi, \tilde{\xi}].$$

If $\xi_1, \xi_2, \xi_3$ are three vector fileds on a smooth manifold $M$ and $\phi_t$ is a one parameter group of local diffeomorphisms generated by $\xi_3$, then differentiating the following relation:

$$\phi_{t*}[\xi_1, \xi_2] = [\phi_{t*}\xi_1, \phi_{t*}\xi_2]$$

w.r. to $t$ at $t = 0$ yields the *Jacobi identity*:

$$[[\xi_1, \xi_2], \xi_3] + [[\xi_2, \xi_3], \xi_1] + [[\xi_3, \xi_1], \xi_2] = 0.$$

Vector fields can be identified with derivations on $M$ and Lie brackets with operator brackets of the derivations. By a *derivation* on $M$ we mean a linear map $L : C^\infty(M, I\!R) \to C^\infty(M, I\!R)$ which obeys the Leibniz rule

$$L(fg) = L(f)g + fL(g) \quad \forall f, g \in C^\infty(M, I\!R).$$

The set $Der(M)$ of derivations on $M$ is a vector space over $I\!R$. To a given vector field $\xi$ on $M$ we associate the map:

$$\begin{aligned} L_\xi : C^\infty(M, I\!R) &\to C^\infty(M, I\!R) \\ f &\mapsto Df(\xi) \end{aligned}$$

which is clearly a derivation.

Let $\xi \neq 0$, then there is some $x \in M$ such that $\xi(x) \neq 0$. Let $(U, \phi)$ be a local chart around $x$ and $u$ a representative of $\xi(x)$ in this chart. Since $u \neq 0$, by the Hahn-Banach theorem there is some linear form $L$ on the model space $E$ such that $L(u) \neq 0$. Thus $L \circ D\phi = D(L \circ \phi)$ does not vanish on $D\phi^{-1}(u)$ which can be identified with $\xi(x)$. Patching up the locally defined maps $f := L \circ \phi : U \to I\!R$ using a smooth partition of unity on $M$ (provided there is one) shows the existence of a function $f \in C^\infty(U, I\!R)$ such that $L_\xi f(x) = Df(\xi(x)) \neq 0$ so that $L_\xi \neq 0$. Thus, provided there is a smooth partition of unity on $M$, there is a one to one correspondence:

$$\begin{aligned} \Xi(M) &\to Der(M) \\ \xi &\mapsto L_\xi : f \to Df(\xi). \end{aligned}$$

The following identification holds:

**Proposition 4** *Given two vector fields $\xi, \tilde{\xi}$ on a smooth manifold $M$ and $f \in C^\infty(M, I\!R)$:*

$$[\xi, \tilde{\xi}]f := [L_\xi, L_{\tilde{\xi}}]f$$

*where the bracket on the r.h.s is an operator bracket.*

This identification yields the antisymmetry of the bracket together with the Jacobi identity which hold for the operator bracket.

## 1.5 Lie groups and Lie algebras

Useful references are [Ad1], [Br], [KN].

**Definition 14** *A Banach (resp. Hilbert) Lie group modelled on a Banach (resp. Hilbert) space $E$ is a $C^\infty$-manifold modelled on $E$, equipped with a group structure such that the multiplication and inverse maps*

$$\begin{aligned} G \times G &\to G \\ (g, h) &\mapsto gh \end{aligned}$$

*and*

$$\begin{aligned} G &\to G \\ g &\mapsto g^{-1} \end{aligned}$$

*are smooth.*

In fact this second property follows from the former using the global inverse mapping theorem (see Section 1.2).

A finite dimensional Lie group is one that has a finite dimensional manifold structure. Examples of finite dimensional Lie groups are the group $GL(n, \mathbb{R})$ of invertible $n \times n$ real matrices, the subgroup $O(n)$ of orthogonal matrices both of which arise as structure groups of frame bundles, the unitary groups $U(n) = \{A \in GL(n, \mathbb{C}), AA^* = 1\}$ and the special unitary groups $SU(n) = \{A \in U(n), \det A = 1\}$, that play an important role in gauge field theory.

**Definition 15** *Let $H, G$ be two Lie groups and $f : H \to G$ an embedding which is homomorphism of Lie groups and a homeomorphism onto $f(H)$. Then $f(H)$ is called a Lie subgroup of $H$.*

**Remark:** Notice that $f : H \to G$ being an immersion, $D_e f(T_e H)$ splits $T_e G$.

**Definition 16** *A Lie algebra $A$ is a vector space equipped with an antisymmetric bilinear map:*

$$\begin{array}{rcl} A \times A & \to & A \\ (a, b) & \mapsto & [a, b] \end{array}$$

*that satisfies the Jacobi identity:*

$$[[a, b], c] + [[b, c], a] + [[c, a], b] = 0 \quad \forall a, b, c \in A.$$

*A Banach, resp. Hilbert Lie algebra is a Banach, resp. Hilbert vector space equipped with a continuous antisymmetric bilinear map which satisfies the Jacobi identity.*

On the grounds of the above remark, we call *Lie subalgebra* of a Banach Lie algebra $A$ any closed linear subspace of $A$ that splits $A$ and that is stable under the Lie bracket, i.e.

$$a, b \in B \Rightarrow [a, b] \in B.$$

In particular, a Lie subalgebra of a Hilbert Lie algebra $A$ is a closed linear subspace of $A$ stable under the Lie bracket of $A$.

**Definition 17** *A left- (resp. right-) action of a Lie group $G$ on a smooth manifold $M$ is a map:*

$$\begin{array}{rcl} \Theta : G \times M & \to & M \\ (g, x) & \mapsto & \Theta(g, x) \end{array}$$

*such that:*

$$\Theta(e, x) = x \quad \forall x \in M$$

*and*

$$\Theta(gh, x) = \Theta\left(g, \Theta(h, x)\right) \quad \forall g, h \in G$$

*(resp.*

$$\Theta(gh, x) = \Theta\left(h, \Theta(g, x)\right) \quad \forall g, h \in G.)$$

*Such an action is smooth if the map $\Theta$ is smooth.*

It is convenient to denote a right action by $\Theta(g, x) := x \cdot g$ and a left action by $\Theta(g, x) := g \cdot x$.

A Lie group acts on itself by a right and a left action via the multiplication maps:

$$R_g(h) := hg, \quad L_g(h) := gh \quad \forall h, g \in G.$$

It also acts on itself via the *adjoint action*:

$$\begin{array}{rcl} G \times G & \to & G \\ ((g, h) & \mapsto & Ad_g(h) := L_g R_{g^{-1}} h = R_{g^{-1}} L_g h. \end{array}$$

A *right invariant field* on $G$ is a vector field $\xi$ such that:

$$\xi(hg) = D_g R(\xi(h)) \quad \text{i.e.} \quad R_{g*}\xi = \xi \quad \forall h, g \in G$$

and a *left invariant field* on $G$ is a vector field $\xi$ such that:

$$\xi(gh) = D_g L(\xi(h)) \quad \text{i.e.} \quad L_{g*}\xi = \xi \quad \forall h, g \in G.$$

Let $\Xi_L(G)$, resp. $\Xi_R(G)$ denotes the space of left, resp. right invariant vector fields on $G$. For any $u \in T_eG$ the vector field $g \to \xi_u^L(g) := D_e L_g(u)$ is left invariant, $g \mapsto \xi_u^R(g) := D_e R_g(u)$ is right invariant, and we can build two maps:

$$\begin{array}{rcl} \Xi_L : T_eG & \to & \Xi_L(G) \\ u & \mapsto & D_e L_g(u) \end{array} \quad \text{and} \quad \begin{array}{rcl} \Xi_R : T_eG & \to & \Xi_R(G) \\ u & \mapsto & D_e R_g(u), \end{array}$$

which are one to one and onto.

If the manifold $M$ is a Lie group, given two left invariant vector fields $\xi^L$ and $\tilde{\xi}^L$ on $G$, their Lie bracket is also left invariant for we have:

$$[L_{g*}\xi, L_{g*}\tilde{\xi}] = L_{g*}[\xi^L, \tilde{\xi}^L] \quad \forall g \in G$$

and a similar property holds for right invariant vector fields on $G$. Thus the Lie bracket on vector fields induces two brackets on $T_eG$:

$$[u, v]_L := [\xi_u^L, \xi_v^L], \quad [u, v]_R := [\xi_u^R, \xi_v^R] \quad \forall u, v \in T_eG.$$

The map:

$$\begin{array}{rcl} J : G & \to & G \\ g & \mapsto & g^{-1} \end{array}$$

satisfies $gJ(g) = e$, i.e. $R_{J(g)}(g) = e$ (or equivalently $J(g)g = e$ i.e. $L_g(J(g)) = e$ for any $g \in G$). Differentiating this relation at point $h \in G$ yields $D_h R_{g^{-1}} u + \left(D_{hg^{-1}} L_g\right) D_h J u = 0$, and gives the expression for its differential map at $h \in G$:

$$\begin{array}{rcl} D_h J : T_hG & \to & T_hG \\ v & \mapsto & -\left(D_{hg^{-1}} L_g\right)^{-1} D_h R_{g^{-1}} v = -D_{hg^{-1}} R_{g^{-1}} \left(D_h L_g\right)^{-1} v \end{array}$$

since $DL_g$ commutes with $DR_g$. Hence $DJ$ takes a left invariant vector field $\xi_u^L(g) = D_e L_g(u) \in T_g G$ to a right invariant vector field $D_g J \xi_u^L(g) = -D_e R_{g^{-1}} \left(D_g L_g\right)^{-1} \xi_u^L(g) = -\xi_u^R(g^{-1})$, so that:

$$D_g J(\xi_u^L) = -\xi_u^R \circ J \quad \text{i.e} \quad J_* \xi_u^R = -\xi_u^L.$$

Since $J$ is a diffeomorphism, it follows that:

$$[\xi_u^L, \xi_v^L] = \left[J_* \xi_u^R, J_* \xi_v^R\right] = J_* \left[\xi_u^R, \xi_v^R\right]$$

and hence

$$[u,v]_L = [u,v]_R = [\xi_u^R, \xi_v^R](e) = [\xi_u^L, \xi_v^L](e).$$

The tangent space $T_e G$ equipped with this Lie bracket becomes a Lie algebra denoted henceforth by $Lie(G)$.

Every automorphism $\phi$ of the Lie group $G$ induces an automorphism $\phi_*$ of its Lie algebra $Lie(G)$ for if $\xi$ is a left invariant vector field, then so is $\phi_* \xi$ and

$$[\phi_* \xi, \phi_* \tilde{\xi}] = \phi_* [\xi, \tilde{\xi}].$$

In particular, for any $g \in G$, the automorphism

$$\begin{array}{rcl} Ad(g) : G & \to & G \\ h & \mapsto & ghg^{-1} \end{array}$$

induces an automorphism of $Lie(G)$ also denoted by $Ad_g$.

To the left invariant and right invariant vector fields $\xi_u^L$ and $\xi_u^R$ built from an element $u \in Lie(G)$, we can associate two local flows $\phi_u^L$ and $\phi_u^R$ defined by

$$\frac{d\phi_u^L(t)}{dt} = \xi_u^L(\phi_u^L(t)), \quad \frac{d\phi_u^R(t)}{dt} = \xi_u^R(\phi_u^R(t)).$$

Let us assume such a flow $\phi_u$ is defined up to time $t_1$. For simplicity we drop the superscript $L$ and set $g_1 = \phi_u(t_1)$. Then $\xi_u^L$ being left invariant $\psi_u(t) := g_1 \phi_u(t)$ verifies:

$$\begin{array}{rcl} \frac{d}{dt}\psi_u(t) & = & D_{\phi_u(t)} L_{g_1} \frac{d}{dt} D_{\phi_u(t)} = D_{\phi_u(t)} L_{g_1} \xi_u^L(\phi_u(t)) \\ & = & \xi_u^L(\phi_u(t))(g_1 \phi_u(t)) = \xi_u^L(u)(\psi_u(t)) \end{array}$$

and $\psi_u(0) = g_1$. As before, $\psi_u$ can be defined on an interval $[0, t_1[$ thus extending the flow $\phi_u$ defined on $[0, t_1[$ to a flow on, $[0, 2t_1[$. Iterating this procedure shows that the flow can be extended to all $I\!R$. The same holds for the flow $\phi_u^R$.

The *left invariant and right invariant integral flows* $\phi_u^L$ and $\phi_u^R$ of some vector $u$ in the Lie algebra of a Lie group are therefore *complete*.
Let us compare these two flows:

$$\begin{array}{rcl} \frac{d}{dt}(J \circ \phi_u^L)(t) & = & D_{\phi_u^L(t)} J \left(\xi_u^L(\phi_u^L(t))\right) \\ & = & -\xi_u^R \left(J \circ \phi_u^L(t)\right) \end{array}$$

and hence
$$\frac{d}{dt}\left(\phi_u^L(t)\right) = \frac{d}{dt}\left(J \circ \phi_u^L(-t)\right) = \xi_u^R \left(J \circ \phi_u^L\right)(-t)$$
so that $\phi_u^L(t)$ satisfies the same differential system as $\phi_u^R(t)$ with the same initial conditions.

From the uniqueness of such a solution, we conclude that
$$\phi_u^L(t) = \phi_u^R(t) := \phi_u(t).$$

**Definition 18** *The map:*
$$\begin{array}{rcl} \exp : Lie(G) & \to & G \\ \xi & \mapsto & \phi_u(1) \end{array}$$
*is called the exponential map on the Lie group $G$.*

Since $D_e \exp = Id$, by the local inverse mapping theorem recalled in section 1.1, it induces a local diffeomorphism:
$$\exp : U \subset Lie(G) \to V \subset G$$
from an open neighborhood of 0 to an open neighborhood of $e \in G$.
On $GL(n, I\!R)$ it coincides with the exponential map of matrices $\exp A = \sum_0^\infty \frac{1}{k!} A^k$.

Given $a \in Lie(G)$, we can define a one parameter family $g_t := \exp(ta), \quad t \in I \subset I\!R$ where $I$ is a (small enough open) interval containing 0, and define the adjoint action of $Lie(G)$ on itself by differentiating that of $G$ on $Lie(G)$:
$$\begin{array}{rcl} ad_a : Lie(G) & \to & Lie(G) \\ b & \mapsto & \frac{d}{dt}_{|t=0} Ad_{g_t}(b) = [a, b] \end{array}$$
thus recovering the Lie bracket of $Lie(G)$.

The exponential map is not a morphism from the vector space $(Lie(G), +)$ to the group $(G, \cdot)$ as can be seen from the Lie Campbell-Hausdorff formula:
$$\exp a \cdot \exp b = \exp\left(\sum_{k=0}^{\infty} C^{(k)}(a, b)\right)$$
using the Banach topology on $Lie(G)$ and where $C^{(1)}(a, b) = a + b$ and for $k > 1$, $C^{(k)}(a, b)$ is a linear combination of Lie monomials of degree $k$ in $a$ and $b$ given by:
$$C^{(k)}(a, b) = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{(j+1)} \sum \frac{(ad\, a)^{\alpha_1}(ad\, b)^{\beta_1} \cdots (ad\, a)^{\alpha_j}(ad\, b)^{\beta_j} b}{(1 + \sum_{l=1}^{j} \beta_l)\alpha_1! \cdots \alpha_j! \beta_1! \cdots \beta_j!}.$$
Here we have set $ad\, a(c) := [a, c]$ and the inner sum is over all $j$-tuples of pairs of nonnegative integers $(\alpha_l, \beta_l)$ with $\alpha_l + \beta_l > 0$ and $\alpha_1 + \cdots \alpha_j + \beta_1 + \cdots \beta_j + 1 = k$ (terms with $\beta_j \neq 0$ vanish).

# 2 Vector bundles and tensor fields

## 2.1 Definition and first properties

Useful references are [HUs], [KN], [Na], [S], [Ts], [Wu].

**Definition 19** *A fibre bundle of class $C^k$ with typical fibre a $C^k$ Banach manifold $V$ (we shall also say modelled on $V$ and call $V$ the model space) is a triple $(F, B, \pi)$ often denoted by $\pi : F \to B$ where*

- *$F$ and $B$ are differentiable manifolds of class $C^k$, called total space and base space respectively,*
- *$\pi : F \to B$ is a map of class $C^k$, called canonical projection , such that there is a set of local charts $(U_i, \phi_i)_{i\in I}$ covering $B$ and $C^k$ diffeomorphisms*

$$\tau_i : \pi^{-1}(U_i) \to \phi_i(U_i) \times V$$

*satisfying the following requirements:*

*i) the fibre $F_b = \pi^{-1}(b)$ is a Banach manifold and*

*ii) $\tau_i(b) := \tau_{i|_{F_b}}$ is a diffeomorphism from $F_b$ to $F$.*

*A triple $(U_i, \phi_i, \tau_i)$ is called a local trivialisation of the bundle.*
*Two local trivializations $(U_i, \phi_i, \tau_i)$ and $(U_j, \phi_j, \tau_j)$ give rise to maps $\tau_{ij} := \tau_i \circ \tau_j^{-1}$ called transition maps of the form:*

$$\begin{array}{rcl} \tau_{ij} : \phi_i(U_i \cap U_j) \times V & \to & \phi_j(U_i \cap U_j) \times V \\ (b, v) & \mapsto & (b, \tau_{ij}(b)(v)) \end{array}$$

*where the $\tau_{ij}(b)$ are diffeomorphisms of class $C^k$ of $V$.*

Transition maps satisfy the following properties:

$$\tau_{ii}(b) = id_V \quad \forall b \in U_i$$

$$\tau_{ij}(b) \circ \tau_{ji}(b) = id_V \quad \forall b \in U_i \cap U_j$$

$$\tau_{ij}(b) \circ \tau_{jk}(b) \circ \tau_{ki}(b) = id_V \quad \forall b \in U_i \cap U_j \cap U_k.$$

The family $\{\tau_{ij}\}$ is called a *cocycle* associated to the trivialization $\{U_i, \tau_i, i \in I\}$, and the last relation mentioned above a *cocycle* relation. From a covering of a manifold $B$ together with a set of transition maps satisfying these relations one can reconstruct the fibre bundle on $B$.

In the following we mainly consider smooth manifolds and smooth bundles as well as smooth sections.

**Definition 20** *A morphism of $C^k$ fibre bundles $\pi : F \to B$ and $\pi' : F' \to B'$ is a couple $(f_0, f)$ of $C^k$ morphisms $f_0 : B \to B'$ and $f : F \to F'$ such that $\pi' \circ f = f_0 \circ \pi$ and the induced map on the fibres $f_x : \pi^{-1}(x) \to (\pi')^{-1}(x)$ is a morphism of the fibres.*

In what follows we shall often take $B = B'$.

Two fibre bundles are *isomorphic* if there is a diffeomorphism from one to the other. A *trivial fibre bundle* is a fibre bundle isomorphic to the bundle $\pi : F = B \times V \to V$. It can be shown (see [At], [1]) that a finite rank vector bundle $V$ over a closed manifold $M$ can be completed to a trivial bundle i.e., that there exists a vector bundle $W$ over $M$ such that $V \oplus W$ is trivial.

**Definition 21** *Let $B' \to B$ be a $C^k$ morphism of Banach manifolds, and let $F \to B$ be a $C^k$-fibre bundle on $B$. The pull-back $\phi^* F$ of $F$ by $\phi$ is a fibre bundle $\phi^* \pi : \phi^* F \to B'$ with total space:*

$$\phi^* F := \{(b', v(\phi(b'))) \in B' \times F_{\phi(b'))}\}$$

*where $V$ is the model space of $F$ and with fibres $(\phi^* \pi)^{-1}(b') = \pi^{-1}(\phi(b'))$ at a point $b'$ in $B$.*

A (real or complex) *vector bundle of class $C^k$* is a fibre bundle of class $C^k$ with typical fibre a (real or complex) vector space $V$, and such that there is a local trivialization inducing automorphisms $\tau_{ij}(x)$ of the Banach vector space $V$, i.e. $\tau_{ij} \in GL(V)$.

When $V = \mathbb{R}^d$ (resp. $\mathbb{C}^d$), the vector bundle has *rank* $d$. If $d = 1$ it is called a *line bundle.*

**Example 2** *The Grassmann bundle $\gamma_k^n$ over the Grassmann manifold $G_k^n$ is the vector bundle with fibre above the vector space $V \subset \mathbb{R}^n$ given by the pairs $(V, x)$ such that $x \in V$*

Operations on linear spaces such as the direct sum carry out to vector bundles; the direct sum of two vector bundles $E_1 \to B$ and $E_2 \to B$ over the same base space $B$ is a vector bundle $E_1 \oplus E_2 \to B$ over $B$ whose model space is the direct sum of the model spaces of $E_1$ and $E_2$.

**Definition 22** *A $C^k$ section of a fibre bundle $\pi : F \to B$ is a map $s : B \to F$ of class $C^k$ such that $\pi \circ s = Id_B$. It is smooth when it is of class $C^k$ for all $k \in \mathbb{N}$.*

The set of $C^k$-sections (resp. $C^\infty$-sections) of a vector bundle $E$ forms a vector space denoted by $C^k(E)$ (resp. $C^\infty(E)$).

A real finite rank vector bundle is *orientable* provided it has a trivialization with transition maps $\tau_{ij}(b)$ with positive determinant. A manifold is orientable whenever its tangent bundle is orientable.

**Example 3** *Given a manifold $M$ of class $C^{k+1}$ (resp. of class $C^\infty$) modelled on a Banach space $V$, the tangent bundle $TM$ is a $C^k$ (resp. $C^\infty$)-vector bundle with fibres modelled on that same space $V$; given a local trivialization $(U_i, \phi_i)$ on $M$, a local trivialization $(U_i, \phi_i, \tau_i)$ on $TM$ is given by $(U_i, \phi_i, D\phi_i)$ and $D\phi_i \circ D\phi_j^{-1}$ is of class $C^{k-1}$.*

Vector fields on a smooth manifold $M$ are smooth sections of the tangent vector bundle so that the space $\Xi(M)$ is now viewed as the vector space of smooth sections $C^\infty(TM)$ of the tangent bundle $TM$.

**Definition 23** *A morphism of $C^k$ vector bundles $\pi : E \to B$ and $\pi' : E' \to B'$ is a couple $(f_0, f)$ of $C^k$ morphisms $f_0 : B \to B'$ and $f : E \to E'$ such that $\pi' \circ f = f_0 \circ \pi$ and the induced map on the fibres $f_x : \pi^{-1}(x) \to (\pi')^{-1}(x)$ is a linear map.*

An isomorphism of vector bundles clearly preserves the rank of the vector bundle. The direct sum of vector bundles induces a direct sum of isomorphic classes of vector bundles so that isomorphism classes of vector bundles form a semi-group. Grothendieck suggested a "symmetrisation" procedure similar to the one which yields $\mathbb{Z}$ from $\mathbb{N} \times \mathbb{N}$ by as the set of equivalent classes $(a, b) \sim (c, d) \Leftrightarrow \exists e \in \mathbb{N}, \quad a + d + e = c + b + e$, to build the K-theory group $K(B)$ from the semi-group of complex vector spaces over $B$ for the direct sum (see [At], [1]).

## 2.2 Tensor, dual and morphism bundles

We refer the reader to the same references as the previous section.
The (topological) tensor product of two Banach spaces is built from their algebraic tensor product as follows.

**Definition 24** *Given two Banach vector spaces $V_1$ and $V_2$, the tensor product $V_1 \hat{\otimes} V_2$ is the unique Banach vector space $V$ such that the following map:*

$$\begin{array}{rcl} \mathcal{L}(V, W) & \to & \mathcal{B}(V_1 \times V_2, W) \\ f & \mapsto & ((u_1, u_2) \mapsto f(u_1 \otimes u_2)) \end{array}$$

*is continuous for any Banach space $W$. Here $\mathcal{B}(V_1 \times V_2, W)$ denotes the set of continuous bilinear forms on $V_1 \times V_2$ with values in $W$.*

If $\| \cdot \|_i$ denotes the norm on $V_i$ for $i = 1, 2$ then $V_1 \hat{\otimes} V_2$ coincides with the closure of the tensor product for the norm on $V_i$ defined by:

$$\|v_1 \hat{\otimes} v_2\| = \|v_1\|_1 \cdot \|v_2\|_2.$$

If both $V_1$ and $V_2$ are finite dimensional, then the tensor product $\hat{\otimes}$ coincides with the ordinary tensor product $\otimes$. In what follows we shall drop the explicit mention of the completion $\hat{}$.

**Definition 25** *Let $\pi_1 : E_1 \to B$ and $\pi_2 : E_2 \to B$ be two vector bundles of class $C^k$ with fibres modelled on $V_1$ and $V_2$ respectively. The tensor product $\pi_1 \otimes \pi_2 : E_1 \otimes E_2 \to B$ is a vector bundle of class $C^k$ modelled on $V_1 \otimes V_2$ with fibre $\pi_1^{-1}(b) \otimes \pi_2^{-1}(b)$ above $b \in B$ and the local trivializations of which are built from the tensor product of local trivializations $(U_i, \phi_i, \tau_i^1)$, $(U_i, \phi_i, \tau_i^2)$ and $(U_i, \phi_i, \tau_i^1 \otimes \tau_i^2)$.*

Transition functions are given by tensor products $\tau_{ij}^1 \otimes \tau_{ij}^2$ where $\tau_{ij}^k, k = 1, 2$ are transition maps for the bundles $E_k, k = 1, 2$.
Whenever $E_1$ and $E_2$ have ranks $d_1$ and $d_2$, their tensor product has rank $d_1 d_2$.

Given a topological vector space $V$, the dual space $V^*$ is the space of continuous linear forms on $V$.

**Definition 26** *Let $\pi : E \to B$ be a $C^k$ vector bundle with fibres modelled on a Banach space $V$. The dual bundle $\pi^* : E^* \to B$ is a vector bundle of class $C^k$ modelled on $V^*$ with fibre $\left(\pi^{-1}(b)\right)^*$ above $b \in B$ and local trivializations $(U_i, \phi_i, \left(\tau_i^{-1}\right)^*)$ induced by some local trivialization $(U_i, \phi_i, \tau_i)$ of $E$.*

The transition maps are given by $\left(\tau_{ij}^{-1}\right)^*$, where the $\tau_{ij}$ are transition maps for $E$.

Combining duals and tensor products yields different types of bundles which are useful for geometric purposes. The homomorphism bundle is one of them:

**Definition 27** *Given two vector bundles $E \to B$ and $F \to B$, we can build the bundle $Hom(E,F) := E^* \otimes F$ of linear morphisms from $E$ to $F$.*

Also we shall use the notion of symmetrised and antisymmetrised tensor products of vector bundles:
Given vector bundles $E_1, \cdots, E_k$ based on some manifold $B$, we can build symmetric sections of their tensor product from sections $\sigma_1, \cdots, \sigma_k$ of $E_1, \cdots, E_k$:

$$\sigma_1 \otimes_s \sigma_2 \otimes_s \cdots \otimes_s \sigma_k := \frac{1}{k!} \sum_{\alpha \in \Sigma_k} \sigma_{\alpha(1)} \otimes \sigma_{\alpha(2)} \otimes \cdots \otimes \sigma_{\alpha(k)},$$

and similarly antisymmetric sections:

$$\sigma_1 \wedge \sigma_2 \wedge \cdots \wedge \sigma_k := \frac{1}{k!} \sum_{\alpha \in \Sigma_k} (-1)^{sign(\alpha)} \sigma_{\alpha(1)} \otimes \sigma_{\alpha(2)} \otimes \cdots \otimes \sigma_{\alpha(k)}$$

where $sign(\alpha)$ is the signature of the permutation. Another useful class of bundles is that of tensor bundles on a manifold:

**Definition 28** *Given a Banach manifold $X$ of class $C^k$ modelled on a Banach vector space $E$ then:*

- *The dual bundle $T^*X$ to the tangent bundle $TX$ is a vector bundle called the cotangent bundle. It is a vector bundle of class $C^{k-1}$ based on $B$ and with fibres modelled on $E^*$. Its sections are called cotangent vector fields.*
- *The tensor bundle $TX^q := \otimes^q TX$, $q \in I\!N^*$ is a vector bundle of class $C^{k-1}$ based on $B$ and with fibres modelled on $\otimes^q E$. Its sections are called contravariant $q$-tensor fields.*
- *The tensor bundle $(TX^*)^p := \otimes^p TX$, $p \in I\!N^*$ is a vector bundle of class $C^{k-1}$ based on $B$ and with fibres modelled on $\otimes^p T^*X$. Its sections are called covariant $p$-tensor fields. Antisymmetric sections are called $p$-forms.*
- *A $(p,q)$ tensor field is a section of the bundle $(\otimes^q TX) \otimes (\otimes^p T^*X)$.*

In finite dimensions, one often writes a $(p,q)$ tensor $T$ in local coordinates as $T_{i_1 \cdots i_p}^{j_1 \cdots j_q}$.
The exterior product $\alpha \wedge \beta$ of a $p$-form $\alpha$ and a $q$-form $\beta$ is a $p+q$ form:

$$\alpha \wedge \beta = \sum_{\sigma \in \mathrm{Sh}(p,q)} (-1)^{|\sigma|} \alpha(x_{\sigma(1)}, \cdots, x_{\sigma(p)}\, \beta(x_{\sigma(p+1)}, \cdots, x_{\sigma(p+q)},$$

where $\mathrm{Sh}(p,q)$ is the subset of $(p,q)$ *shuffles*, namely permutations of the set $\{1, \ldots, p+q\}$ such that $\sigma() < \sigma(2) \cdots < \sigma(p)$ and $\sigma(p+1) < \cdots < \sigma(p+q)$.
Pull-backs can be extended to covariant tensor fields.

Given a morphism $\phi : X \to Y$ between two $C^k$ manifolds $X$ and $Y$, the pull-back by $\phi$ of a covariant $p$-tensor field $T$ on $Y$ is given by:

$$(\phi^*T)_x(U_1, \cdots, U_p) := T_{\phi(x)}(D_x\phi(U_1), \cdots, D_x\phi(U_p)) \quad \forall U_1, \cdots, U_p \in T_xX.$$

In particular, the pull-back of a $p$-form is also a $p$-form. It is easy to check that

$$\phi^*(T_1 \otimes T_2) = \phi^* T_1 \otimes \phi^* T_2$$

and that given two morphisms $\phi, \psi$ we have:

$$(\phi \otimes \psi)^* = \psi^* \otimes \phi^*.$$

If $\phi$ is a diffeomorphism, the pull-back can be extended to contravariant vector fields:

$$\phi^*(\xi_1 \otimes \cdots \xi_q) := (\phi^{-1})_* \xi_1 \otimes \cdots \otimes (\phi^{-1})_* \xi_q.$$

## 2.3 Examples of tensors: metrics and almost complex structures

Important examples of covariant tensor fields are the Riemannian (resp. Hermitian) metrics.

**Definition 29** *A weak (resp. strong) Riemannian metric on a smooth real vector bundle with fibres modelled on a Banach space and based on a manifold $B$, is a smooth section $g$ of $E^* \otimes E^*$ such that for any $b \in B$, $g_b$ induces a symmetric positive definite form on each fibre $E_b$, producing a weaker topology than the Banach topology on the fibre (respectively the same topology as the Banach topology on the fibre).*

**Definition 30** *A weak (resp. strong) Hermitian metric on a smooth complex vector bundle with fibres modelled on a Banach space and based on a manifold $B$, is a smooth section $h$ of $E^* \otimes E^*$ such that for any $b \in B$, $h_b$ induces a Hermitian positive definite form on each fibre $E_b$, producing a weaker topology than the Banach topology on the fibre (respectively the same topology as the Banach topology on the fibre).*

In the following when there is no other explicit mention, we shall be thinking of strong metrics.

A weak (resp. strong) Riemannian (Hermitian) metric on a Banach manifold is a weak (resp. strong) Riemannian (Hermitian) metric on the tangent bundle $TB$.

If $M$ is a manifold of dimension $n$, then weak and strong topologies coincide and one only requires that $g_x$ (resp. $h_x$) at a point $x \in M$ be a positive definite symmetric (resp. Hermitian) form on the fibres, locally represented by an $n \times n$ matrix $(g_{ij})$ (resp. $(h_{ij}(x))$ whose inverse is denoted by $(g^{ij})$. In a local orthonormal system of coordinates $g_{ij}(x) := g_x(e_i, e_j) = \delta_{ij}$, i.e. the matrix representing $g_x$ in this coordinate system is the identity matrix.

Given a diffeomorphism $\phi : N \to M$ between two manifolds and a Riemannian (resp. Hermitian) metric $g$ (resp. $h$) on a vector bundle based on $M$, the pull-back $\phi^* g$ (resp. $\phi^* h$) yields a Riemannian (resp. Hermitian) metric on the pull-back vector bundle $\phi^* E$ based on $N$.

In particular, if $\xi$ is a vector field on a Riemannian manifold $(M, g)$ the local one parameter group of diffeomorphisms $\phi_t$ generated by $\xi$ acts on the metric by pull-back $\phi_t^* g$. A *Killing vector field* also called an infinitesimal isometry, is a vector field $\xi$ such that the Lie derivative of the metric in the direction $\xi$ vanishes, i.e.:

$$L_\xi g := \left(\frac{d}{dt}\right)_{t=0} \phi_t^* g = 0.$$

If $\xi, \tilde{\xi}$ are two Killing vector fields, then so is their bracket $[\xi, \tilde{\xi}]$.

A vector bundle equipped with a (strong) Riemannian (resp. Hermitian) metric is called a *Riemannian* (resp. *Hermitian*) vector bundle. A manifold $M$ such that $TM$ is equipped with a (strong) Riemannian (resp. Hermitian) metric is called a Riemannian (resp. Hermitian) manifold.

Notice that a Banach vector bundle equipped with a strong Riemannian metric becomes a Hilbert bundle since the fibres become Hilbert spaces when equipped with the inner product induced by the metric. This is of course not the case anymore if the Riemannian structure is weak.

Metrics do not always exist on a manifold; however, provided there is a smooth partition of unity on the manifold, one can always build a Riemannian metric patching up locally defined positive definite forms. Also, if $M$ is a Riemannian manifold, tensor bundles over $M$ can be equipped with a metric structure induced from that of $M$.

The existence of a Riemannian metric on a manifold $M$ provides explicit isomorphisms between the tangent and cotangent vector fields called *musical isomorphisms:*

$$\begin{array}{ccc} T_x M & \to & T_x^* M \\ V & \mapsto & V^\flat \end{array}$$

defined by

$$V^\flat(W) = \langle V, W \rangle_x, \quad \forall W \in T_x M$$

where $\langle \cdot, \cdot \rangle_x$ is the scalar product on the fibre $T_x M$ of the tangent bundle above $x \in M$ induced by the Riemannian metric. Similarly, using the Riesz theorem, one defines:

$$\begin{array}{ccc} T_x^* M & \to & T_x M \\ \alpha & \mapsto & \alpha^\sharp \end{array}$$

by

$$\alpha(W) = \langle \alpha^\sharp, W \rangle_x, \quad \forall W \in T_x M.$$

**Definition 31** *An almost complex structure on an oriented Banach vector bundle $\pi : E \to B$ is a smooth section $J$ of $E^* \otimes E \simeq \mathrm{End}(E)$ preserving orientation and such that $J^2 = -Id$. An almost complex structure on an oriented manifold $M$ is one on the tangent bundle $TM$, i.e. it is a $(1,1)$ tensor $J$ inducing a morphism $J$ on $TM$ which preserves orientation and satisfies $J^2 = -Id$.*

An almost complex structure $J$ on a real vector bundle $E$ extends linearly to its complexification $J^{\mathbb{C}} : E^{\mathbb{C}} \to E^{\mathbb{C}}$. This complexified vector bundle splits $E^{\mathbb{C}} = E^{1,0} \oplus E^{0,1}$ where $E^{1,0}$ is the vector bundle over $B$ with fibre $Ker(J(b)-i) := \{u_b \in E_b^{\mathbb{C}}, J(b)(u_b) = iu_b\}$ above $b \in B$, resp. $E^{0,1}$ the vector bundle over $B$ with fibre $Ker(J(b)+i) := \{v_b \in E_b^{\mathbb{C}}, J(b)(v_b) = -iv_b\}$ above $b \in B$.

**Definition 32** *Let $M$ be a manifold equipped with an almost complex structure $J$ which induces a splitting $TM^{\mathbb{C}} = T^{1,0}M \oplus T^{0,1}M$. If $T^{1,0}M$ is stable under brackets of vector fields, then $J$ is said to be integrable.*

**Proposition 5** *An almost complex structure $J$ on $M$ is integrable if and only if the Nuijenhuis tensor field $N : C^\infty(TM) \times C^\infty(TM) \to C^\infty(TM)$ defined by*

$$N(U,V) = [U,V] + J[JU,V] + J[U,JV] - [JU,JV]$$

*vanishes for any $U \in C^\infty(TM), V \in C^\infty(TM)$.*

**Proof 4** *Extending the Nuijenhuis tensor to complex vector fields, $W = U + iV, Z = X + iY$, we can write:*

$$N(W,Z) = N(U,X) - N(V,Y) + i\left(N(V,X) + N(U,Y)\right).$$

*Thus if $N$ vanishes on real vector fields, it also vanishes on complex vector fields. Assume that $W, Z \in C^\infty(T^{1,0}M)$. Then $JW = iW$ and $JZ = iZ$ so that $N(W,Z) = 2\left([W,Z] + iJ[W,Z]\right)$. Hence $N(W,Z) = 0 \Rightarrow J[W,Z] = -i[W,Z]$, i.e. $[W,Z] \in C^\infty(T^{1,0}M)$. It follows that $J$ is integrable.*

*Conversely, let us write $W = W^+ + W^-$ and $Z = Z^+ + Z^-$ according to the splitting $C^\infty(T^{\mathbb{C}}M) = C^\infty(T^{1,0}M) \oplus C^\infty(T^{0,1}M)$. Then*

$$N(W,Z) = N(W^+,Z^+) - N(W^-,Z^-) + i\left(N(Z^-,W^+) + N(W^+,Z^-)\right).$$

*Since $JW^+ = iW^+$, $JW^- = -iW^-$, $JZ^+ = iZ^+$, $JZ^- = -iZ^-$, it follows that $N(W^+,Z^-) = N(W^-,Z^+) = 0$ and $N(W^+,Z^+) = N(W^-,Z^-) = 0$ so that $N$ finally vanishes on all complex tangent fields.*

**Definition 33** *A complex manifold is a manifold $M$ equipped with a complex structure i.e., with an atlas $(U_i, \phi_i)$ with transition maps given by holomorphic maps.*

A complex manifold inherits from the local charts an almost complex structure. Let us comment on the finite dimensional case; if $M$ has finite real dimension $2n$ then, in a local chart $(x_1, \cdots, x_n, y_1, \cdots, y_n)$ the complex structure is given by

$$J_x\left(\frac{\partial}{\partial x_k}\right) = \frac{\partial}{\partial y_k}, \quad J\left(\frac{\partial}{\partial y_k}\right) = -\frac{\partial}{\partial x_k}$$

at point $x$ in $M$. This defines a $(1,1)$ tensor on $M$ independently of the choice of local coordinates. Indeed, given another system of local coordinates $\{z'_k := x'_k + iy'_k, k = 1, \cdots, n\}$, the Cauchy-Riemann equations

$$\frac{\partial x_i}{\partial x'_j} = \frac{\partial y_i}{\partial y'_j}, \quad \frac{\partial x_i}{\partial y'_j} = \frac{\partial y_i}{\partial x'_j}$$

lead to a similar expression

$$J\left(\frac{\partial}{\partial x'_k}\right) = \frac{\partial}{\partial y'_k}, \quad J\left(\frac{\partial}{\partial y'_k}\right) = -\frac{\partial}{\partial x'_k}$$

so that we obtain an almost complex structure $J$ on $M$. Conversely we have:

**Theorem 5** *(Newlander and Nirenberg) Let $M$ be an (even dimensional) real manifold equipped with an almost complex structure $J$. If $J$ is integrable, it yields a complex structure on the manifold with associated almost complex structure $J$.*

*If $M$ and $N$ are two complex manifolds, a map $f : M \to N$ is called holomorphic if it is holomorphic in any local chart, this requirement being independent of the choice of local chart since the transition maps are holomorphic.*

Let $M$ be a real even dimensional manifold equipped with a complex structure $c$. Given another real even dimensional manifold $N$, a diffeomorphism $f : N \to M$ induces a complex structure $f^*c := \{f^{-1}(U_i), \phi_i \circ f\}$ on $N$ called the *pull-back* of $c$ by $f$. If $N = M$ then $f^*c$ is a priori different from the initial complex structure $c$ in the sense that the charts are not only different from the initial ones but also incompatible with them. Yet $(M, c)$ and $(M, f^*c)$ are holomorphically equivalent in the sense that $f : (M, c) \to (M, f^*c)$ is a holomorphic map and so is its inverse.

A Riemannian metric $h$ on a complex manifold $M$ with complex structure $J$ is called a *Hermitian metric* if $J$ is isometric:

$$h(\cdot, \cdot) := h(J\cdot, J\cdot)$$

and the pair $(M, h)$ is called a *Hermitian manifold* .
Given a h Hermitian manifold $(M, h)$ with complex structure $J$, we build a covariant tensor

$$\omega(\cdot, \cdot) = h(J\cdot, \cdot),$$

called the *Kähler form* of $h$. The two-form $\omega$ is antisymmetric since $J^2 = -Id$ and invariant under $J$:

$$\omega(J\cdot, J\cdot) = h(J^2\cdot, J\cdot) = h(J^3\cdot, J^2\cdot) = h(J\cdot, \cdot) = \omega.$$

**Remark 1** *On a complex manifold of complex dimension $n$ with complex structure $J$, the form $\omega \wedge \cdots \wedge \omega$ (n-times) is nowhere vanishing since with the notation of the above remark we have*

$$\begin{aligned} \omega \wedge \cdots \omega(e_1, Je_1, e_2, Je_2, \cdots, e_n, Je_n) &= \sum_{\sigma \in \Sigma_n} \omega(e_{\sigma(1)}, Je_{\sigma(1)}) \cdots \omega(e_{\sigma(n)}, Je_{\sigma(n)}) \\ &= n!\Omega(e_1, Je_1) \cdots \Omega(e_n, Je_n) = n!. \end{aligned}$$

*It serves as a volume element on $M$, which is therefore orientable.*

## 2.4 Bundle valued forms

Useful references are [BGV], [MT].

**Definition 34** *Let $\pi : E \to B$ be a smooth fibre bundle. An $E$-valued $p$-form $\alpha$ on $B$ is a smooth section of the tensor product $\otimes^p T^*B \otimes E$ such that:*

$$\alpha\left(U_{\sigma(1)}, \cdots, U_{\sigma(p)}\right) = (-1)^{\varepsilon(\sigma)}\alpha(U_1, \cdots, U_p) \quad \forall U_1, \cdots, U_p \in T_bB \quad \forall \sigma \in \Sigma_p$$

*where $\varepsilon(\sigma)$ is the signature of $\sigma$.*

In particular, such an expression vanishes whenever two vectors $U_i$ and $U_j$ coincide so that if the manifold $B$ is $n$-dimensional, using the multilinearity property, one can show that a $p$-form with $p > n$ vanishes identically.

We denote by $\Omega^p(E)$ the space of smooth $E$-valued $p$-forms and $\Omega(E) := \oplus_{p=0}\Omega^p(E)$, which becomes a finite sum when $B$ is finite dimensional. For $p = 0$ we get back the space of smooth sections of $E$. The degree of a $p$-form $\alpha$ is the integer $p$ also denoted by $|\alpha|$.

If $E$ is the trivial vector bundle $E = B \times I\!R$ (or $B \times \mathbb{C}$) we set $\Omega^p(B) := \Omega^p(E)$, and $\Omega(B) := \oplus\Omega^p(B)$ which is in fact a finite sum as soon as $B$ is finite dimensional. Given a local system of coordinates $(x_1, \cdots, x_n)$ around a point $x$ of an $n$-dimensional manifold, a one form $\alpha(x)$ reads $\alpha(x) := \sum_{i=1}^n \alpha_i(x)dx^i$.

Whenever $\mathcal{A}$ is a fibration of algebras, the space $\Omega(\mathcal{A})$ can be equipped with the *exterior product* or *wedge product* which sends $\alpha \in \Omega^p(\mathcal{A})$ and $\beta \in \Omega^q(\mathcal{A})$ to $\alpha \wedge \beta \in \Omega^{p+q}(\mathcal{A})$:

$$\begin{aligned}&(\alpha \wedge \beta)\left(U_1, \cdots, U_p, U_{p+1}, \cdots, U_{p+q}\right)\\ &:= \frac{1}{p!q!}\sum_{\sigma\in\Sigma_{p+q}} (-1)^{sign(\sigma)}\alpha(U_{\sigma(1)}, \cdots, U_{\sigma(p)}) \cdot \beta(U_{\sigma(p+1)}, \cdots, U_{\sigma(p+q)}).\end{aligned}$$

In particular, for two one-forms $\alpha$ and $\beta$ and two vector fields $U, V$ we have $\alpha \wedge \beta(U, V) = \alpha(U) \cdot \beta(V) - \alpha(V) \cdot \beta(U)$. Here the dot denotes the product of sections of $\mathcal{A}$. Thus $\Omega(\mathcal{A})$ becomes a graded algebra with the grading given by the degree on forms. here are two important examples:

- Starting from the bundle $E = B \times K$ where $K$ is a field, yields a graded algebra structure on $\Omega(B, K)$ using the product on $K$.
- Starting from a vector bundle $E$ based on $B$, the bundle $\mathcal{A} = Hom(E)$ yields a fibration of algebras on $B$ and $\Omega(Hom(E))$ can be equipped with a graded algebra structure using the composition of homomorphisms.

We introduce two operators on forms which are useful to construct a Clifford multiplication on forms later in these notes.

- Given a Riemannian manifold $M$, the exterior multiplication $\varepsilon(V) : \Omega^*(M) \to \Omega^{*+1}(M)$ is the operator defined by:

  $$\varepsilon(V)\alpha = V^\sharp \wedge \alpha \tag{1}$$

  where $V^\sharp$ is the 1-form associated to the vector field $V$ by the musical isomorphisms.

- Given a fibration of algebras $\mathcal{A}$ and a vector field $V$ on $B$, the *contraction* operator $\iota(V) : \Omega^*(\mathcal{A}) \to \Omega^{*-1}(\mathcal{A})$ is the unique operator such that:

$$\iota(V)\alpha = \alpha(V) \quad \forall \alpha \in \Omega^1(\mathcal{A}) \tag{2}$$

  extended to higher order forms by the Leibniz rule:

$$\iota(V)(\alpha \wedge \beta) = \iota(V)\alpha \wedge \beta + (-1)^{|\alpha|}\alpha \wedge \iota(V)\beta \quad \forall \beta \in \Omega(\mathcal{A}).$$

On a smooth oriented closed $n$-dimensional manifold $M$, any smooth $n$-form $\omega$ can be integrated to give a complex (or real) number $\int_M \omega$. Given a smooth map $f : N \to M$ between two closed oriented $n$-dimensional smooth manifolds $N$ and $M$, the pull-back $f^*\omega$ by $f$ of this form can be integrated on $N$ and we have:

$$\int_N f^*\omega = \deg(f) \cdot \int_M \omega$$

where $\deg(f)$ is an integer called the *degree* of the map $f$. The bilinear map:

$$\begin{array}{rcl} \Omega^p(M) \times \Omega^{n-p}(M) & \to & \mathbb{C} \\ (\alpha, \beta) & \mapsto & \int_M \alpha \wedge \beta \end{array}$$

will later yield a dual pairing between $p$-th and $n-p$-th cohomology groups. Notice that when $n = 2k$, for two $k$-forms $\alpha$ and $\beta$ we have $\int_M \alpha \wedge \beta = (-1)^{\frac{k}{2}} \int \beta \wedge \alpha$, so that this bilinear map yields a symmetric bilinear form on $\Omega^k(M)$whenever $k$ is even.

A Riemannian structure on a finite $n$-dimensional oriented manifold $M$ yields a particular $n$-form, the *volume form* given in a local system of coordinates $(x_1, \cdots, x_n)$ at a point $x$ by:

$$dvol(x) = \sqrt{\det g_x}\, dx_1 \wedge \cdots \wedge dx_n = e_1^* \wedge \cdots \wedge e_n^*$$

where $\det g_x$ is the positive determinant of the matrices representing the metric locally at point $x$ and where $\{e_1^*(x), \cdots, e_n^*(x)\}$ is an orthonormal basis of $T_x^*M$ equipped with the inner product induced by the Riemannian metric.
Given an $n$-dimensional Riemannian manifold $(M, g)$, the *Hodge star* operator $\star$ is defined pointwise by the linear operator

$$\star_x : \Lambda^p T_x^* M \to \Lambda^{n-p} T_x^* M \tag{3}$$

on a positively oriented orthonormal local basis $\{e_1^*, \cdots, e_n^*\}$ of $T_x^*M$ by:

$$e_{i_1}^* \wedge \cdots \wedge e_{i_p}^* \wedge \star_x(e_{i_1}^* \wedge \cdots \wedge e_{i_p}^*) = dvol(x)$$

for any $i_1 < \cdots < i_p$. This definition is independent of the choice of oriented orthonormal basis and one can check that $\star^2 = (-1)^{p(n-p)}$ on $\Lambda^p T_x^* M$. The Hodge $\star$ operator induces a duality on forms $\Omega^p(M) \simeq \Omega^{n-p}(M)$ called *Hodge duality.* When $M$ is closed, the above bilinear form on differential forms yields the following bilinear form on $\Omega^p(M)$:

$$\langle \alpha, \beta \rangle = \int_M \langle \alpha, \beta \rangle_x dvol(x) = \int_M \alpha(x) \wedge \star\beta(x).$$

When $M$ is a *complex manifold*, just as $TM^{\mathbb{C}} = T^{1,0}M \oplus T^{0,1}M$, the complexified space of forms $\Omega^r(M) \otimes \mathbb{C}$ splits:

$$\Omega^r(M) \otimes \mathbb{C} = \sum_{p+q}^{r} \Omega^{p,q}(M),$$

where $\Omega^{p,q}(M)$ is the space of smooth antisymmetric sections of the tensor bundle $\left(\left(T^{1,0}M\right)^*\right)^{\otimes p} \otimes \left(\left(T^{0,1}M\right)^*\right)^{\otimes q}$.

A Hermitian metric $h$ on $M$ is a $(1,1)$ covariant two tensor which, if $M$ is $n$-dimensional reads in local coordinates:

$$h(z) = \sum_{1\leq j,k\leq n} h_{jk} dz_j \otimes d\bar{z}_k.$$

Given a Hermitian manifold $(M,h)$ with complex structure $J$, we build a covariant tensor

$$\omega(\cdot,\cdot) = h(J\cdot,\cdot),$$

called the *Kähler form* of $h$. The two form $\omega$ is antisymmetric since $J^2 =_I d$ and invariant under $J$:

$$\omega(J\cdot, J\cdot) = h(J^2\cdot, J\cdot) = h(J^3\cdot, J^2\cdot) = h(J\cdot,\cdot) = \omega.$$

**Remark 2** *On a complex manifold of complex dimension $n$ with complex structure $J$, the form $\omega \wedge \cdots \wedge \omega$ ($n$-times) is nowhere vanishing since with the notation of the above remark we have*

$$\begin{aligned}\omega \wedge \cdots \wedge \omega(e_1, Je_1, e_2, Je_2, \cdots, e_n, Je_n) &= \sum_{\sigma\in\Sigma_n} \omega(e_{\sigma(1)}, Je_{\sigma(1)}) \cdots \omega(e_{\sigma(n)}, Je_{\sigma(n)}) \\ &= n!\Omega(e_1, Je_1)\cdots\Omega(e_n, Je_n) = n!.\end{aligned}$$

*It serves as a volume element on $M$, which is therefore orientable.*

# 3 Differential forms and connections

The exterior differential Useful references are [BGV], [MT].

*Exterior differentiation* on forms on a given smooth manifold $M$ is defined as follows:

**Proposition 6** *The derivation $f \to Df$ defined on the space of smooth functions on a manifold $M$ extends to a unique derivation map $d : \Omega(M) \to \Omega(M)$ such that*

*i) $d$ sends $\Omega^p(M)$ to $\Omega^{p+1}(M)$.*

*ii) $(d \circ d)(f) = 0 \quad \forall f \in C^\infty(M)$.*

*We set $df = Df$ for $f \in C^\infty(M) = \Omega^0(M)$.*

As a derivation map, exterior differentiation is linear and satisfies the (graded) *Leibniz rule*, which reads here:

$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^{|\alpha|} \alpha \wedge d\beta \quad \forall \alpha, \beta \in \Omega(M).$$

On a form $\alpha \in \Omega^p(M)$, and for smooth vector fields $U_0, \cdots, U_p$ on $M$, the exterior differentiation reads:

$$\begin{aligned} d\alpha(U_0, \cdots, U_p) &= \sum_{k=0}^{p} (-1)^k U_k \left( \alpha(U_0, \cdots, \hat{U}_k, \cdots, U_p) \right) \\ &+ \sum_{1 \leq k, l \leq p} (-1)^{k+l} \alpha \left( [U_k, U_l], U_0, \cdots, \hat{U}_k, \cdots, \hat{U}_l, \cdots, U_p \right) \end{aligned}$$

where the "hat" above the vector fields means we have deleted them. It is important to notice that the second requirement that $d \circ d$ vanishes on functions in fact implies, using the other two requirements, that it vanishes on all forms.

On a compact finite dimensional oriented Riemannian manifold, one can define the adjoint $d^*$ of $d$ setting:

$$\langle d\alpha, \beta \rangle = \langle \alpha, d^* \beta \rangle \quad \forall \alpha \in \Omega^p(M), \beta \in \Omega^{p+1}(M), \tag{4}$$

so that we have:

$$\begin{aligned} \langle \alpha, d^* \beta \rangle &= \int_M \alpha \wedge d^* \beta \\ &= \int_M d\alpha \wedge \star \beta \\ &= \int_M d(\alpha \wedge \star \beta) - (-1)^p \int_M \alpha \wedge d * \beta \\ &= (-1)^{p+1} \int_M \alpha \wedge d \star \beta \\ & \quad \left( \text{since} \int_M d\gamma = 0 \right) \\ &= (-1)^{p+1} (-1)^{(n-p)(n-(n-p))} \int_M \alpha \wedge \star \star d * \beta \end{aligned}$$

$$
\begin{aligned}
&= (-1)^{np+1}\int_M \alpha\wedge\star(\star d * \beta)\\
&= (-1)^{np+1}\langle\alpha, \star d\star\beta\rangle
\end{aligned}
$$

Thus on $\Omega^p(M)$ we get:

$$d^* = (-1)^{np+1}\star d\star .$$

## 3.1 Connections and geodesics

Useful references are [GHL], [J], [K], [KN], [L].

Covariant derivatives extend the exterior differentiation to sections of vector bundles.

**Definition 35** *Given a vector bundle $\pi : E \to B$ based on a manifold $B$, a covariant derivative (also abusively called connection) on $E$ is a differential operator:*

$$\nabla : C^\infty(E) \to C^\infty(T^*B \otimes E)$$

*which satisfies the Leibniz rule:*

$$\nabla(f\sigma) = df \otimes \sigma + f\nabla\sigma.$$

It extends in a unique way to the space $\Omega(B, E)$ of $E$-valued forms on $B$ in such a way that:

$$\nabla(\alpha\wedge\theta) := d\alpha\wedge\theta + (-1)^{|\alpha|}\alpha\wedge\nabla\theta \quad \forall\alpha\in\Omega(B), \theta\in C^\infty(E).$$

Notice that setting $\nabla_U := \iota(U)\circ\nabla$ for $U \in C^\infty(TM)$ we have:

$$\nabla_{fU}\sigma = f\nabla_U\sigma$$

and

$$\nabla_{U+V}\sigma = \nabla_U\sigma + \nabla_V\sigma \quad \forall\sigma\in C^\infty(E), f\in\Omega^0(B), U, V \in TB.$$

A covariant derivation $\nabla$ on a vector bundle $E$ induces a *dual connection* $\nabla^*$ on the dual bundle $E^*$, given by the Leibniz rule using the duality product $\langle\cdot,\cdot\rangle : C^\infty(E^*)\times C^\infty(E) \to C^\infty(M)$:

$$d\langle\rho^*, \sigma\rangle = \langle\nabla^*\rho^*, \sigma\rangle + \langle\rho^*, \nabla\sigma\rangle, \quad \forall\sigma\in C^\infty(E), \rho\in C^\infty(E^*),$$

and a connection $\nabla^{End}$ on the bundle $\mathrm{End}(E) \simeq E^*\otimes E$ defined by:

$$\nabla^{\mathrm{End}} := \nabla^*\otimes 1 + 1\otimes\nabla.$$

On a trivial vector bundle $E \to B$, a connection is given by an End$E$-valued one form $\theta$ via the formula $\nabla = d+\theta$. As a consequence, a connection on a general vector bundle End$E$ can locally be described by $\nabla = d+\theta_U$ where now $\theta_U$ is a End$E$-valued one form on an open subset $U \in B$ over which we have trivialised the bundle. Another consequence is that two connections on $\pi : E \to B$ differ by a (globally defined) End$E$-valued one form on $B$. An easy computation yields that if $\nabla = d+\theta_U$ locally, then $\nabla^* = d-\theta_U$ and $\nabla^{\mathrm{End}} = d + [\theta_U, \cdot]$.

A similar formula to that of the differentiation on ordinary forms holds for a covariant derivative on $E$-valued forms:

$$\begin{aligned}\nabla\alpha(U_0,\cdots,U_p) &= \sum_{k=0}^{p}(-1)^k\nabla_{U_i}\Big(\alpha(U_0,\cdots,\hat{U}_i,\cdots,U_k)\Big)\\ &+ \sum_{0\le k<l\le p}(-1)^{k+l}\alpha(\nabla_{U_k}U_l-\nabla_{U_l}U_k,U_0,\cdots,\hat{U}_k,\cdots,\hat{U}_l,\cdots,U_p)\end{aligned}$$

where $\hat{U}_i$ means that we have left out the vector field $U_i$.

A connection $\nabla$ on a Riemannian (resp. Hermitian) bundle $\pi: E\to B$ based on a manifold $B$ is *Riemannian* provided it is compatible with the Riemannian (resp. Hermitian) metric in the following sense:

$$d\langle\sigma,\rho\rangle_b=\langle\nabla\sigma,\rho\rangle_b+\langle\sigma,\nabla\rho\rangle_b \quad \forall\sigma,\rho\in C^\infty(E) \quad \forall b\in B$$

where $\langle\cdot,\cdot\rangle_b$ is the inner product on the fibre above $b$.

Given a connection $\nabla$ on a finite $n$-dimensional manifold $M$, and given a local system of coordinates $(x_1,\cdots,x_n)$ at a point $x\in M$, we define the *Christoffel symbols:*

$$\nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_j}=\sum_{k=1}^{n}\Gamma_{ij}^k\frac{\partial}{\partial x_k}.$$

These extend to the Banach setting as follows. Let $\pi: E\to B$ be a vector bundle with base $B$ modelled on a linear Banach space $V$ and fibre modelled on a linear Banach space $V_1$. Let $(U,\phi,\Phi)$ be a local trivialization of the bundle $E$ over an open subset $U$ of $B$. A Christoffel coefficient corresponding to this trivialization is given by a map:

$$\Gamma_\Phi:\Phi(\pi^{-1}(U))\to\mathcal{L}(V\times V_1,V_1)$$

with the following property. If $(W,\psi,\Psi)$ is another trivialization then

$$D(\Psi\circ\Phi^{-1})\Gamma_\Phi(\phi_*X,\tau\sigma)=D^2((\Psi\circ\Phi^{-1})(\phi_*X,\Phi\sigma)+\Gamma_\psi\circ(D(\psi\circ\phi^{-1})X,D(\Psi\circ\Phi^{-1})$$

where $X$ is a vector at a point of $U\cap W$ and $\sigma$ a section of $E$. Under this assumption, it makes sense to define a connection $\nabla: C^\infty(E)\to C^\infty(T^*M\otimes E)$ in a local trivialization $(U,\phi,\Phi)$ using the Christoffel symbol $\Gamma_\Phi$ by:

$$\Phi(\nabla_X\sigma)=D(\Phi\sigma).\phi_*X+\Gamma_\Phi(\phi_*X,\Phi\sigma)$$

since the latter definition is independent of the choice of local trivialization.

**Definition 36** *The torsion of a connection on the tangent bundle $TM$ to a manifold $M$ is given by:*

$$T(U,V):=\nabla_UV-\nabla_VU-[U,V] \quad \forall U,V\in TM.$$

In a system of local coordinates $(x_1,\cdots,x_n)$ around a point $x$ of a finite $n$-dimensional manifold $M$, setting $e_i=\frac{\partial}{\partial x_i}$ we have $T(e_i,e_j)=\nabla_{e_i}e_j-\nabla_{e_j}e_i$ so that if the torsion vanishes then $\nabla_{e_i}e_j=\nabla_{e_j}e_i$, i.e. the Christoffel symbols are symmetric $\Gamma_{ij}^k=\Gamma_{ji}^k$.

**Remark 3** *In the absence of torsion, i.e. when* $T = 0$ *the covariant derivative on forms reads:*

$$\begin{aligned}\nabla\alpha(U_0, \cdots, U_p) &= \sum_{k=0}^{p}(-1)^k \nabla_{U_i}\Big(\alpha(U_0, \cdots, \hat{U}_i, \cdots, U_k)\Big) \\ &+ \sum_{0\le k<l\le p}(-1)^{k+l}\alpha([U_k, U_l], U_0, \cdots, \hat{U}_k, \cdots, \hat{U}_l, \cdots, U_p).\end{aligned}$$

**Proposition 7** *There is a unique connection on a Riemannian manifold which has vanishing torsion and is compatible with the (strong) Riemannian metric; it is called the Levi-Civita connection.*

**Proof 5** *(Idea of the proof) We first write*

$$U\langle V, W\rangle = d\langle V, W\rangle(U) = \langle \nabla_U V, W\rangle + \langle V, \nabla_U W\rangle$$

*as well as circular combinations of this expression. Using the fact that the torsion vanishes yields the following expression of* $\langle \nabla_U V, W\rangle$*:*

$$\begin{aligned}2\langle \nabla_U V, W\rangle &= \langle [U, V], W\rangle - \langle [V, W], U\rangle + \langle [W, U], V\rangle \\ &+ U\langle V, W\rangle + V\langle W, U\rangle - W\langle U, V\rangle.\end{aligned}$$

*in terms of differentials of the inner product* $\langle U, V\rangle$*,* $\langle V, W\rangle$ *and* $\langle U, W\rangle$*. The existence and uniqueness of* $\nabla_U V$ *then follows from Riesz's theorem.*

A torsion free connection relates to the exterior differentation:

**Proposition 8** *If* $\nabla$ *is a torsion free connection on* $M$ *then the exterior differential coincides with* $\varepsilon \circ \nabla$ *where* $\varepsilon$ *is the exterior multiplication. In particular, if* $\nabla$ *is the Levi-Civita connection on a Riemannian manifold* $M$*, then* $d = \varepsilon \circ \nabla$*.*

**Proof 6** *(Idea of the proof) Setting* $\tilde{d} = \varepsilon \circ \nabla$ *one proves that* $\tilde{d}^2 f = -\langle T, df\rangle$ *for any smooth function* $f$ *on* $M$ *where* $T$ *is the torsion. Since the torsion vanishes by assumption, this will prove that* $\tilde{d}^2 f = 0$*. One is then left to check the Leibniz property for* $\tilde{d}$ *and the fact that it coincides with the ordinary differentiation on smooth functions in order to conclude that it coincides withg* $\tilde{d}$ *on all differential forms.*

A Hermitian complex manifold $(M, h)$ can be equipped with a Riemannian metric $g(\cdot, \cdot) := h(\cdot, J\cdot)$ where $J$ is the almost complex structure on $M$ induced by the complex structure.

**Proposition 9** *A Hermitian complex manifold* $M$ *is Kählerian provided the bundles* $T^{1,0}M$ *and* $T^{0,1}M$ *are preserved by the Levi-Civita connection* $\nabla$*, or equivalently provided the Levi-Civita connection* $\nabla$ *is compatible with the complex structure* $J$ *i.e.* $[\nabla, J] = 0$*.*

**Proof 7** *(Idea of proof) Recall that if* $M$ *is a complex manifold with a Hermitian metric* $h$*, the real part of* $h$ *restricted to the tangent bundle* $TM$ *is a Riemannian metric* $g$ *on* $M$*, while the imaginary part* $\omega$ *restricted to* $TM$ *is a two form on* $M$*. For any two vector fields on* $M$*, we have* $g(U, V) = \omega(JU, V)$ *where* $J$ *is the almost* *complex structure on $M$. Letting $\langle\cdot,\cdot\rangle$ denote the Riemannian scalar product, we have:*

$$\begin{aligned}
\langle(\nabla_U J)V,W\rangle &= \langle\nabla_U(JV),W\rangle+\omega(\nabla_U V,W\rangle\\
&= -U\omega(V,W)+\omega(\nabla_U V,W)+\omega(V,\nabla_V W)\\
&= -(\nabla_U\omega)(V,W).
\end{aligned}$$

*Since $\nabla$ is torsion free, $d=\varepsilon\circ\nabla$ where $\varepsilon$ is the exterior product and $d\omega(U,V,W)=(\nabla_U)\omega(V,W)-(\nabla_V)(W,U)+(\nabla_W)(U,V)$, which vanishes as a consequence of the condition $\nabla J=0$. Hence $\nabla J\Rightarrow d\omega=0$. On the other hand, the formula for the Levi-Civita connection applied to the holomorphic coordinate system $z_i$ yields:*

$$\begin{aligned}
2\langle\nabla_{\partial_{z^j}}\partial_{z^k},\partial_{z^l}\rangle &= 0\\
2\langle\nabla_{\partial_{\bar z^j}}\partial_{z^k},\partial_{z^l}\rangle &= i\,d\omega\left(\partial_{z^k},\partial_{z^l},\partial_{\bar z^j}\right)\\
2\langle\nabla_{\partial_{\bar z^j}}\partial_{z^k},\partial_{\bar z^l}\rangle &= id\omega\left(\partial_{\bar z^j},\partial_{z^k},\partial_{\bar z^l}\right),1
\end{aligned}$$

*from which it follows that if $d\omega=0$, then the Levi-Civita connection preserves $T^{1,0}M$.*

**Definition 37** *A geodesic on a Banach Riemannian manifold $M$ is a smooth curve $c:I\to M$ on $M$ solution of the second order differential equation:*

$$\nabla_{\dot c(t)}\dot c(t)=0$$

*where $I$ is some open interval in $\mathbb{R}$.*

Such a solution exists locally by the theory of differential equations on Banach spaces and there is a unique solution $c_{x,u}$ determined by the initial conditions $c(0)=x,\quad \dot c(0)=u\in T_xM$ provided $0\in I$.

Taking $u$ in a small enough neighborhood of 0 ensures the existence of the geodesic up to time 1 and we define the *exponential map:*

$$\begin{aligned}
\exp:U\subset T_xM &\to M\\
u &\mapsto c_{x,u}(1)
\end{aligned}$$

which yields, by the local inverse mapping theorem (see section 1.1), a local diffeomorphism from $U$ onto its range.

The Riemannian manifold is *complete* provided all geodesics are defined on $\mathbb{R}$, in which case the exponential map is defined on the whole tangent bundle. A compact Riemannian manifold is complete.

The exponential map defined on Lie groups can in some cases be described as an exponential map built from geodesics, choosing an adapted left invariant metric on the group, e.g. on $GL(n,\mathbb{R})$ the one given by the inner product $\langle A,B\rangle:=\mathrm{tr}(A^tB)$ on $gl(n,\mathbb{R})$.

### 3.2 De Rham and Dolbeault cohomology

Combining the interior and exterior products on $\Omega(M)$, where $M$ is a Riemannian manifold, yields a Clifford multiplication from which we shall later build a Dirac operator using the Levi-Civita connection.

**Proposition 10** *1. $\varepsilon(v)^* = \iota(v) \quad \forall v \in T_xM, x \in M$,*

*2. $c = \varepsilon - \iota$ acting on $\Omega(M)$ satisfies the Clifford relations i.e.*

$$c(v)c(w) + c(w)c(v) = -2\langle v, w\rangle_x, \quad \forall v, w \in T_xM, \tag{5}$$

*where $\langle \cdot, \cdot, \rangle_x$ is the inner product on $T_xM$ induced by the metric structure.*

*3. $[\nabla, c] = 0$.*

*4. $d = \sum_{i=1}^n \varepsilon(e_j)\nabla_{e_j}$,*

*5. $d^* = -\sum_{j=1}^n i(e_j)\nabla_{e_j}$,*
*where $(e_1, \cdots, e_n)$ is an orthonormal basis of $T_xM$.*

**Proof 8** *(Partial) To avoid technicalities, we prove the results on one forms only.*

1. *Given $v \in T_xM$, $f \in \Omega^0(M)$ and $\alpha \in T_x^*M$ we have:*

$$\langle \iota(v)\alpha, f(x)\rangle_x = \langle \alpha(v), f(x)\rangle_x = \alpha(v)f(x).$$

*On the other hand*

$$\langle \alpha, \varepsilon(v)f(x)\rangle_x = \langle \alpha, f(x)v^\flat\rangle = \langle \alpha(x), v^\flat\rangle_x f(x) = \alpha(v)\, f(x).$$

*Hence $\varepsilon^* = \iota$ on 1-forms.*

2. *Let $v, w \in T_xM$. We first observe that*

$$\varepsilon(v)\iota(w) + \iota(w)\varepsilon(v) = \langle v, w\rangle_x \quad \forall v, w \in T_xM.$$

*Here again, we check the property on a one-form $\alpha$.*

$$(\varepsilon(v)\iota(w) + \iota(w)\varepsilon(v))\,\alpha = \alpha(w)v^\flat + i(w)(v^\flat \wedge \alpha) = \alpha(w)v^\flat + v^\flat(w)\alpha - v^\flat\alpha(w) = v^\flat(w)\alpha = \langle v, w\rangle_x\,\alpha.$$

*As a consequence we have:*

$$c(v)c(w) + c(w)c(v) = \varepsilon(v)\varepsilon(w) + \varepsilon(w)\varepsilon(v) + \iota(v)\iota(w) + \iota(w)\iota(v) - 2\,(\varepsilon(v)\iota(w) + \iota(w)\varepsilon(v)) = -2\langle v, w\rangle_x,$$

*where we have used the fact that $\varepsilon(v)\varepsilon(w) + \varepsilon(w)\varepsilon(v) = \iota(v)\iota(w) + \iota(w)\iota(v) = 0$.*

3. *For any $u, v \in T_xM$,$\alpha \in \Omega^1(M)$ we have*

$$\begin{aligned}(\nabla_u\, c(v))\,(\alpha(x)) - c(\nabla_u v)\alpha &= \nabla_u(c(v)\alpha) - c(v)\,(\nabla_u \alpha)\\ &= \nabla_u\left(v^\flat \wedge \alpha - \alpha(v)\right) - \left(v^\flat \wedge \nabla_u v - (\nabla_u\alpha)(v)\right)\\ &= c(\nabla_u v)(\alpha).\end{aligned}$$

4. *Let us set $\tilde d = \sum_{i=1}^n \varepsilon(e_j)\nabla_{e_j}$ and show that $\tilde d$ satisfies the requirements i), ii), iii) which define $d$ uniquely:*

i) *Since $\nabla_{e_j}$ sends $\Omega^p(M)$ to $\Omega^p(M)$, and $\varepsilon(e_j)$ increases the degree of the form by 1, $\tilde{d}$ sends $\Omega^p(M)$ to $\Omega^{p+1}(M)$.*

ii) *$\tilde{d} \circ \tilde{d}(f) = 0 \quad \forall f \in C^\infty(M, \mathbb{C})$. We prove that $\tilde{d}^2 f = -\langle T, df \rangle$ where $T$ is the torsion. Since the torsion of the Levi-Civita connection vanishes by definition, this will prove that $\tilde{d}^2 = 0$. To simplify notations we set $\nabla_j = \nabla_{\frac{\partial}{\partial x_j}} = \nabla_{e_j}$ where $e_j = \frac{\partial}{\partial x_j}$.*

$$\begin{aligned}
\tilde{d}^2 f &= \tilde{d}(df) \\
&= \sum_{ij} \varepsilon(dx_i) \nabla_i (\partial_j f dx_j) \\
&= \sum_{ij} \partial_i \partial_j f dx_i \wedge dx_j + \sum_{ij} \varepsilon(dx_i) \partial_j f \nabla_i (dx_j) \\
&= \sum_{ij} \varepsilon(dx_i) \partial_j f \nabla_i (dx_j).
\end{aligned}$$

*By Leibniz's rule:*

$$0 = \frac{\partial}{\partial x_i} \langle dx_j, e_k \rangle = \langle \nabla_i dx_j, e_k \rangle + \langle dx_j, \nabla_i e_k \rangle$$

*so that*

$$\begin{aligned}
\tilde{d}^2 f &= -\sum_{ijk} \varepsilon(dx_i) \frac{\partial}{\partial x_j} f \langle dx_j, \nabla_i e_k \rangle dx_k \\
&= -\sum_{ijk} \frac{\partial}{\partial x_j} f \langle dx_j, \nabla_i e_k \rangle_x \, dx_i \wedge dx_k \\
&= -\sum_{i<k} \langle df, \nabla_i e_k - \nabla_k e_i \rangle_x \, dx_i \wedge dx_k \\
&= -\sum_{i<k} \langle df, T(e_i, e_k) \rangle_x \, dx_i \wedge dx_k \\
&= -\langle T, df \rangle.
\end{aligned}$$

iii) *$\tilde{d}$ is a derivation. Indeed, the Levi-Civita connection on the tangent bundle $TM$ extends to a connection on the exterior cotangent bundle $\Lambda T^* M$ and satisfies the following rule:*

$$\nabla_X (\alpha \wedge \beta) = \nabla_X \alpha \wedge \beta + \alpha \wedge \nabla_X \beta \quad \forall \alpha, \beta \in \Omega(M), \forall X \in C^\infty(TM).$$

*Hence $\tilde{d} = \sum_i \varepsilon(e_i^*) \nabla_{e_i}$ satisfies a graded Leibniz rule:*

$$\tilde{d}(\alpha \wedge \beta) = \tilde{d}\alpha \wedge \beta + (-1)^{|\alpha|} \alpha \wedge \tilde{d}\beta \quad \forall \alpha, \beta \in \Omega(M)$$

*and therefore yields a (graded) derivation.*

5. *Given $\alpha \in \Omega^p(M)$ and $\beta \in \Omega^{p+1}(M)$ we want to check that $\langle \varepsilon(dx_i) \nabla_i \alpha, \beta \rangle = \langle \alpha, \iota(dx_i) \nabla_i \beta \rangle$. Differentiating the one form defined on $v \in T_x M$ by $\alpha(v) = \langle \alpha, \iota(v)\beta \rangle_x$ and using Leibniz's rule yields:*

$$\sum_i (e_i \alpha(e_i) - \alpha(\nabla_i e_i)) = \langle \nabla_i \alpha, \iota(e_i)\beta \rangle_x + \langle \alpha, \nabla_i \iota(e_i)\beta \rangle_x = \langle \varepsilon(e_i) \nabla_i \alpha, \beta \rangle_x + \langle \alpha, i(e_i) \nabla_i \beta \rangle$$

*where we have used the fact that $\varepsilon^* = i$. On the other hand since the divergence is given by $d^*\alpha = -\mathrm{tr}(\nabla\alpha)$ for a one form $\alpha$, it follows from Stokes' theorem that $\mathrm{tr}(\nabla\alpha) := \sum_{i=1}^n \nabla\alpha(e_i, e_i) = \sum_i (e_i\alpha(e_i) - \alpha(\nabla_i e_i))$ integrates to 0 on $M$, i.e.*

$$\int_M \mathrm{tr}(\nabla\alpha)\, d\mathrm{vol} = -\int_M d^*\alpha = 0.$$

*Thus*

$$\langle \varepsilon(e_i)\nabla_i\alpha, \beta\rangle_x + \langle \alpha, i(e_i)\nabla_i\beta\rangle = 0$$

*so that $d^* = -\iota \circ \nabla$.*

Combining $d$ and $d^*$ yields the operator

$$D := d + d^* = \sum_{j=1}^n c(e_j)\nabla_{e_j} \tag{6}$$

whose square is the *Laplacian on forms* given by

$$\Delta = d^* \circ d + d \circ d^*$$

whose restriction to $p$-forms reads $\Delta_p = d_p^* \circ d_p + d_{p-1} \circ d_{p-1}^*$ where $d_p : \Omega^p(M) \to \Omega^{p+1}(M)$.

**Remark 4** *The operator $D$ yields an example of Dirac type operator introduced later in these notes; it is of the form $\sum_{j=1}^n c(e_j)\nabla_{e_j}$ for some connection $\nabla$ (here the Levi-Civita connection) which commutes $[c, \nabla] = 0$ with the Clifford multiplication (here $c = \varepsilon - \iota$), a property characteristic of Clifford connections.*

A form $\alpha$ is *closed* whenever $d\alpha = 0$, and *exact* whenever there is a form $\beta$ such that $\alpha = d\beta$. Since $d \circ d = 0$, exact forms are closed but closed forms are not expected to be exact, they are only locally exact by the Poincaré lemma. The obstruction to their global exactness is measured by the de Rham cohomology groups:

$$H^p(M) := \mathrm{Ker}(d|_{\Omega^p(M)})/\mathrm{R}(d|_{\Omega^{p-1}(M)})$$

where $\mathrm{R}(d|_{\Omega^{p-1}(M)})$ denotes the range of the map $d|_{\Omega^{p-1}(M)}$. The theory of elliptic operators on closed manifolds which we describe later in these notes shows that these cohomology groups are finite dimensional, the dimension of $H^p(M)$ corresponds to the Betti number of $M$ introduce later in the notes.

When $M$ is a complex manifold, the exterior differentiation splits $d = \partial + \bar\partial$ where $\partial : \Omega^{p,q}(M) \to \Omega^{p+1,q}(M)$ and $\bar\partial : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M)$ and it follows from the relation $d \circ d = 0$ that $\partial \circ \bar\partial + \bar\partial \circ \partial = 0$, $\partial^2 = 0$, $\bar\partial^2 = 0$.
Since $\bar\partial^2 = 0$,a $\bar\partial$-exact form $\alpha$ (i.e.$\alpha = \bar\partial\beta$) is $\bar\partial$-closed (i.e. $\bar\partial\alpha = 0$) and there is an associated complex

$$0 \to \Omega^{0,0}(M) \to \Omega^{0,1}(M) \to \Omega^{0,2}(M) \to \cdots,$$

called the *Dolbeault complex.* A $\bar\partial$-closed form is however generally not $\bar\delta$-exact and the obstruction to the exactness of closed forms is measured by the *Dolbeaut cohomology* groups:

$$H^{p,q}(M) := \mathrm{Ker}(\bar\partial|_{\Omega^{p,q}(M)})/\mathrm{R}(\bar\partial|_{\Omega^{p,q-1}(M)})$$

where $\mathrm{R}(d|_{\Omega^{p,q-1}(M)})$ denotes the range of the map $\bar{\partial}|_{\Omega^{p,q-1}(M)}$. The *Hodge decomposition theorem* gives a relation between the de Rham and the Dolbeault cohomology groups:

$$H^k(M) = \oplus_{p+q=k} H^{p,q}(M).$$

Using the theory of elliptic operators on closed manifolds one can show that these cohomology groups are finite dimensional; their dimensions are called the *Hodge numbers* $h^{p,q} := \dim H^{p,q}(M)$.

From the Hodge decomposition theorem it follows that the Betti numbers relate to the Hodge numbers as follows:

$$b_k = \sum_{p+q=k} h^{p,q}.$$

**Definition 38** *A complex manifold $(M, J)$ is Kähler if and only if the Kähler form $\omega$ is closed $d\omega = 0$.*

**Proposition 11** *A complex Hermitian manifold $(M, h)$ is Kähler if and only if $\nabla J = 0$ where $\nabla$ is the Levi-Civita connection on $M$ and $J$ the almost-complex structure on $TM$ i.e., if and only if the holomorphic and anti-holomorphic tangent bundles are preserved by covariant differentiation.*

**Remark 5** *Consequently (using the fact that the Levi-Civita connection is torsion free) the Lie bracket preserves the holomorphic and anti-holomorphic tangent bundles on a Kähler manifold.*

**Proof 9** *Let $\omega = h(J\,\cdot, \dot{)}$ be the Kähler form. For any three vector fields $U, V$ and $W$ we have:*

$$U\omega(V, W) = (\nabla_U \omega)(V, W) + \omega(\nabla_U V, W) + \omega(V, \nabla_U W).$$

*Using the fact that the Levi-Civita connection is torsion free (i.e., $\nabla_U V - \nabla_V U = [U, V]$ for any tangent vector fields $U$ and $V$) and the antisymmetry of $\omega$ we write:*

$$\begin{aligned}
d\omega(U, V, W) &= U\omega(V, W) - V\omega(U, W) + W\omega(U, V) \\
&- \omega([U, V], W) - \omega([V, W], U) + \omega([U, W], V) \\
&= (\nabla_U \omega)(V, W) + \omega(\nabla_U V, W) + \omega(V, \nabla_U W) \\
&- \nabla_V \omega(U, W) - \omega(\nabla_V U, W) - \omega(U, \nabla_V W) \\
&+ \nabla_W \omega(U, V) + \omega(\nabla_W U, V) + \omega(U, \nabla_W V) \\
&- \omega([U, V], W) - \omega([V, W], U) + \omega([U, W], V) \\
&= (\nabla_U \omega)(V, W) + \omega(\nabla_U V, W) + \omega(V, \nabla_U W) \\
&- (\nabla_V \omega)(U, W) - \omega(\nabla_V U, W) - \omega(U, \nabla_V W) \\
&+ (\nabla_W \omega)(U, V) - \omega(V, \nabla_W U) + \omega(U, \nabla_W V) \\
&- \omega([U, V], W) + \omega(U, [V, W]) - \omega(V, [U, W]) \\
&= \nabla\omega(U, V, W).
\end{aligned}$$

*Hence $d\omega = \nabla\omega$. Furthermore, for any vector fields $U, V$ and $W$ we have*

$$\begin{aligned}
\nabla_U \omega(V, W) &= U\omega(V, W) - \omega(\nabla_U V, W) - \omega(V, \nabla_U W) \\
&= Uh(JV, W) - h(J\nabla_U V, W) - h(JV, \nabla_U W) \\
&= h(\nabla_U JV, W)
\end{aligned}$$

*and hence*

$$U\omega(V,W) = \nabla_U \omega(V,W) = h(\nabla_U JV, W)$$

*holds for any vector fields* $U, V, W$. *Consequently* $d\omega = 0 \Longleftrightarrow \nabla J = 0$.

**Example 4** *The projective space* $P_n(\mathbb{C})$ *has a natural Kählerian metric called the Fubini Study metric defined by:*

$$\pi^*\omega = \frac{1}{2\pi}\partial\bar{\partial}\log\left(|\zeta_0|^2 + |\zeta_1|^2 + \cdots + |\zeta_n|^2\right)$$

*where the* $\zeta_i, i = 0, \cdots, n$ *are the coordinates on* $\mathbb{C}^{n+1}$ *and where* $\pi : \mathbb{C}^{n+1}/\{0\} \to P_n(\mathbb{C})$ *is the canonical projection. Let* $z = (\frac{\zeta_1}{\zeta_0}, \cdots, \frac{\zeta_n}{\zeta_0})$ *be the homogeneous coordinates of the chart* $\mathbb{C}^n \subset P_n(\mathbb{C})$ *then:*

$$\omega = \frac{1}{2\pi}\partial\bar{\partial}\log(1 + |z|^2).$$

*Using the Hodge decomposition theorem, on a closed kähler manifold* $M$, *one can relate the de Rham cohomology groups to the Dolbeault cohomology groups by:*

$$H^k(M) = \oplus_{p+q=k} H^{p,q}(M).$$

### 3.3 The curvature and characteristic classes

Useful references are [BGV], [LM], [MS], [Na], [S].

**Definition 39** *The curvature of a covariant derivation is given by the* $Hom(E)$*-valued two form* $\Omega^E = \left(\nabla^E\right)^2 \in \Omega^2(B, Hom(E))$*:*

$$\Omega^E(U,V) := [\nabla^E_U, \nabla^E_V] - \nabla^E_{[U,V]} \quad \forall U, V \in C^\infty(B, TB).$$

*An easy computation shows that the curvature is a local operator, meaning by this that* $\Omega^E(U,V)f = f\Omega^E(U,V)$, *although one could expect a priori from the above formula that* $f$ *might get differentiated.*

It is clear from the definition of the curvature that the *Bianchi identity*

$$[\nabla^E, \Omega^E] = 0 \tag{7}$$

holds.
Writing the connection on a vector bundle in a trivialization over an open subset $U$ of the base manifold $\nabla^E = d + \theta^E_U$, the curvature reads

$$\Omega^E = d\theta^E_U + \theta^E_U \wedge \theta^E_U.$$

**Lemma 3** *Let* $E$ *be a real Riemannian vector bundle equipped with a connection* $\nabla^E$ *which is compatible with the metric. Its curvature* $\Omega^E$ *is an* $so(E)$*-valued* 2*-form on* $M$ *where* $so(E)$ *is the subbundle of* $Hom(E)$ *of antisymmetric morphisms of* $E$.

**Proof 10** *Let* $U, V$ *be two vector fields on the base manifold:*

$$\begin{aligned} 0 &= (UV - VU - [U,V])\langle\sigma, \rho\rangle \\ &= U\langle\nabla_V \sigma, \rho\rangle + U\langle\sigma, \nabla_V \rho\rangle \end{aligned}$$

$$
\begin{aligned}
- &\quad V\langle \nabla_U \sigma, \rho\rangle - V\langle \sigma, \nabla_U \rho\rangle \\
- &\quad \langle \nabla_{[U,V]}\sigma, \rho\rangle - \langle \sigma, \nabla_{[U,V]}\rho\rangle \\
= &\quad \langle \nabla_U \nabla_V \sigma, \rho\rangle + \langle \nabla_V \sigma, \nabla_U \rho\rangle \\
+ &\quad \langle \nabla_U \sigma, \nabla_V \rho\rangle + \langle \sigma, \nabla_U \nabla_V \rho\rangle \\
- &\quad \langle \nabla_V \nabla_U \sigma, \rho\rangle - \langle \nabla_U \sigma, \nabla_V \rho\rangle \\
- &\quad \langle \nabla_V \sigma, \nabla_U \rho\rangle - \langle \sigma, \nabla_V \nabla_U \rho\rangle \\
- &\quad \langle \nabla_{[U,V]}\sigma, \rho\rangle - \langle \sigma, \nabla_{[U,V]}\rho\rangle \\
= &\quad \langle \Omega(U,V)\sigma, \rho\rangle + \langle \sigma, \Omega(U,V)\rho\rangle
\end{aligned}
$$

*so that* $\langle \Omega^E(U,V)\sigma, \rho\rangle = -\langle \sigma, \Omega^E(U,V)\rho\rangle$ *which shows that* $\Omega^E(U,V)$ *is antisymmetric.*

Let now $E = TM$ where $M$ is a Riemannian manifold. We drop the upper index $E$ in the notation.

The *Ricci tensor* of a connection $\nabla$ on a Riemannian manifold $M$ is defined by

$$R(X,Y,W,Z) := \langle \Omega(X,Y)W, Z\rangle$$

where $X, Y, W, Z$ are vector fields on $M$ and $\langle \cdot, \cdot \rangle$ the inner product induced by the Riemannian structure. We have:

$$R(X,Y,W,Z) = -R(Y,X,W,Z) = -R(X,Y,Z,W)$$

and

$$R(X,Y,W,Z) = R(W,Z,X,Y).$$

When $M$ is finite dimensional, the *Ricci curvature* is given by the trace of the operator $\Omega(X,\cdot)Y$, i.e. $\mathrm{Ricc}(X,Y) := \mathrm{tr}(\Omega(X,\cdot)Y)$. The *scalar curvature* is the trace of the Ricci curvature $s(x) = \sum_{i=1}^n \mathrm{Ricc}(e_i(x), e_i(x))$, where $(e_i(x))_{i\in\{1,\cdots,n\}}$ is any local orthonormal frame of $T_xM$.

A connection with vanishing curvature is called a *flat* connection. When the Ricci curvature vanishes, the manifold is called *Ricci flat.*

The ordinary differentiation on sections of a trivial bundle is flat since $d \circ d = 0$.

**Characteristic classes:**

- *Complex vector bundles:* Recall that the trace $\mathrm{tr} : gl_n(\mathbb{C}) \to \mathbb{C}$ on matrices has the following invariance property:

  $$\mathrm{tr}(C^{-1}AC) = \mathrm{tr}(A) \quad \forall C \in Gl_n(\mathbb{C}).$$

  As a consequence it extends to a morphism of complex vector bundles:

  $$\mathrm{tr} : \mathrm{End}(E) \to B \times \mathbb{C}$$

where $E$ is a complex vector bundle over $B$. It furthermore extends to a $\mathbb{Z}_2$-graded (the grading is given by the parity of the forms) trace on $\mathrm{End}(E)$-valued forms on $B$ setting:

$$\mathrm{tr}(\alpha \otimes A) := \alpha \mathrm{tr}(A) \quad \forall \alpha \in \Omega(B), A \in C^\infty(\mathrm{End}(E))$$

with the following (graded) *cyclicity* property:

$$\mathrm{tr}\left([\alpha \otimes A, \beta \otimes B]\right) = 0 \quad \forall \alpha, \beta \in \Omega(B), \quad \forall A, B \in C^\infty(\mathrm{End}(E)), \tag{8}$$

using the graded brackets on endomorphism valued forms:

$$[\alpha \otimes A, \beta \otimes B] = (-1)^{|\alpha|} \alpha \wedge \beta \otimes [A, B].$$

**Remark 6** *If $E$ is $\mathbb{Z}_2$-graded, one equips* $\mathrm{End}(E)$ *with a $\mathbb{Z}_2$-graded super-trace which then extends (see e.g. [BGV] Definition 1.32 and Paragraph 1.5) to a $\mathbb{Z}_2$-graded super-trace on* $\mathrm{End}(E)$*-valued forms on the base manifold.*

Combining the cyclicity of the (graded) trace with the Bianchi identity, provides closed forms.

**Proposition 12** *Let $E \to M$ be a finite rank complex vector bundle over a closed finite dimensional manifold $M$ equipped with a connection $\nabla^E$. For any non negative integer $k$*

$$r_k(\nabla^E)) := tr\left(\left(\Omega^E\right)^k\right)$$

*defines a de Rham cohomology class which is independent of the choice of connection $\nabla^E$.*

**Remark 7** *In the above formula, the product $\Omega^k$ uses both the exterior product and the composition in $Hom(E)$ since $\Omega$ is a $Hom(E)$-valued form.*

**Proof 11** *For convenience we drop the upper index $E$ in the notation. The local description $\nabla = d + [\theta, \cdot]$ of a connection on $Hom(E)$ induced by a connection $\nabla = d + \theta$ on $E$ combined with the cyclicity of the trace $tr([A, B]) = 0$ yields*

$$\mathrm{tr}\left([\nabla, \Omega^j]\right) = \mathrm{tr}\left(d(\Omega^j)\right) + \mathrm{tr}\left([\theta, \Omega^j]\right) = \mathrm{tr}\left(d(\Omega^j)\right) \quad \forall j \in \mathbb{N}.$$

*On the other hand, by the Bianchi identity (7) we have*

$$[\nabla, \Omega^j] = \sum_{i=1}^{j} \Omega^{j-1} [\nabla, \Omega] \Omega^{i-j} = 0 \quad \forall j \in \mathbb{N}, \quad \forall i \in [[1, j]].$$

*Hence,*

$$d\, tr(\Omega^k) = k\, tr(d\, \Omega^k) = k\, tr(\Omega^{k-1} d\Omega) = k\, tr(\Omega^{k-1}[\nabla, \Omega]) = 0.$$

*Let now $\nabla_t$ be a differentiable one-parameter family of connections on $E$, meaning by this that in a local trivialising chart $\nabla_t = d + \theta_t$ where $\theta_t$ is a differentiable one parameter family of local one-forms. The differential $\dot{\nabla}_t = \dot{\theta}_t$ defines a family of globally defined one-forms on the manifold and we have*

$\frac{d}{dt}\Omega_t = \nabla_t \dot{\nabla}_t + \dot{\nabla}_t \nabla_t = [\nabla_t, \dot{\theta}_t]$ *where the bracket is now an anticommutator. Similar arguments to the ones above then yield*

$$\begin{aligned}\frac{d}{dt}\, tr(\Omega_t^k) &= tr(\frac{d}{dt}\,\Omega_t^k) \\ &= k\, tr(\Omega_t^{k-1}\frac{d}{dt}\Omega_t) \\ &= k\, tr(\Omega_t^{k-1}\,[\nabla_t, \dot{\theta}_t]) \\ &= k\, tr\left([\nabla_t, \Omega_t^{k-1}\dot{\theta}_t]\right) \\ &= k\, dtr(\Omega_t^{k-1}\dot{\theta}_t).\end{aligned}$$

This formula extends replacing the $k$-th power by any analytic function $f$ so that $\mathrm{tr}(f(\Omega))$ (which is in fact a polynomial expression in $\Omega$ of degree $\left[\frac{n}{2}\right]$, the integer part of half the dimension of the manifold $M$) is closed in the de Rham cohomology. Its cohomology class, called *Chern-Weil cohomology class*, is independent of the choice of connection.

Different Chern-Weil classes carry different names according to the choice of the function $f$. As an example, the *first Chern form* is obtained from $f(x) := x$,

$$r_1(\nabla) := \mathrm{tr}(\Omega),$$

the *Chern character* is obtained from $f(x) := e^{-x}$,

$$\mathrm{ch}(\nabla) := \mathrm{tr}(e^{-\Omega})$$

where we have set $\Omega = \nabla^2$ for the curvature. The exponential map involves wedge products as well as composition of morphisms since $\Omega$ is a $Hom(E)$-valued two-form. Notice that $r_1(\nabla) = -[\mathrm{ch}(\nabla)]_{[2]}$, namely minus the part of degree 2 of the form $\mathrm{ch}(\nabla)$.
Choosing $f(z) = \frac{z}{e^z - 1}$ on a complex bundle $E$ yields the *Todd genus*

$$\mathrm{Td}(\nabla) = e^{\mathrm{tr}\log\left(\frac{\Omega}{e^{\Omega}-1}\right)}.$$

- *Real vector bundles:* Since the trace vanishes on antisymmetric matrices, the trace is not very useful to define characteristic classes from real vector bundles for which the curvature is an antisymmetric tensor. We therefore use another tool to define characteristic classes form real vector bundles, namely the Pfaffian, which in turn is related to another very useful tool, namely Berezin integration.

  Let $G = SO_n(I\!R)$ and $\gamma = so_n(I\!R)$, there is a one to one correspondence:

$$\begin{aligned}\Lambda^2 I\!R^n &\leftrightarrow so_n(I\!R) \\ a_{ij} e_i \wedge e_j &\leftrightarrow (a_{ij})\end{aligned}$$

  where $e_i, i = 1, \cdots, n$ is an orthonormal basis for the canonical scalar product on $I\!R^n$.

**Definition 40** *Berezin integration on* $\Lambda \mathbb{R}^n$ *is the linear map defined by:*

$$\begin{array}{rcl} T : \Lambda \mathbb{R}^n & \longrightarrow & \mathbb{R} \\ \alpha & \longmapsto & e_i^* \wedge \cdots e_n^*(\alpha) \end{array}$$

*where* $e_i^*, i = 1, \cdots, n$ *is the dual basis* $e_i^*(e_j) = \delta_{ij}$.

Notice that $T$ vanishes on $\Lambda^p \mathbb{R}^n$ for any $p < n$ so that for any $v \in \mathbb{R}^n$ and any $\alpha \in \Lambda \mathbb{R}^n$, $T(\iota(v)\alpha) = 0$ where $\iota(v)$ is the interior product. The fact that $T$ yields a linear map which vanishes on derivations justifies the terminology "integral" (analogy with Stokes' theorem).

Given a real oriented Riemannian vector bundle $E$ of rank $n$ based on a manifold $M$, Berezin integration generalizes to a vector bundle morphism:

$$\begin{array}{rcl} T : \Lambda E & \longrightarrow & M \times \mathbb{R} \\ \alpha & \mapsto & e_i^* \wedge \cdots e_n^*(\alpha) \end{array}$$

where $e_i^*, i = 1, \cdots, n$ is now an orthonormal frame of $E$. $T$ in turn induces a map on sections (denoted by the same symbol) $T : C^\infty(M, \Lambda E) \to C^\infty(M, \mathbb{R})$ in an obvious way.

**Definition 41** *Under the above assumptions on* $E$*, the Pfaffian of* $A = (a_{ij}) \in C^\infty(M, \Lambda^2 E \simeq so(E))$ *is the real valued function on* $M$ *defined by:*

$$\mathrm{Pf}(A) := T\left(e^{\frac{1}{2}\sum_{i,j=1}^n a_{ij} e_i \wedge e_j}\right) = T\left(e^{\sum_{i<j;i,j=1}^n a_{ij} e_i \wedge e_j}\right).$$

In some cases, the Pfaffian is identified to the top form $\mathrm{Pf}(A) e_1 \wedge \cdots e_n$.

We state the following result without proof, leaving the proof as an exercise.

**Lemma 4** *Given* $A = (a_{ij}) \in C^\infty(M, \Lambda^2 E \simeq so(E))$*, if the rank* $n$ *of* $E$ *is even, setting* $n = 2k$ *we have:*

$$\mathrm{Pf}(A) = \frac{(-1)^k}{2^k k!} \sum_{\sigma \in \Sigma_n} \varepsilon(\sigma) a_{\sigma(1)\sigma(2)} \cdots a_{\sigma_{2k-1}\sigma_{2k}}$$

*and the Pfaffian vanishes if the rank of* $E$ *is odd. Here* $\varepsilon(\sigma)$ *denotes the signature of* $\sigma$.

Given a function with Taylor expansion at all orders at 0, namely $f(z) = \sum_{k=0}^K \frac{f^{(k)}}{k!}(0) z^k + o(z^K) \quad \forall K \in \mathbb{N}$ and an oriented metric real vector bundle $(E, \nabla^E)$ equipped with a connection compatible with the metric, similarly to the construction of characteristic classes via the trace, here again, using the Bianchi indentity and the properties of the Pfaffian, one can show that $P(\Omega) = \mathrm{Pf}\left(f(\Omega^E)\right)$ defines a closed form with cohomology class independent of the choice of connection.

Choosing $f(z) = -z$ yields the *Euler class*

$$e(\nabla^E) = \mathrm{Pf}(-\Omega^E) \in \Omega^N(M) \tag{9}$$

where $N$ is the rank of $E$. The Euler class vanishes if $N$ is odd as a consequence of the vanishing of the Pfaffian in odd dimensions. Moreover, as a consequence of the multiplicativity of the Pfaffian on tensor products, this characteristic class obeys the following property:

$$e(\nabla^{E\oplus F}) = e(\nabla^E) \wedge e(\nabla^F).$$

**Remark 8** *If $M$ is an oriented Riemannian surface, and $(E = TM, \nabla^{TM})$ is the tangent bundle equipped with the Levi-Civita connection, then*

$$e(\nabla^{TM}) = K\, d\mathrm{vol}$$

*where $K$ is the Gaussian curvature (see (40)).*

Choosing $f(z) = \frac{\frac{z}{2}}{\mathrm{sh}\frac{z}{2}}$ yields the $\hat{A}$-*genus*

$$\hat{A}(\nabla^E) = \mathrm{Pf}\left(\frac{\frac{\Omega^E}{2}}{\mathrm{sh}\frac{\Omega^E}{2}}\right)$$

and $f(z) = \frac{\frac{z}{2}}{\mathrm{th}\frac{z}{2}}$ the $L$-*genus*

$$L(\nabla^E) = \mathrm{Pf}\left(\frac{\frac{\Omega^E}{2}}{\mathrm{th}\frac{\Omega^E}{2}}\right). \tag{10}$$

As a consequence of the multiplicativity of the Pfaffian on tensor products, these characteristic classes obey the following property:

$$\hat{A}(\nabla^{E\oplus F}) = \hat{A}(\nabla^E) \wedge \hat{A}(\nabla^F); \quad L(\nabla^{E\oplus F}) = L(\nabla^E) \wedge L(\nabla^F).$$

The Todd genus and $\hat{A}$ genus are cllosely related: for any oriented real vector bundle $E$ we have:

$$\mathrm{Td}(E \otimes \mathbb{C}) = \hat{A}(E)^2.$$

# 4 Principal bundles

## 4.1 Classification of principal bundles

Useful references are [MM], [Gr], [S].

**Definition 42** *A (Banach) $C^k$-principal $G$-bundle based on a $C^k$-manifold $B$, where $G$ is a Banach Lie group is a (Banach ) $C^k$-fibre bundle $P$ based on $B$ with typical fibre $G$, such that if $\Phi$ and $\Psi$ are two trivialisations above some open subset $U \in B$, there exists a local map $\gamma : U \subset B \to G$ verifying:*

$$\Phi_b \circ \Psi_b^{-1} = \gamma(b) \quad \forall b \in B.$$

*$G$ is called the structure group of $P$.*

If the group $G$ acts on itself by left translation $L_g : h \to g \cdot h$ and if $(U, \phi, \Phi)$ and $(W, \psi, \Psi)$ are two local trivialization with $b \in U \cap W$, then we have:

$$\Phi_b(p_b) = g \cdot \Phi_b(q_b) \Rightarrow \Psi_b(p_b) = \left(\gamma(b)^{-1} g \gamma(b)\right) \Psi_b(q_b)$$

for any $g \in G$, $p_b, q_b \in E_b$ where $E_b$ is the fibre over $b$. Thus, a change of trivialization induces an inner automorphism $g \mapsto \gamma(b)^{-1} g \gamma(b)$ of $G$.

Given a $C^k$-morphism $\phi : B' \to B$ between two $C^k$-manifolds, the pull-back $\phi^* P$ to $B'$ of a $C^k$-principal $G$-bundle on $B$ is a $C^k$-principal $G$-bundle on $B'$.

Let us now restrict ourselves to $C^0$-bundles. One can show that two homotopic maps $\phi : B' \to B$ and $\psi : B' \to B$ give rise to equivalent principal $G$-bundles $\phi^* P \simeq \psi^* P$. One can therefore associate to the homotopy class $[\phi] \in [B', B]$ of a map $\phi : B' \to B$ the equivalence class of $\phi^* P$. This leads to the following definition.

**Definition 43** *A classifying space for a Lie group $G$ is a connected topological space $BG$ together with a principal $G$-bundle $PG \to BG$, such that for any compact Hausdorff space $X$, there is a one to one correspondence between the homotopy classes $[\phi]$ of maps $\phi : X \to BG$ and equivalence classes of principal $G$-bundles on $X$. A principal $G$-bundle $PG$ on $BG$ yields a pull-back bundle $\phi^* PG$ on $X$. The base space $BG$ is defined up to homotopy type and the bundle $PG \to BG$ is called the universal principal $G$-bundle.*

A principal $G$-bundle $P \to B$ with the property that the total space is contractible yields a classifying space $B$ for $G$. An important example is the Grassmannian $G_n(\mathbb{C}^\infty) := \cup_{N=n}^\infty G_n(\mathbb{C}^N)$ which yields a classifying space for the unitary group $U(n)$ so that $BU(n) = G_n(\mathbb{C}^\infty)$.

Letting $\pi_n(G) := [S^n, G]$ denote the $n$-th homotopy group of $G$, the long exact sequence of homotopy groups:

$$\cdots \to \pi_n(P) \to \pi_n(B) \to \pi_{n-1}(G) \to \pi_{n-1}(P) \to \cdots$$

yields $\pi_n(B) \simeq \pi_{n-1}(G)$, using the fact that $\pi_n(P) = \{1\}$. *Singular cohomology* is needed for further information on the principal bundle (we refer the reader to any classical text on algebraic topology). A *universal characteristic class* for a principal $G$-bundle is a non zero class in the singular cohomology $H^*(BG, \Lambda)$ with coefficients in a ring $\Lambda$. Given a class $c \in H^k(BG, \Lambda)$, and any principal $G$-bundle $P \to B$, there is a map $\phi : B \to BG$ such that $P \simeq \phi^* PG$ and $c(P) := \phi^*(c) \in H^k(B, \Lambda)$ is the $c$-characteristic class of $P$. In particular the cohomology ring $H^*(BU(n), \mathbb{Z})$ is a $\mathbb{Z}$-polynomial ring with canonical generators $c_k \in H^{2k}(BU(n), \mathbb{Z})$, called the universal $k$-th Chern class. Thus to any $U(n)$-principal bundle $P \to B$, classified by a map $\phi : B \to BU(n)$, one can associate the $k$-th Chern class $c_k(P) := \phi^*(c_k)$. The relation to the Chern classes described at the end of the previous chapter will become clear once we have set up a correspondence between vector bundles and principal bundles; via this correspondence, the Chern class $c_k(P)$ can be seen as a Chern class on a complex rank $n$ vector bundle $E$.

## 4.2 From group actions to principal bundles

Useful references in view of the applications we have in mind for quantum field theory are [AM], [Br], [E], [KR], [FU], [Tr]. Foundations for this type of slice theorem were set up in [P].

## 4.3 The slice theorem in the Hilbert setting

Foundations for this type of slice theorem were set up in [P].

**Definition 44** *A $C^k$-manifold (resp. $C^\infty$-manifold) modelled on a topological (real) vector space $V$ is a Hausdorff topological space $X$ together with a family of charts $(U_\alpha, \phi_\alpha), \alpha \in A$, such that*

1. *$U_\alpha$ are open subsets of $X$ which cover $X$ i.e., $X \subset \cup_{\alpha \in A}$;*
2. *$\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha) \subset V$ are homeomorphisms onto open sets $\phi_\alpha(U_\alpha), \alpha \in A$*
3. *$\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \longrightarrow \phi_\beta(U_\alpha \cap U_\beta)$ are $C^k$ (resp. $C^\infty$) maps.*

We recall that a locally convex vector space is said to be convenient [KM] if a curve $c : \mathbb{R} \to V$ is smooth whenever $\lambda \circ c : \mathbb{R} \to \mathbb{R}$ is smooth for any $\lambda \in V^*$ the topological dual of $V$. A map $f : U \subset V \longrightarrow W$ from an open subset $U$ of a convenient vector space $V$ to another convenient vector space $W$ is said to be smooth if $f \circ c : \mathbb{R} \to W$ is smooth for any smooth curve $c$ in $U \subset V$. For maps on Fréchet spaces this notion of smoothness coincides with all other reasonable definitions and multilinear mappings are smooth if and only if they are bounded.

**Definition 45** *If in the above definition, $V$ is (in increasing level of generality) a Banach, resp. Hilbert vector space, resp. Fréchet vector space, resp. convenient locally convex vector space, then $X$ is called a Banach, resp. Hilbert, resp. Fréchet, resp. convenient locally convex manifold.*

**Remark 9** *If $V$ is finite-dimensional, $V \simeq \mathbb{R}^n$ for some $n \in \mathbb{N}$, we recover the definition of a finite $n$-dimensional manifold.*

**Example 5** • *$C^k$ diffeomorphisms of a closed manifold $M$*

$$\mathrm{Diff}^k(M) = \{f \in C^k(M, M),\ f\, bijective,\ f^{-1} \in C^k(M, M)\}$$

*form a Banach manifold.*

- *An important Fréchet manifold is the diffeomorphism group of a closed manifold $M$:*
$$\mathrm{Diff}(M) = \{f \in C^\infty(M,M),\, f\, bijective,\, f^{-1} \in C^\infty(M,M)\}.$$

**Definition 46** *A set $G$ endowed with a group structure and a smooth Banach, resp. Hilbert , resp. Fréchet, resp. convenient manifold structure, in such a way that these two structures are compatible i.e., the mapping $G \times G \ni (g,h) \longmapsto g\,h^{-1} \in G$ is smooth, is called a* **Banach, resp. Hilbert , resp. Fréchet, resp. convenient Lie group**. *If the manifold is finite dimensional, this concept coincides with the usual concept of a Lie group.*

**Example 6** $\mathrm{Diff}(M)$ *is a Fréchet Lie group yet* $\mathrm{Diff}^k(M)$ *is a Banach topological group but not a Banach Lie group, due to the lack of smoothness of the composition map.*

The tangent space of a Banach, resp. Hilbert , resp. Fréchet, resp. convenient manifold is a Banach, resp. Hilbert , resp. Fréchet, resp. convenient vector space. A **strong Riemannian structure** on a Hilbert manifold is a (smooth) choice of inner products which induces a Hilbert space structure on each tangent space.

**Definition 47** *Let $G$ be a Hilbert Lie group acting on the right on a Hilbert manifold $X$ via a smooth action:*

$$\begin{aligned} \Theta : G \times X &\to X \\ (g,x) &\mapsto x.g := R_g(x) \end{aligned}$$

*i.e. $R_{g\cdot g'} = R_g \circ R_{g'}$ for $g, g' \in G$.*

1. *The action $\Theta$ is* **proper** *provided the map*
$$\begin{aligned} \Xi : G \times X &\to X \times X \\ (g,x) &\mapsto (x \cdot g, x) \end{aligned}$$
*is proper, i.e. preimages of compact sets have compact closure.*
2. *The action $\Theta$ is* **free** *provided it has no fixed points:*
$$\exists x \in X, \quad x \cdot g = x \Rightarrow g = e.$$
3. *If $X$ equipped with a strong Riemannian metric, we say that the action is* **isometric** *provided it leaves the metric (given by inner products $\langle \cdot, \cdot \rangle_x$ on the fibre $T_xX$) invariant:*
$$\langle DR_g U, DR_g V\rangle_{x\cdot g} = \langle U, V\rangle_x \quad \forall U, V \in T_xX.$$
*The metric is said to be compatible with the group action.*

**Remark 10**
- *The freedom of the action $\Theta$ corresponds to the injectivity of the map $\theta_x$, that for any $x \in X$ sends an element of $G$ to an element of the orbit $O_x = \{x\cdot, g, g \in G\}$:*
$$\begin{aligned} \theta_x : G &\to O_x \\ g &\mapsto x \cdot g. \end{aligned}$$

- *If $G$ is a compact Lie group the action is proper. To see this, we show that one can extract a convergent subsequence from any sequence $(x_n, g_n) \in G \times X$ such that $\Xi((x_n, g_n)) = (x_n \cdot g_n, x_n) \in K$ where $K$ is a compact subset of $X \times X$. $K$ being compact, so is its projection onto the second component and we can extract from $(x_n)$ a convergent subsequence $(x_{\phi(n)})$. $G$ being compact, there is a subsequence of $(g_{\phi(n)})$ which we denote by the same symbol for simplicity, converging to some $g \in G$. The subsequence $(x_{\phi(n)}, g_{\phi(n)})$ therefore does the job.*

**Theorem 6** *(***The slice theorem***) Let $G$ be a Hilbert Lie group acting on the right on a (strong) Riemannian (Hilbert) manifold $X$ via an isometric action:*

$$\Theta : G \times X \to X$$

*which is smooth, free and proper.*
*If for any $x \in X$ the tangent map $\tau_x := D_e\theta_x$ has a closed range, then*

1) *the orbits are closed submanifolds of $X$ and $\theta_x : G \to O_x$ is a diffeomorphism of manifolds,*

2) *the quotient space $X/G$ has a smooth Hilbert structure,*

3) *the projection $\pi : X \to X/G$ yields a principal fibre bundle.*

**Remark 11** *In the finite dimensional case, there is no need for the splitting condition ($R(\tau_x)$ is closed) on $\tau_x$, which is automatically fulfilled. In the more general Hilbert setting, a Fredholm operator $\tau_x$ fulfills the additional requirement that the range be closed.*

**Proof 12** • *We first check item 1).*

i) *$\Theta$ being proper, $\theta_x$ is a closed mapping. Indeed, if $\theta_x(g_n)$ converges to $y$, then $(x, g_n \cdot x)$ converges to $(x, y)$ and the properness of the action implies the existence of a subsequence $g_{\phi(n)}$ converging to some $g \in G$. The action being continuous, $\theta_x(g_{\phi(n)}) = g_{\phi(n)} \cdot x$ converges to $y = g \cdot x$. It follows that $\theta_x(g_n)$ converges to $y = g \cdot x$. Thus $\theta_x$ is a homeomorphism onto its range $O_x$.*

ii) *Let us check that $D_g\theta_x$ is injective. Otherwise, there is some $u \neq 0 \in \mathfrak{g}$ such that $D_g\theta_x(u \cdot g) = 0$. But since $D_g\theta_x = DR_g \circ \tau_x \circ DR_g^{-1}$, this would imply that $\tau_x u = 0$. Then, for any $t_0 \in I\!R$*

$$\begin{aligned} \frac{d}{dt}_{|t=t_0} x \cdot e^{tu} &= \frac{d}{dt}_{|t=0} x \cdot e^{tu} \cdot e^{t_0 u} \\ &= DR_{g_0} \frac{d}{dt}_{|t=0} x \cdot e^{tu} \\ &= DR_{g_0}(\tau_x u) \\ &= 0 \end{aligned}$$

*where we have set $g_0 := \exp t_0 u$. This would imply that $\theta_x(g_t)$ is constant which contradicts the freedom of the action.*

*iii) Let us check that the range of the map $D\theta_x$ is closed. This follows from $R(D_g\theta_x) = DR_g R(\tau_x)$ (see the second preliminary remark) combined with the fact that $\tau_x$ has closed range. Moreover, since $R_g$ is an isometry, it preserves orthogonality and*
*$R(D_g\theta_x)^\perp = DR_g\,(R(\tau_x))^\perp$ so that we have the following orthogonal splitting:*

$$T_{x\cdot g}X = R(D_g\theta_x) \oplus DR_g\,(R(\tau_x))^\perp\,.$$

*iv) Thus $\theta_x$ is an injective immersion which is also a homeomorphism onto its image. The inverse mapping theorem for Hilbert manifoldss then implies it is a diffeomorphism $G \simeq O_x$ and the orbit $O_x$ is a submanifold of $X$. The tangent space of $O_x$ at point $y = x\cdot g$ is $R(D_g\theta_{xg})$. This finishes the proof of point 1) of the Theorem.*

- *Let us now check items 2) and 3).*

*i) Let $U_x$ be an open neighborhood of $x$ in $R(\tau_x)^\perp$, small enough to build the submanifold $S_x := \exp_x(U)$ of $X$ using the exponential map $\exp_x : U \to V_x \subset X$ at point $x$, where $U$ is an open neighborhood of the zero section of the tangent bundle $TX$ and $V_x$ an open neighborhood of $x \in X$. Since the exponential map defines a local diffeomorphism, $S_x$ inherits a manifold structure which by construction has tangent space at point $x$ given by $N_x(O_x) := R(\tau_x)^\perp$ where $N$ stands for normal, $N_x(O_x)$ being the fibre above $x$ of the normal bundle to the orbit $O_x$.*

*ii) The action being continuous, free and proper, one can chose $U$ small enough so that*

$$(S_x)\,.g \cap S_x \neq \phi \Leftrightarrow g = e. \tag{11}$$

*Indeed, otherwise, we could find a sequence $(u_n) \in N_x(O_x)$ with norm $\|u_n\| \leq \frac{1}{n}$ such that both $(x_n)$ and $(x_n \cdot g_n)$ converge to some $x$. But in that case, the properness of the action yields the existence of a subsequence $(g_{\phi(n)})$ converging to some $g \in G$. The continuity of the action then implies that in the limit $x \cdot g = g$. But the action being free, this implies in turn that $g = e$.*

*iii) It follows from (11) that the local slice $S_x$ is in one to one correspondence with a subset $U_{\bar{x}}$ of the quotient space $X/G := B$. Equipping $B$ with the quotient topology turns the projection map $\pi : X \to B$ into a continuous map and yields a homeomorphism $\pi : S_x \to U_{\bar{x}}$. The manifold structure on $S_x$ then yields a local chart over the neighborhood $U_{\bar{x}}$ of $\bar{x} \in B$. Patching up such local trivializations yields a smooth atlas on $B$ with transition maps obtained from the exponential maps.*

*iv) This quotient manifold inherits a metric structure from the $G$-invariant structure on $X$. Given $\bar{U}, \bar{V} \in T_{\bar{x}}B$ we set:*

$$\langle \bar{U}, \bar{V}\rangle_{\bar{x}} := \langle U, V\rangle_x$$

*for any $x$ in the fibre above $\bar{x}$ and any $U, V \in T_xX$ such that $D\pi(U) = \bar{U}, D\pi(V) = \bar{V}$. Since the metric is $G$-invariant, this is independent of the choice of $x$ and of $U$ and $V$.*

*v) The above local charts induce local trivializations for the projection $\pi : X \to B$ so that we can equip $X$ with a $G$-principal bundle structure over $B$. Locally we have:*

$$X_{|U_{\bar{x}}} \simeq S_{\bar{x}} \times G.$$

### 4.4 From vector bundles to principal bundles and back

Useful references are [BGV], [KN].

To a smooth vector bundle $E \to B$ based on a manifold $B$ with typical fibre a Banach space $V$, we associate a principal bundle $GL(E) \to B$ called the associated *frame bundle* with structure group $G := GL(V)$ and fibre above $b \in B$ given by:

$$GL_b(E) := \{L_b : V \to E_b, \quad \text{continuous and one to one}\}.$$

(Recall from the open mapping theorem that it is a homeomorphism).
Letting

$$\begin{array}{rcl} (\phi, \Phi) : E_{|U} & \to & U \times V \\ (b, u) & \mapsto & (\phi(b), \Phi_b u) \end{array}$$

be a local trivialization of $E$ above an open subset $U$ of $B$, a local trivialization $(\phi, \bar{\Phi})$ of $GL(E)$ above $U$ is given by

$$\begin{array}{rcl} GL(E)_{|U} & \to & U \times GL(V) \\ (b, L_b) & \mapsto & \left(\phi(b), \bar{\Phi}_b(L_b) := \Phi_b \circ L_b\right). \end{array}$$

Given two local trivializations $(\phi, \Phi)$ and $(\psi, \Psi)$ of $E$ and hence two induced trivializations $(\phi, \bar{\Phi})$, $(\psi, \bar{\Psi})$ of $GL(E)$ we can build a map:

$$\begin{array}{rcl} \rho : U & \to & GL(V) \\ b & \mapsto & \bar{\Psi}_b \circ \bar{\Phi}_b^{-1} : L \to \Psi_b \circ \Phi_b^{-1} \circ L_b \end{array}$$

A Banach vector bundle $E$ of class $C^k$ (resp. $C^\infty$) is trivial if and only if $GL(E)$ admits a global section of class $C^k$ (resp. $C^\infty$). Indeed, a global $\alpha$ section of class $C^k$ (resp. $C^\infty$) yields a diffeomorphism:

$$\begin{array}{rcl} E & \to & B \times V \\ (b, u_b) & \mapsto & (b, \alpha(b)(u_b)). \end{array}$$

If $E$ is a rank $n$ vector bundle, a global section of class $C^k$ (resp. $C^\infty$) of the principal bundle $GL(E)$ corresponds to a family of frames $(e_1(b), \cdots, e_n(b))$ of class $C^k$ (resp. $C^\infty$) parametrized by $B$ and the section $\alpha$ yields the coordinates $\alpha_i(b), i = 1, \cdots, n$ of the vector $u_b$ in the basis $(e_1(b), \cdots, e_n(b))$ of the fibre $E_b$ above $b$.

When $E = TM$, the tangent bundle to a manifold $M$ of class $C^k$ (resp. $C^\infty$), the existence of a global section of $GL(E)$ is a constraint on the manifold $M$ and we say that $M$ is $C^k$- (resp. $C^\infty$-) *parallelizable*. If we only require this section to be continuous, it is a topological constraint. A Lie group is clearly $C^\infty$-paralellizable since left (or right) action $L_g : h \mapsto g \cdot h$ (or $R_g : h \mapsto h \cdot g$) of the group on itself induces a smooth parallelization $L_g : Lie(G) \to T_g G$ (resp. $R_g : Lie(G) \to T_g G$).

A result by Kuiper [K] tells us that given a $C^k$ (resp. smooth) Hilbert vector bundle $E \to B$, the associated frame bundle $GL(E)$ admits a global $C^0$ section. Thus any Hilbert manifold is $C^0$-parallelizable.

Conversely, given a principal bundle with structure group $G$ and a representation $\rho : G \to Diff(V)$ on a Banach space $V$, we can build the *associated vector bundle*:

$$P \times_\rho V := P \times V/_\sim$$

where $\sim$ is the equivalence relation defined by

$$(p, v) \sim (p', v') \Leftrightarrow \exists g \in G, p = p' \cdot g \quad \text{and} \quad v' = \rho(g)v$$

so that $(p \cdot g, v)$ and $(p, \rho(g)v)$ get identified. Locally, above an open subset $U$ of the base manifold, we have:

$$P \times_\rho V_{|U} \simeq (U \times G) \times_\rho V \simeq U \times V.$$

In particular, a vector bundle $E$ with typical fibre $V$ is associated to its frame bundle $GL(E)$ with structure group $GL(V)$:

$$GL(E) \times_\rho V = E$$

where $\rho$ is the natural the action of $GL(V)$ on $V$.

## 4.5 Connections on a principal bundle

Useful references are [BGV], [KN], [MM], [Ts].

Given a principal bundle $\pi : P \to B$ with structure group $G$, and the induced map $D\pi : TP \to TB$, we call a vector field $\xi$ *vertical* provided $D\pi(\xi) = 0$. Let us denote by $VTP$ the subbundle of vertical vector fields with fibre above $p \in P$ given by

$$VT_pP := \{v \in T_pP, D_p\pi(v) = 0\}.$$

Given a point $p \in P$, the right action

$$\begin{array}{rcl} \theta_p : G & \longrightarrow & P \\ g & \longmapsto & p \cdot g \end{array}$$

induces a map:

$$\begin{array}{rcl} \tau_p : Lie(G) & \to & T_pP \\ u & \mapsto & \frac{d}{dt}_{|t=0}(p \cdot \exp(tu)), \end{array}$$

which in turn gives rise to a vertical vector field:

$$p \mapsto \xi(p) := \bar{u}_p := \tau_p(u),$$

called the *canonical vector field* associated to $u$. $\tau_p : u \to \bar{u}_p$ is an isomorphism of $Lie(G)$ onto $VT_pP$.

**Definition 48** *A connection on the principal bundle $P$ with structure group $G$ is a smooth splitting*

$$T_pP := VT_pP \oplus HT_pP, \quad \forall p \in P$$

*with an equivariance property* $HT_{p \cdot g}P = DR_g(HT_pP) \quad \forall p \in P, \forall g \in G$. *$HTP$ is called a horizontal distribution and $HT_pP$ the horizontal tangent space to $P$ at point $p$. A horizontal distribution $HTP$ turns the map $D\pi : HTP \to TB$ into an ismorphism and we call $\tilde{\xi}$ the horizontal lift of a vector field $\xi$ on $B$.*

Equivalently,

**Definition 49** *A connection on the principal bundle $P$ is given by a Lie algebra valued one form on $P$, $\omega \in \Omega^1(P, Lie(G))$ such that:*

*(i)* $\omega_p(\bar{u}_p) = u \quad \forall u \in Lie(G)$

*(ii)* $\omega_{p \cdot g} = R_g^* \omega_p \quad \forall p \in P$ *where $R_g : p \to p \cdot g$ corresponds to the action of $G$ on $P$.*

The two formulations are equivalent. Indeed, given a smooth horizontal distribution $HTP$, any tangent vector field $\xi$ splits in a unique way $\xi = \xi^v \oplus \xi^h$ into a vertical part $\xi^v := \tau_p u$ for some unique $u \in Lie(G)$, and a horizontal part $\xi^h$. $\omega(\xi) := u$ defines a unique $Lie(G)$ valued one form $\omega$ on $P$ satisfying requirements (i) and (ii). The curvature of the connection reads $\Omega := d\omega + \omega \wedge \omega$. it measures in how far the splitting $TP = VTP \oplus HTP$ does not respect the Lie algebra structure on vector fields, $\Omega(U, V) = [\tilde{U}, \tilde{V}] - \widetilde{[U, V]}$ where $\tilde{U}$, $\tilde{V}$ are the horizontal lifts of the vector fields $U$ and $V$.

Conversely, such a one form $\omega$ defines a vector bundle $HTP := \{\xi \in TP, \omega(\xi) = 0\}$ which has the required invariance property by property (ii).

There is a natural horizontal distribution $HTP = (VTP)^\perp$ whenever there is a (strong) Riemannian metric compatible with the action of the structure group $G$ on $P$.
There is a one to one correspondence between covariant derivations defined on vector bundles $E$ and connections defined on principal bundles in the following sense.

A connection $\omega$ on the principal bundle $\pi : P \to B$, seen as a horizontal distribution on $P$, yields a covariant derivation on the associated vector bundle $E$. Let $X \in T_bB, b \in B$ and let $\tilde{X}$ be its horizontal lift. Letting $\rho : G \to Aut(V)$ be an action of the gauge group $G$ of $P$ on a Banach vector space $V$, a section $\sigma$ of the associated vector bundle $P \times_G V$ can be seen locally as a map from an open subset of $B$ to the vector space $V$ so that it makes sense to set

$$\nabla_X \sigma := \left(p, \tilde{X}\sigma\right).$$

Notice that we implicitly have used the equivariance of the horizontal distribution in this definition. $\nabla$ yields a connection on the associated vector bundle $E := P \times_G V$.

The covariant derivation $\nabla$ is compatible with the metric whenever the horizontal distribution is given by the orthogonal supplement to the vertical bundle.

Conversely, a covariant derivation $\nabla^E$ on a vector bundle $E$ with typical fibre $V$ yields a connection on the frame bundle $GL(E)$ (recall that its structure group is $GL(V)$ with Lie algebra $Lie(G) = Hom(V)$). Let us set $P = GL(E)$, the canonical projection $\pi : P \to B$ induces a map $D_p\pi : T_pB \to T_{\pi(b)}B$ so that given a tangent vector $X \in T_pP$, we can set:

$$\omega(X) = \nabla^{Hom(E)}_{D_p\pi X},$$

where $\nabla^{Hom(E)}$ is the covariant derivation induced by $\nabla^E$ on $Hom(E)$. Here $TGL(E)$, on which the form $\omega$ is defined, is locally seen as $U \times Hom(V)$. Since $Hom(E)$ is also locally seen as $U \times Hom(V)$, it makes sense to let the covariant derivation on the r.h.s. act on a section of $TGL(E)$.

## 4.6 Reducing and lifting principal bundles: spin and spin$^c$ structures

Useful references are [H], [LM], [Ma], [Mo].

**Definition 50** *Let $H$ be a closed subgroup of a Banach Lie group $G$. A principal bundle $P$ based on $B$ with structure group $G$ reduces to a principal bundle with structure group $H$ whenever there is an atlas of charts $(U_i, \Phi_i)$ for $P$ such that the transition maps have values in the subgroup $H$. Let $\{1\} \to H \to \tilde{G} \to G \to \{1\}$ be an exact sequence of Banach Lie groups. A $G$-principal bundle $P$ based on $B$ lifts to a $\tilde{G}$-principal bundle $\tilde{P}$ whenever $\tilde{P}$ reduces to $P$, where we view $G$ as a subgroup of $\tilde{G}$ via the isomorphism $\tilde{G} \simeq H \times G$.*

A principal bundle reduces to a bundle with structure group $H = \{1\}$ whenever the bundle is trivial.

Reducing the structure group is a way to impose geometric constraints on the bundle. In particular, a real (resp. complex) vector bundle $E \to B$ with typical fibre $V$ can be equipped with a (strong) Riemannian (resp. Hermitian) structure whenever the associated frame bundle $GL(E)$ with structure group $GL(V, \mathbb{R})$ (resp. $GL(V, \mathbb{C})$) reduces to the orthonormal frame bundle, a principal bundle with structure group $O(V) := \{g \in GL(V, \mathbb{R}), g^*g = I\}$ (resp. $U(V) := \{g \in GL(V, \mathbb{C}), g^*g = 1\}$). In particular, when $E := TM$ where $M$ is an $n$-dimensional real (resp. complex manifold), then $V = \mathbb{R}^n$ (resp. $V = \mathbb{C}^n$) and $M$ can be equipped with a Riemannian (resp. Hermitian) metric whenever the frame bundle $GL(M) := GL(TM)$, with structure group $GL(n, \mathbb{R})$ (resp. $GL(n, \mathbb{C})$), reduces to the orthonormal (resp. unitary) frame bundle $O(M)$ (resp. $U(M)$) with structure group $O(n)$ (resp. $U(n)$). Furthermore a real rank $n$ Riemannian vector bundle $E$ is orientable whenever its frame bundle $GL(M)$ reduces to a principal bundle with structure group $SO(n) := \{g \in O(n), detg > 0\}$.

Lifting principal bundles is not always possible, as we shall see shortly, when trying to define spin and spin$^c$ structures.

**Definition 51** *Let $V$ be a real Euclidean vector space. The algebra $C\ell(V)$ over $\mathbb{R}$ generated by $V$ with the relations:*

$$v \cdot w + w \cdot v = -2\langle v, w\rangle$$

*where $\langle \cdot, \cdot\rangle$ denotes the inner product on $V$ is called the Clifford algebra of $V$.*

The Clifford algebra of $V$ can also be seen as a quotient space $C\ell(V) = \mathcal{T}(V)/\{v \otimes v + 2\|v\|^2\}$ of the tensor algebra $\mathcal{T}(V) = \oplus_{k=1}^{\infty} V^{\otimes^k}$ by the relation $v \otimes v = -2\|v\|^2$. The $\mathbb{Z}$-grading on $\mathcal{T}(V)$ induces a natural $\mathbb{Z}_2$ -grading on $C\ell(V)$:

$$C\ell(V) =: C\ell_0(V) + C\ell_1(V)$$

into even and odd (Clifford) products.
A linear map $c : V \to \mathrm{Emd}(W)$ for some vector space $W$ extends to an algebra morphism $c : C\ell(V) \to \mathrm{End}(W)$. whenever $c(v)^2 = -1_W$.

**Example 7** *Given $v \in V$, let $c(v)$ act on the exterior algebra $W := \Lambda V$ by*

$$c(v)\alpha := \varepsilon(v)\alpha - \iota(v)\alpha$$

*where $\varepsilon(v)\alpha := v \wedge \alpha$ is the adjoint of the contraction operator $\iota(v)(u) := \langle u, v\rangle$ for any $u \in V$ extended to $\Lambda V$ by the Leibniz rule:*

$$\iota(v)(\alpha \wedge \beta) := \iota(v)\alpha \wedge \beta + (-1)^{|\alpha|}\alpha \wedge \iota(v)\beta \quad \forall \alpha, \beta \in \Lambda V.$$

*Since it satisfies the relation*

$$c(v)c(w) + c(w)c(v) = -2\langle v, w\rangle$$

*it extends to an action of $C\ell(V)$ on $\Lambda V$, which makes $\Lambda V$ a $C\ell(V)$-module.*
*The symbol map $\sigma : C\ell(V) \to \Lambda V$ is defined by:*

$$\sigma(a) := c(a)1 \in \Lambda V.$$

These constructions can be extended fibrewise to vector bundles. Given a Riemannian bundle $E$ based on $B$ with typical fibre $V$, one can define the bundle $C\ell(E)$ of Clifford algebras based on $B$ with typical fibre $C\ell(V)$ defined fibrewise by $C\ell(E_b)$ where $E_b$ is the fibre above $b \in B$ equipped with the inner product induced by the Riemannian structure.

**Definition 52** *A Clifford module on a Riemannian manifold $M$ is a vector bundle $E \to M$ with a Clifford action of $C\ell(M) := C\ell(TM)$ on it:*

$$\begin{array}{ccc} C\ell(M) \times E & \to & E \\ (v, \sigma) & \mapsto & c(v)\sigma. \end{array}$$

**Example 8** *Take $E := \Lambda TM$, a vector field $v \in C^\infty(TM)$ acts on a form $\alpha \in \Omega(M)$ by the following Clifford action:*

$$c(v)\alpha := \varepsilon(v)\alpha - \iota(v)\alpha,$$

*which by Proposition 10 extends to an action of sections of the bundle $C\ell(TM)$ on $\Omega(M)$. Thus $E := \Lambda T^*M$ the exterior bundle on $M$ yields a Clifford module over $M$. The symbol map sends a section $a$ of $C\ell(M)$ to $c(a) \in \Omega(M)$.*

Going back to the algebraic setting, let us assume that $V$ is finite dimensional. The space $C\ell^2(V) := c(\Lambda^2 V)$ is a Lie subalgebra of $C\ell(V)$ with bracket given by the commutator of $C\ell(V)$. The *spin group* Spin $(V)$ is the group generated by elements in $C\ell_0(V)$ with norm 1. It can also be seen as the group obtained by exponentiating the Lie algebra $C\ell^2(V)$ inside the Clifford algebra $C\ell(V)$. Letting $V := \mathbb{R}^n$, we simply write $C\ell(n) := C\ell(\mathbb{R}^n)$ and Spin $(n) := \mathrm{Spin}(\mathbb{R}^n)$.

Here is a very classical result which we do not prove here since it is purely algebraic and can be found in any text book on spin structures.

**Proposition 13** *If dim* $V > 1$*, there is an exact sequence of groups:*

$$1 \to \mathbb{Z}_2 \to Spin(V) \to SO(V) \to 1.$$

*Spin*$(V)$ *is therefore a double covering of* $SO(V)$.

*Similarly,* $Spin^c(V)$ *is the subgroup of* $C\ell_0(V) \otimes_{\mathbb{R}} \mathbb{C}$ *generated by* $Spin(V)$ *and the unit circle of complex scalars. It yields a double covering of* $SO(V) \times S^1$ *and there is an exact sequence of groups:*

$$1 \to \mathbb{Z}_2 \to Spin^c(V) \to SO(V) \times S^1 \to 1.$$

$Spin(V)$ *is then naturally identified with the subgroup* $Spin(V) \times_{\pm 1} \{\pm 1\}$ *of* $Spin^c(V)$ *and*

$$Spin^c(V) \simeq Spin(V) \times_{\pm 1} S^1.$$

*Letting* $V := \mathbb{R}^n$*, we simply write* $Spin^c(n) := Spin^c(\mathbb{R}^n)$.

**Definition 53** *An oriented Riemannian rank n-bundle* $E \to B$ *admits a spin (resp. spin*$^c$*) structure whenever the bundle* $SO(E)$ *of oriented orthonormal frames lifts to a principal bundle* $P_{spin}(E)$ $(\tilde{P}_{spin^c}(E))$ *with structure group Spin* $(n)$ *(resp. Spin* $^c(n)$*). In particular an n-dimensional oriented Riemannian manifold M is spin (resp. spin*$^c$*) whenever its frame bundle* $GL(TM)$ *admits a spin (resp. spin*$^c$*) structure.*

The obstruction to the existence of a spin structure on a vector bundle is measured by the second *Stiefel Whitney class* in $H^2(M, \mathbb{Z}_2)$. The obstruction to the existence of a spin$^c$ structure is weaker since such a structure exists whenever this second Stiefel-Whitney class is a reduction modulo 2 of an integral class $c \in H^2(M, \mathbb{Z}_2)$, i.e. if its third Stiefel-Withney class vanishes. In particular, any spin manifold is spin$^c$.

Let us make a short comment on Stiefel-Whitney classes. To any rank $n$ real Riemannian vector bundle $E \to B$ classified by a map $f_E : B \to BO(n)$, one can associate the $k$-th *Stiefel-Whitney class* $w_k(E) := f_E^*(w_k) \in H^k(B, \mathbb{Z}_2)$ where $w_k \in H^k(BO(n), \mathbb{Z}_2)$ are the canonical generators of the $\mathbb{Z}_2$-polynomial ring $H^*(BO(n), \mathbb{Z}_2)$. The first Stiefel-Whitney class measures an obstruction to the orientability of a Riemannian bundle, and the second Stiefel-Whitney class measures the obstruction to the existence of a spin structure on an orientable Riemannian bundle.

Back again to the algebraic setting, let us set $\mathbb{C}\ell(n) := C\ell(n) \otimes \mathbb{C}$. Then Spin$(n) \subset C\ell(n) \subset \mathbb{C}\ell(n)$ so that any complex representation of the complexified Clifford algebra $\mathbb{C}\ell(n)$ on some vector space $S$ reduces to a complex representation of Spin$(n)$. There are essentially two types of representations according to the parity of the manifold which we briefly describe in the following proposition, referring the reader to any classical text on spin structures for a proof.

**Proposition 14** *When n is odd all irreducible complex representations* $\mathbb{C}\ell(n) \to Hom_{\mathbb{C}}(S, S)$ *restrict to a unique irreducible representation of* $Spin(n)$*. When n is even, a complex representation* $\mathbb{C}\ell(n) \to Hom_{\mathbb{C}}(S, S)$ *yields a representation* $\Delta_n$ *of* $Spin(n)$ *which decomposes into a direct sum of two inequivalent irreducible complex representations* $\Delta_n^+$ *and* $\Delta_n^-$ *on* $S^+$ *and* $S^-$ *respectively. Such representations are called spinor representations and the corresponding representation spaces* $S$, $S^+$,$S^-$ *are called spinor spaces.*

These spinor spaces give rise to *spinor bundles*:

$$S(E) := P_{spin}(E) \times_{\mathrm{Spin}(n)} S,$$

$$S^{+}_{-}(E) := P_{spin}(E) \times_{\mathrm{Spin}(n)} S^{+}_{-},$$

where $E \to B$ is some vector bundle with a spin structure. In fact, any *Clifford module* $\mathcal{M}$ based on a odd (resp. even) dimensional spin manifold $B$, i.e. any (resp. $\mathbb{Z}_2$-graded) vector bundle $\mathcal{M}$ with an (graded) action of the bundle $C\ell(B)$ of Clifford algebras on it

$$\begin{array}{rcl} C^{\infty}(C\ell(B)) \times C^{\infty}(\mathcal{M}) & \to & C^{\infty}(\mathcal{M}) \\ (a, s) & \mapsto & c(a) \cdot s \end{array}$$

is of the form:

$$\mathcal{M} := S(TB) \otimes W,$$

$$(\text{resp.} \quad \mathcal{M}^{+}_{-} := S^{+}_{-}(TB) \otimes W)$$

where $W$ is an exterior vector bundle based on $B$.

Any complex representation $\rho : Spin(V) \to GL_{\mathbb{C}}(W)$ extends in an unique way to a representation $\tilde{\rho} : Spin^c(V) \to GL_{\mathbb{C}}(W)$. In particular the complex representations $\Delta_n$, $\Delta_n^-$, $\Delta_n^+$ uniquely extend to $\tilde{\Delta}_n$, $\tilde{\Delta}_n^-$, $\tilde{\Delta}_n^+$ on $\tilde{S}$, $\tilde{S}^+$, $\tilde{S}^-$. The corresponding spinor spaces give rise to spinor bundles:

$$\tilde{S}(E) := \tilde{P}_{spin^c}(E) \times_{\mathrm{Spin}^c(n)} \tilde{S},$$

$$\tilde{S}^{+}_{-}(E) := \tilde{P}_{spin^c}(E) \times_{\mathrm{Spin}^c(n)} \tilde{S}^{+}_{-}$$

where $E \to B$ is some vector bundle with a spin$^c$ structure.

# 5 Bounded and unbounded linear operators

## 5.1 Bounded linear operators

Useful references are [B], [Bre], [Mu], [KR], [RLL], [RS], [Ru], [We].

Let $\mathbb{E}$ and $\mathbb{F}$ be two complex Banach spaces with norm $\|\cdot\|_{\mathbb{E}}$ and $\|\cdot\|_{\mathbb{F}}$ respectively.

**Definition 54** *The space of bounded linear operators from* $\mathbb{E}$ *to* $\mathbb{F}$

$$\mathcal{B}(\mathbb{E},\mathbb{F}) := \{A : \mathbb{E} \to \mathbb{F}, \quad \exists C > 0, \|Au\|_{\mathbb{F}} \leq C\|u\|_E \quad \forall u \in \mathbb{E}\}$$

*equipped with the norm* $|||A||| \equiv sup_{u\in\mathbb{E},x\neq 0} \frac{\|Au\|_{\mathbb{F}}}{\|u\|_{\mathbb{E}}}$ *is a Banach space.*

When $\mathbb{F} = \mathbb{E}$ we set $\mathcal{B}(\mathbb{E}) := \mathcal{B}(\mathbb{E},\mathbb{E})$ which is an example of Banach algebra, a notion we briefly recall. Let us first give a useful example of bounded operator.

**Example 9** *The left shift operator* $L : \ell_2 \to \ell_2$ *on the set* $\ell_2 := \{(u_n), \quad \sum_{n=0}^{\infty} |u_n|^2 < \infty\}$ *convergent sequences of complex numbers defined by*

$$L((u_n)) := (v_n), \quad v_n = u_{n+1}$$

*is a bounded linear operator.*

Let $= \mathbb{R}$ or $= \mathbb{C}$. A **-algebra** $\mathcal{A}$ is a -vector space equipped with a bilinear map:

$$\begin{array}{rcl} \mathcal{A} \times \mathcal{A} & \to & \mathcal{A} \\ (a,b) & \to & a\,b \end{array}$$

such that $a(bc) = (ab)c$. It is a **normed algebra** whenever it can be equipped with a map $\|\cdot\| : A \to \mathbb{R}_{\geq 0}$ with the following **submultiplicativity** property

$$\|ab\| \leq \|a\| \cdot \|b\| \quad \forall (a,b) \in \mathcal{A}^2.$$

It is a **unital algebra** whenever it admits a unit 1, i.e. $1\,a = a\,1 = a \quad \forall a \in \mathcal{A}$.
A **Banach algebra** is a complete normed algebra.

A $C^*$**-algebra** is a Banach $\mathbb{C}$-algebra $\mathcal{A}$ [1] equipped with a map $* : \mathcal{A} \to \mathcal{A}$ fulfilling the following properties:

1. (**conjugate linearity**) $(a+b)^* = a^* + b^*; \quad (\lambda\, a)^* = \bar{\lambda}\, a^*$ for any $\lambda \in \mathbb{C}$ and any $(a,b) \in \mathcal{A}^2$;
2. (**involution**) $(a^*)^* = a$ for any $a \in \mathcal{A}$;
3. (**compatibility with the product**) $(a\,b)^* = b^*\,a^*$ for any $(a,b) \in \mathcal{A}^2$,

and such that

$$(C^* - \textbf{identity})\|a^*a\| = \|a\|^2 \quad \forall a \in A. \tag{12}$$

[1] The real case $= \mathbb{R}$ requires a special treatment; $*$ is the identity operator so that only the relation $\|a^2\| = \|a\|^2$, one needs an extra assumption, namely $1 + a^*a$ is invertible in $\mathcal{A}$.

**Example 10** *The space $\mathcal{C}_b(U)$ of bounded continuous functions on an open subset $U$ of $I\!R^n$ equipped with $* : f \longmapsto \overline{f}$ and the supremum norm $\|f|vert = \sup_{x\in U}|f(x)|$, is a $C^*$-algebra.*

**Remark 12** *Let $\mathcal{A}$ be a $C^*$-algebra. We have*

1. *(**compatibility with the norm**) $\|a^*\| = \|a\| \quad \forall a \in \mathcal{A}$. Indeed, using the submultiplicativity, we have*

$$\|a\|^2 = |a^*\, a\| \leq \|a\|\, \|a^*\| \Longrightarrow \|a\| \leq \|a^*\| \leq \|(a^*)^*\| = \|a\| \Longrightarrow \|a*\| = \|a\|.$$

2. *$\|a^*\, a\| = \|a\|\, \|a^*\| \forall a \in \mathcal{A}$. Indeed, the $C^*$- identity reads $\|a^*\, a\| = \|a\|^2 = \|a\|\, \|a^*\|$ as a consequence of the first item.*

There is another characterisation of bounded operators on Hilbert spaces using the Hermitian products. Letting $\mathbb{H}_1, \mathbb{H}_2$ be two Hilbert spaces equipped with the Hermitian products $\langle\cdot,\cdot\rangle_1, \langle\cdot,\cdot\rangle_2$, $A$ an operator in $\mathcal{B}(\mathbb{H}_1, \mathbb{H}_2)$, for any $u \in \mathbb{H}_1, v \in \mathbb{H}_2$ we have $| < Au, v >_2 | \leq |\|A\||\, \|u\|_1 \|v\|_2$ by the Cauchy-Schwartz inequality so that the following inequality holds

$$|\|A\|| \geq \sup_{\|u\|_1 = \|v\|_2 = 1} |\langle Au, v\rangle_2|.$$

To prove the equality, we observe that by the very definition of $|\|A\||$, there is a sequence $(u_n)$ in $\mathbb{H}_1$ such that $\|u_n\| = 1$ and $(\|Au_n\|)$ converges to $|\|A\||$. If $A$ is not identically zero, the sequence $(Au_n)$ in $\mathbb{H}_2$ does not identically vanish so that (extracting a subsequence if necessary) we can assume that $Au_n \neq 0$ for any integer $n$. The sequence $(v_n) := \left(\frac{Au_n}{\|Au_n\|_2}\right)$, whose general term has norm 1 is such that $|\langle Au_n, v_n\rangle_2| = \|Au_n\|_2$ tends to $|\|A\||$ so that

$$|\|A\|| = \sup_{\|u\|_1 = \|v\|_2 = 1} |\langle Au, v\rangle_2|. \tag{13}$$

## 5.2 Self-adjoint bounded operators

Given two Hilbert spaces $(\mathbb{H}_1, \langle\cdot,\cdot\rangle_1)$ and $(\mathbb{H}_2, \langle\cdot,\cdot\rangle_2)$ and an operator $A \in \mathcal{B}(\mathbb{H}_1, \mathbb{H}_2)$, by the Riesz Lemma the relation

$$\langle Au, v\rangle_2 := \langle u, A^* v\rangle_1 \quad \forall u \in \mathbb{H}_1, v \in \mathbb{H}_2$$

uniquely defines an operator $A^* \in \mathcal{B}(\mathbb{H}_2, \mathbb{H}_1)$ called the *adjoint* of $A$.

**Example 11** *In Example 12, it is easy to check that the adjoint $L^*$ of the left shift operator corresponds to the right shift operator*

$$\begin{aligned} R : l^2 &\to l^2 \\ (u_n) &\mapsto R((u_n)) = (u_{n-1}) \end{aligned}$$

*where we have set $u_{-1} = 0$ by convention. Hence we have $LL^* = Id$ but $L^* L \neq Id.$.*

# 6 Bounded and unbounded linear operators

## 6.1 Bounded linear operators

Useful references are [B], [Bre], [Mu], [KR], [RLL], [RS], [Ru], [We].

Let $\mathbb{E}$ and $\mathbb{F}$ be two complex Banach spaces with norm $\|\cdot\|_{\mathbb{E}}$ and $\|\cdot\|_{\mathbb{F}}$ respectively.

**Definition 55** *The space of bounded linear operators from $\mathbb{E}$ to $\mathbb{F}$*

$$\mathcal{B}(\mathbb{E},\mathbb{F}) := \{A : \mathbb{E} \to \mathbb{F}, \quad \exists C > 0, \|Au\|_{\mathbb{F}} \leq C\|u\|_E \quad \forall u \in \mathbb{E}\}$$

*equipped with the norm $|||A||| \equiv sup_{u\in\mathbb{E},x\neq 0} \frac{\|Au\|_{\mathbb{F}}}{\|u\|_{\mathbb{E}}}$ is a Banach space.*

When $\mathbb{F} = \mathbb{E}$ we set $\mathcal{B}(\mathbb{E}) := \mathcal{B}(\mathbb{E},\mathbb{E})$ which is an example of Banach algebra, a notion we briefly recall. Let us first give a useful example of bounded operator.

**Example 12** *The left shift operator $L : \ell_2 \to \ell_2$ on the set $\ell_2 := \{(u_n), \quad \sum_{n=0}^{\infty} |u_n|^2 < \infty\}$ convergent sequences of complex numbers defined by*

$$L((u_n)) := (v_n), \quad v_n = u_{n+1}$$

*is a bounded linear operator.*

An *algebra* $\mathcal{A}$ is a vector space equipped with a bilinear map:

$$\begin{array}{rcl} A \times A & \to & A \\ (a,b) & \to & a\,b \end{array}$$

such that $a(bc) = (ab)c$. It is a *normed algebra* whenever it can be equipped with a submultiplicative norm $\|\cdot\|$ with the following property:

$$\|ab\| \leq \|a\| \cdot \|b\| \quad \forall (a,b) \in \mathcal{A}^2.$$

It is a *unital algebra* whenever it admits a unit 1, i.e. $1\,a = a\,1 = a \quad \forall a \in \mathcal{A}$.
A *Banach algebra* is a complete normed $\mathbb{C}$-algebra.

A $C^*$*-algebra* is a Banach $\mathbb{C}$-algebra $\mathcal{A}$ [2] equipped with an involution $* : \mathcal{A} \to \mathcal{A}$ (i.e. $*$ is a linear map satisfying $*^2 = I$) such that $\|a^*a\| = \|a\|^2$ for any $a \in \mathcal{A}$. Consequently, the involution is isometric, i.e. $\|a^*\| = \|a\| \quad \forall a \in \mathcal{A}$.

There is another characterisation of bounded operators on Hilbert spaces using the Hermitian products. Letting $\mathbb{H}_1, \mathbb{H}_2$ be two Hilbert spaces equipped with the Hermitian products $\langle\cdot,\cdot\rangle_1, \langle\cdot,\cdot\rangle_2$, $A$ an operator in $\mathcal{B}(\mathbb{H}_1,\mathbb{H}_2)$, for any $u \in \mathbb{H}_1, v \in \mathbb{H}_2$ we have $| < Au, v >_2 | \leq |||A|||\,\|u\|_1\|v\|_2$ by the Cauchy-Schwartz inequality so that the following inequality holds

$$|||A||| \geq \sup_{\|u\|_1=\|v\|_2=1} |\langle Au, v\rangle_2|.$$

[2] The real case requires a special treatment.

To prove the equality, we observe that by the very definition of $|||A|||$, there is a sequence $(u_n)$ in $\mathbb{H}_1$ such that $\|u_n\| = 1$ and $(\|Au_n\|)$ converges to $|||A|||$. If $A$ is not identically zero, the sequence $(Au_n)$ in $\mathbb{H}_2$ does not identically vanish so that (extracting a subsequence if necessary) we can assume that $Au_n \neq 0$ for any integer $n$. The sequence $(v_n) := \left(\frac{Au_n}{\|Au_n\|_2}\right)$, whose general term has norm 1 is such that $|\langle Au_n, v_n\rangle_2| = \|Au_n\|_2$ tends to $|||A|||$ so that

$$|||A||| = \sup_{\|u\|_1=\|v\|_2=1}|\langle Au, v\rangle_2|. \tag{14}$$

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Given two Hilbert spaces $(\mathbb{H}_1, \langle\cdot,\cdot\rangle_1)$ and $(\mathbb{H}_2, \langle\cdot,\cdot\rangle_2)$ and an operator $A \in \mathcal{B}(\mathbb{H}_1, \mathbb{H}_2)$, by the Riesz Lemma the relation

$$\langle Au, v\rangle_2 := \langle u, A^*v\rangle_1 \quad \forall u \in \mathbb{H}_1, v \in \mathbb{H}_2$$

uniquely defines an operator $A^* \in \mathcal{B}(\mathbb{H}_2, \mathbb{H}_1)$ called the *adjoint* of $A$.

**Example 13** *In Example 12, it is easy to check that the adjoint $L^*$ of the left shift operator corresponds to the right shift operator*

$$\begin{array}{rcl} R : l^2 & \to & l^2 \\ (u_n) & \mapsto & R((u_n)) = (u_{n-1}) \end{array}$$

*where we have set $u_{-1} = 0$ by convention. Hence we have $LL^* = Id$ but $L^*L \neq Id$..*

**Definition 56** *Given a Hilbert space $H$, an operator $A \in \mathcal{B}(H)$ is self-adjoint if $A = A^*$.*

Here is another characterisation (14) of the norm of a bounded operator when the operator is self-adjoint.

**Lemma 5** *Let $H$ be a Hilbert space equipped with a scalar product $\langle\cdot,\cdot\rangle$ and $A \in \mathcal{B}(H)$ a self-adjoint operator then*

$$|||A||| = \sup_{\|u\|=\|v\|=1}|\langle u, Av\rangle| = \sup_{\|u\|=1}|\langle u, Au\rangle|.$$

**Proof 13** *Taking $v = u$ in (14) we infer that*

$$|||A||| = \sup_{\|u\|=\|v\|=1}|\langle u, Av\rangle| \geq \sup_{\|u\|=1}|\langle u, Au\rangle|.$$

*Conversely, let $C := \sup_{\|u\|=1}|\langle u, Au\rangle|$ and let us assume that $A$ is self-adjoint. We want to show that $\sup_{\|u\|=\|v\|=1}|\langle u, Av\rangle| \leq C$. From*

$$\langle u+v, A(u+v)\rangle - \langle u-v, A(u-v)\rangle = 4\,\mathrm{Re}\langle u, Av\rangle$$

*we infer that*

$$\|u\| = \|v\| = 1 \Longrightarrow |\mathrm{Re}\langle u, Av\rangle| \leq \frac{C}{4}\left(\|u+v\|^2 + \|u-v\|^2\right) \leq \frac{C}{2}\left(\|u\|^2 + \|v\|^2\right) = C.$$

*Choosing $\theta$ such that the image $R_\theta u$ of $u$ under the rotation of angle $\theta$ gives rise to a real number $\langle R_\theta u, Av\rangle$, and applying the above inequality to $R_\theta u$ instead of $u$ we get*

$$\|u\| = \|v\| = 1 \Longrightarrow |\langle u, Av\rangle| \leq C$$

*and consequently the identity*

$$|||A||| = \sup_{\|u\|=\|v\|=1}|\langle u, Av\rangle| = \sup_{\|u\|=1}|\langle u, Au\rangle|.$$

Here are a few useful spectral properties of self-adjoint operators.

**Lemma 6** *Let $H$ be a Hilbert space and $A \in \mathcal{B}(H)$ be a self-adjoint operator. The following properties hold:*

1. *The eigenvalues of $A$ are real.*
2. *The eigenspaces of $A$ associated to different eigenvalues are orthogonal.*
3. *Any non-zero vector $u$ such that*

$$\frac{|\langle Au, u\rangle|}{\|u\|^2} = |||A|||$$

*is an eigenvector of $A$ with eigenvalue in $\{-|||A|||, |||A|||\}$.*

**Proof 14** 1. *Let $\lambda \in \mathbb{C}$ be an eigenvalue of $A$ and $u$ an associated eigenvector, then $< Au, u >= \lambda < u, u >=< u, Au >= \bar{\lambda} < u, u >$. But since $\|u\| \neq 0$, it follows that $\bar{\lambda} = \lambda$ and $\lambda \in \mathbb{R}$.*

2. *Given two eigenvalues $\lambda_i, i = 1, 2$, we have*

$$\lambda_1 \neq \lambda_2 \Rightarrow \mathrm{Ker}(A - \lambda_1 I) \perp \mathrm{Ker}(A - \lambda_2 I).$$

*Indeed, if $\lambda_1 \neq \lambda_2$ are two eigenvalues associated with the eigenvectors $u_1$ and $u_2$, then $< Au_1, u_2 >= \lambda_1 < u_1, u_2 >=< u_1, Au_2 >= \lambda_2 < u_1, u_2 >$, from which it follows that $< u_1, u_2 >= 0$.*

3. *By the Cauchy-Schwartz inequality, we have*

$$|||A||| = \frac{|\langle Au, u\rangle|}{\|u\|^2} \leq \frac{\|Au\|\,\|u\|}{\|u\|^2} \leq |||A|||$$

*and in particular*

$$|\langle Au, u\rangle| = \|Au\|\,\|u\|.$$

*This leads to the existence of a complex constant $\lambda$ such that $Au = \lambda u$ and $|\lambda| = |||A|||$. The fact that this eigenvalue of $A$ is real then follows from the first item so that $\lambda \in \{-|||A|||, |||A|||\}$.*

**Proposition 15** *Given a Hilbert space $\mathbb{H}$, the algebra $\mathcal{B}(\mathbb{H})$ equipped with the adjoint map $A \mapsto A^*$ is a $C^*$-algebra.*

**Proof 15** *We shall take for granted the fact that $\mathcal{B}(\mathbb{H})$ defines a Banach space for the topology induced by the operator norm.*

1. *The product of two operators $A, B$ in $\mathcal{B}(\mathbb{H})$ satisfies the inequality*

$$|||B\,A||| \leq |||A|||\,|||B|||.$$

2. *The product of two bounded operators is bounded as a consequence of the above inequality and $\mathcal{B}(\mathbb{H})$ is a unital algebra since it contains the identity operator.*

*3. Since*

$$\begin{aligned} |||A||| &= \sup_{\|u\|_1=\|v\|_2=1}|\langle Au, v\rangle_2| \\ &= \sup_{\|u\|_1=\|v\|_2=1}|\langle u, A^*v\rangle_2| \\ &= \sup_{\|u\|_1=\|v\|_2=1}|\langle A^*v, u\rangle_2| \\ &= |||A^*|||, \end{aligned}$$

*the map $A \longmapsto A^*$ preserves the boundedness and defines an involution on $\mathcal{B}(\mathbb{H})$.*

*4. Using Lemma* **??**, *we have*

$$\begin{aligned} |||A^* A||| &= \sup_{\|u\|=1}|\langle A^* Au, u\rangle| \\ &= \sup_{\|u\|=1}|\langle Au, Au\rangle| \\ &= |||A|||^2. \end{aligned}$$

The **Gelfand-Naimark** theorem says that every abstract (resp. commutative) unital $C^*$-algebra is isometrically $*$-isomorphic to a $\|\cdot\|$-closed $*$-subalgebra of $\mathcal{B}(H)$ (resp. to the space $C(K)$ of continuous functions on some compact Hausdorff space $K$) for some Hilbert space $H$. To prove the statement in the general (not necessarily commutative) case, one uses the *Gelfand-Naimark-Segal* or *GNS* construction which produces a representation from a state. To a state $\rho$ on a $C^*$-algebra $\mathcal{A}$, i.e. a positive linear functional $\rho : \mathcal{A} \to \mathbb{C}$ ($\rho(a^*a) \geq 0 \forall a \in \mathcal{A}$), one can associate a positive semi-definite bilinear form $\langle a, b\rangle = \rho(b^*a)$ with kernel $\mathcal{N}_\rho = \{a \in \mathcal{A}, \rho(a^*a) = 0\}$ which is a subvector space of $\mathcal{A}$ and a left ideal in $\mathcal{A}$. This bilinear form therefore induces a positive definite form $\langle\cdot,\cdot\rangle$ on $A/\mathcal{N}_\rho$ and hence a pre-Hilbert space structure on that quotient space, which by completion gives rise to a Hilbert space $\mathbb{H}_\rho$. The left regular representation

$$\begin{aligned} \pi_\rho : \mathcal{A} &\to \mathcal{B}(\mathbb{H}_\rho) \\ b &\mapsto (a \mapsto ba) \end{aligned}$$

is cyclic with cyclic vector $x_\rho := 1_\mathcal{A} + \mathcal{N}_\rho$ and $\rho(a) = \langle \pi_\rho(a)x_\rho, x_\rho\rangle$ for any $a \in \mathcal{A}$.

## 6.3 Closed graph theorem

Useful references are [Br], [Ru].

The operators one comes across in geometry or in physics usually are unbounded and only defined on a dense domain of the Banach space.

**Definition 57** *Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be two Banach space. The graph of an operator*

$$A : D(A) \subset E \to F$$

*defined on a domain $D(A)$ is the set:*

$$Gr(A) := \{(u, Au) \in E \times F, u \in D(A)\}.$$

It can be equipped with the (graph) norm

$$\|(u,v)\| := \|u\|_E + \|v\|_F.$$

Notice that whenever $A$ is invertible and its inverse bounded, then the graph of $A^{-1}$ is the symmetric of the graph of $A$ w.r.to the diagonal axis.

**Definition 58** *The operator $A$ is closed whenever its graph is closed for the graph norm.*

When $E$ and $F$ are separable, there is another characterisation for closed operators.

**Lemma 7** *An operator $A : D(A) \subset E \to F$ is* **closed** *if, given any sequence $(u_n)$ converging to $u \in E$ such that $Au_n$ converges in $F$, then the limit $u$ lies in the domain $D(A)$ and $Au_n \to Au$.*

**Proof 16** *The graph of $A$ is closed whenever for a sequence $(u_n, Au_n)$ in $\mathrm{Gr}(A)$ which converges in $E \times F$ to $(u, v)$ we have that $u \in D(A)$ and $v = Au$ i.e. $Au_n$ converges to $v = Au$, which corresponds to the characterisation in the lemma.*

*We shall henceforth assume that the Banach spaces under consideration are separable.*

**Lemma 8** *Any bounded linear operator defined on a closed domain is closed.*
*Furthermore, a closed linear operator $A : D(A) \subset \mathbb{E} \to \mathbb{F}$ defined on a dense domain $D(A)$ of $\mathbb{E}$ extends in an unique way to a bounded operator on $E$ whenever there is a constant $C > 0$ such that $\|Au\|_{\mathbb{F}} \leq C\|u\|_{\mathbb{E}}, \forall u \in D(A)$.*

**Example 14** *Let $\mathbb{F} = C([0,1])$ equipped with the norm $\|f\|_\infty = \sup_{x\in[0,1]}|f(x)|$ and $\mathbb{E} = C^1([0,1])$ equipped with the norm $\|f\|_{\infty,1} := \|f\|_\infty + \|f'\|_\infty$. The operator $A : f \mapsto f'$ is defined on $D(A) = C^\infty([0,1]$ which is dense in $\mathbb{E}$ and we have $\|A(f)\|_\infty \leq \|f\|_{\infty,1}$ for any $f \in D(A)$. This yields back the well-known fact that $A$ extends to a bounded linear operator $A : \mathbb{E} \to \mathbb{F}$.*

**Proof 17**

- *Let $A$ be a bounded linear operator on a closed domain $D(A)$. A sequence $(u_n, Au_n)$ in $\mathrm{Gr}(A)$ which converges in $\mathbb{E} \times \mathbb{F}$, therefore converges to $(u, v) \in D(A) \times \mathbb{F}$. It follows from the continuity of $A$ that $v = \lim Au_n$ so that the sequence $(u_n, Au_n)$ converges in $\mathrm{Gr}(A)$.*
- *To prove the second one, all we need is to define the image of any element $u \in \mathbb{E}$ by an extension of $A$. Since $D(A)$ is dense in $\mathbb{E}$, $u$ can be seen as a limit $u = \lim_{n\to\infty} u_n$ of a sequence $(u_n)$ in $D(A)$. Since $(u_n)$ is convergent it is a Cauchy sequence, and hence so is the sequence $(Au_n)$ a Cauchy sequence so that it converges to some $v \in \mathbb{F}$ since $\mathbb{F}$ is complete. The operator $A$ being closed, this implies that $u$ lies in the domain $D(A)$ and $Au = v$. This extended operator (also denoted by $A$) is clearly a bounded operator.*
  *Moreover this extension does not depend on the choice of the sequence. For if $(u'_n)$ is another sequence tending to $u$, from the inequality $\|Au_n - Au'_n\| \leq C\|u_n - u'_n\|$, it follows that $Au'_n \to Au$.*

**Proposition 16 (Open mapping Theorem)** *A surjective bounded linear operator $A : \mathbb{E} \to \mathbb{F}$ between two Banach spaces is open i.e., it sends open subsets to open subsets.*
*Consequently, if it is invertible, its inverse is continuous.*

**Proof 18** *It relies on the following result which we take for granted. Under the assumptions of the proposition, the image* $A(B_{\mathbb{E}}(0,r))$ *of an open non void ball in* $E$ *contains a non void ball* $B_{\mathbb{F}}(0,\varepsilon)$ *in* $\mathbb{F}$.
*Assuming this, we want to show that the range* $A(U)$ *of an open subset* $U$ *in* $\mathbb{E}$ *is open i.e., that any* $y = A(x)$ *in* $A(U)$ *is the center of a ball* $B_{\mathbb{F}}(y,\varepsilon)$ *in* $A(U)$.
*The subset* $U - x$ *of* $\mathbb{E}$ *is open and contains 0 so it contains an open non void ball* $B_E(0,r)$. *Thus,* $A(U - x) = A(U) - y$ *which contains* $A(B_{\mathbb{E}}(0,r))$ *contains a non void ball* $B_{\mathbb{F}}(0,\varepsilon)$. *This implies that* $A(U)$ *contains the non void open ball* $B_{\mathbb{F}}(y,\varepsilon) = B_{\mathbb{F}}(0,\varepsilon) + y$.

We have shown that a bounded linear operator is closed; the closed graph theorem provides a converse statement.

**Theorem 7 (Closed graph Theorem)** *Let* $A : \mathbb{E} \to \mathbb{F}$ *be a closed linear operator with closed domain* $D(A)$. *In particular, a closed linear operator with domain* $\mathbb{E}$ *lies in* $A \in \mathcal{B}(\mathbb{E},\mathbb{F})$, *i.e.* $A$ *is bounded on* $\mathbb{E}$.

**Proof 19**

- *Since* $\mathbb{E}$ *and* $\mathbb{F}$ *are complete, the cartesian product* $\mathbb{E}\times\mathbb{E}$ *is complete. Since* $D(A)$ *is closed, its graph* $\mathrm{Gr}(A)$ *is closed in* $\mathbb{E}\times\mathbb{E}$ *and hence complete.*
- *The map* $P : \mathrm{Gr}(A) \to D(A)$ *defined as* $P(x,y) = x$ *is linear and bounded since* $\|u\|_{\mathbb{E}} \leq \|(u,Au)\|_{\mathrm{Gr(A)}}$. *It is a bijection with inverse* $x \longmapsto (x, Ax)$. *The open mapping theorem implies that its inverse is continuous and hence* $\|u\|_{\mathbb{E}} + \|Au\|_{\mathbb{F}} = \|P^{-1}(u)\|_{\mathrm{Gr(A)}} \leq C\,\|u\|_{\mathbb{E}}$ *for some positive constant* $C$ *(chosen large enough). Thus* $\|Au\|_{\mathbb{F}} \leq (C-1)\,\|u\|_{\mathbb{E}}$ *and* $A$ *is bounded.*

*In the following, we assume the operators are closed and defined on a dense domain* $D(A)$.

**Proposition 17** *The inverse of a bijective linear operator* $A : D(A) \subset \mathbb{E} \to \mathbb{F}$ *is a bounded operator* $A^{-1} : \mathbb{F} \to \mathbb{E}$.

**Proof 20** *Since the graph of* $A$ *is closed so is the graph of* $A^{-1}$ *and the result follows from the closed graph theorem.*

**Definition 59** *The resolvent of* $A$ *is the set*

$$\begin{aligned} \rho(A) &:= \{\lambda \in \mathbb{C}, A - \lambda I \quad \textit{is bijective}\ \} \\ &= \{\lambda \in \mathbb{C}, A - \lambda I \quad \textit{is bijective and} \quad (A-\lambda I)^{-1} \in \mathcal{B}(H)\}. \end{aligned}$$

*The spectrum* $\sigma(A)$ *of* $A$ *is the complement of the resolvent:*

$$\sigma(A) := \mathbb{C}/\rho(A).$$

*The point spectrum* $\sigma_p(A)$ *is the set*

$$\sigma_p(A) := \{\lambda \in \mathbb{C}, Ker(A - \lambda I) \neq \{0\}\}.$$

In finite dimensions, $\sigma_p(A) = \sigma(A)$, which is not generally the case in infinite dimensions.

**Example 15** *Letting* $H = l^2$ *be the space of* $l^2$ *convergent sequences and let* $A : l^2 \to l^2$ *be the operator sending a sequence* $(u_n)$ *to a sequence* $(v_n)$ *with* $v_0 = 0, v_i = u_{i-1}$. *0 does not belong to* $\sigma_p(A)$ *but* $0 \in \sigma(A)$.

## 6.4 Adjoint of an (unbounded) operator

Useful references are [Br], [RS].

Let $\mathbb{H}_1$ and $\mathbb{H}_2$ be Hilbert spaces equipped with the Hermitian products $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$ respectively. We extend here the notion of adjoint of an operator to unbounded operators.

**Definition 60** *The adjoint of an operator $A : D(A) \subset \mathbb{H}_1 \to \mathbb{H}_2$ defined on a dense domain $D(A)$ is an operator $A^*$ defined on*

$$D(A^*) := \{v \in \mathbb{H}_2, \exists C(v) > 0, | < Au, v >_2 | \leq C(v)\|u\|_1 \quad \forall u \in D(A)\}$$

*by $< Au, v >_2 = < u, A^* v >_1$ $\forall u \in D(A), v \in D(A^*)$.*

This defines $A^*$ in an unique way. For if $v_1^*$ et $v_2^*$ are elements in $\mathbb{H}_1$ such that $< Au, v >_2 = < u, v_i^* >_1, i = 1, 2$ for any $u \in D(A)$, then $0 = < u, v_1^* - v_2^* >_1$ for any $u \in D(A)$ dense in $E$, which implies $v_1^* = v_2^*$.

**Remark 13** *Provided $R(A) \subset D(A^*)$ we have* $\mathrm{Ker}(A^*A) = \mathrm{Ker}(A)$. *Indeed, clearly the inclusion* $\mathrm{Ker}(A) \subset \mathrm{Ker}(A^*A)$ *holds. Conversely we have*

$$A^*Au = 0 \Longrightarrow \langle A^*Au, u \rangle = 0 \Longrightarrow \|Au\| = 0 \Longrightarrow Au = 0,$$

*so that* $\mathrm{Ker}(A^*A) \subset \mathrm{Ker}(A)$.

Whenever $E = F = H$ is a Hilbert space equipped with the inner product $\langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_E = \langle \cdot, \cdot \rangle_F$, we say $A$ is *self-adjoint* if $A = A^*$, i.e

$$D(A) = D(A^*) \quad \text{and} \quad < Au, v > = < u, Av > \quad \forall u \in D(A).$$

The graph of the adjoint $A^*$ of an operator $A$ is given by

$$(Gr'(-A^*))^\perp = \overline{Gr(A)}$$

where "prime" means the symmetric set w.r.to the diagonal axis. This is an easy consequence of the fact that

$$< (u, Au), (-A^*v, v) > = < u, -A^*v > + < Au, v > = 0$$

for $u \in D(A)$, $v \in D(A^*)$.

**Proposition 18** *The domain $D(A^*)$ of the adjoint $A^*$ of an operator $A : D(A) \subset \mathbb{H}_1 \to \mathbb{H}_2$ is dense in $\mathbb{H}_1$, and the adjoint is also closed.*
*(Recall that as before, $D(A)$ is dense in $\mathbb{H}_1$ and that $A$ is closed.)*

**Proof 21** *Let us assume that the domain of $A^*$ is not dense in $\mathbb{H}_2$. Then there exists a non zero vector $u \in \mathbb{H}_2$ such that $< u, v >_2 = 0$ for any $v \in D(A^*)$. Using the closedness of $A$, this would imply that*

$$\begin{aligned} < u, v >_2 = 0 \quad &\Leftrightarrow \quad < u, v >_2 + < 0, -A^*v >_1 = 0 \\ &\Leftrightarrow \quad (0, u) \in (Gr'(-A^*))^\perp = Gr(A) \end{aligned}$$

*and hence $A(0) = u \neq 0$ which contradicts the linearity of $A$.*
*Let us check that $A^*$ is closed. Let $(v_n, A^* v_n)$ be a Cauchy sequence in $D(A^*) \times \mathbb{H}_1$ converging to $(y, x) \in \mathbb{H}_2 \times \mathbb{H}_1$. For any $u \in D(A)$, we have*

$$< Au, y >_2 = \lim_{n \to \infty} < Au, v_n >_2 = \lim_{n \to \infty} < u, A^* v_n >_1 = < u, x >_1$$

*so that $| < Au, y >_2 | \leq \|x\|_1 \cdot \|u\|_1$ for any $v \in D(A)$, which implies $y \in D(A^*)$ and $A^* y = u$.*

Let $A : D(A) \subset H \to H$. When the space $\mathbb{H}_p(A)$ spanned by the eigenvectors of $A$ coincides with the total Hilbert space $H$, letting $\{e_n, n \in \mathbb{N}\}$ be an orthonormal basis of eigenvectors of $H$, the operator has the following *discrete resolution*:

$$Au = \sum_n \lambda_n < u, e_n > \quad \forall u \in D(A).$$

Given a map $f : \sigma_p(A) \subset \mathbb{C} \to \mathbb{C}$, using this spectral representation of $A$ we can define the map $f(A)$ on the domain:

$$D(f(A)) := \{u \in H, \sum_n f(\lambda_n)^2 < u, e_n >^2 < \infty\}$$

by

$$f(A)u = \sum_n f(\lambda_n) < u, e_n > e_n \quad \text{for} \quad u \in D(f(A)).$$

**Example 16** *When $A$ has a discrete resolution with purely discrete spectrum and positive eigenvalues outside a discrete set of eigenvalues, the function $f(x) = e^{-tx}$ defines a bounded operator $e^{-tA}$ called the heat-operator associated to $A$. Heat-operators will be investigated more thoroughly in the sequel.*

Let $A : D(A) \subset \mathbb{H}_1 \to \mathbb{H}_2$ be a closed operator defined on a dense domain $D(A)$ of a Hilbert space $(\mathbb{H}_1, \langle \cdot, \cdot \rangle_1)$ with values in $(\mathbb{H}_2, \langle \cdot, \cdot \rangle_2)$, then the domain $D(A^*)$ is dense in $\mathbb{H}_2$ and we can define the adjoint $A^{**}$ of $A^*$ which coincides with $A$.

Let us first check that $A \subset A^{**}$. Given $x \in D(A)$, $| < x, A^* y > | = | < Ax, y > | \leq \|Ax\| \|y\|$ for any $y \in D(A^*)$ and hence $x \in D(A^{**})$. Since $< x, A^* y > = < Ax, y >$ the operators $A$ and $A^{**}$ coincide on $D(A)$. Since $A^{**}$ is closed as the adjoint of a closed operator, so is its graph and we have $Gr((A^*)^*) = (Gr'(-A^*))^\perp = Gr(A)$, for $Gr(A)$ is closed. This ends the proof of the identity $A^{**} = A$.

Another useful property for $A : D(A) \subset \mathbb{H}_1 \to \mathbb{H}_2$ is

$$R^\perp(A) = Ker(A^*)$$

and hence

$$Ker(A^*) \oplus \overline{R(A)} = \mathbb{H}_2,$$

which applied to $A^*$ yields:

$$R^\perp(A^*) = Ker(A)$$

and hence

$$Ker(A) \oplus \overline{R(A^*)} = \mathbb{H}_1.$$

Indeed, we have $R^{\perp}(A) \subset Ker A^*$. For if $< y, Ax >= 0 \; \forall x \in D(A)$ then $y \in D(A^*)$ and $< A^*y, x >= 0 \; \forall x \in D(A)$. But $D(A)$ is dense in $H$ so it follows that $y \in Ker A^*$. Let us now check the other inclusion. Given $x \in Ker A^*$, we have $< A^*x, y >= 0$ for any $y \in D(A)$ and hence $< x, A^{**}y >=< x, Ay >= 0$ for any $y \in D(A)$ which shows that $x \in R(A)^{\perp}$.

# 7 Compact and Fredholm operators

## 7.1 Compact operators

Useful references are [Bre], [HL], [Mu]. We also used some notes by B. Driver [Dr].

We first recall equivalent characterisations of compactness.
Let $(\mathbb{E}, d)$ be a metric space. A subset $S$ of $\mathbb{E}$ is compact if it satisfies one of the following equivalent (Theorem of Heine Borel) assertions:

1. Any (infinite) sequence of points in $S$ has a converging subsequence in $S$. In other words any (infinite) sequence of points in $S$ has an accumulation point in $S$.

2. From any open covering of $S$ we can extract a finite open covering.

Let us also recall that a compact subset of a metric space can be covered by a finite number of balls of any given positive radius.
Any compact set is closed and bounded but if a normed space $\mathbb{E}$ contains a closed and bounded subset which is compact, then it is finite dimensional.

**Definition 61** *A relatively compact subset of a metric space $\mathbb{E}$ is a subset whose closure is compact. Equivalently, it is a set in which any infinite sequence of points has a converging subsequence (whose limit does not necessarily lie in the set).*

**Example 17** *The open unit ball $B(0,1)$ in $(\mathbb{R}^n, \|\cdot\|_\infty)$ is relatively compact.*

Let $\mathbb{E}, \mathbb{F}$ be two Banach spaces.

**Proposition 19** *For an operator $A \in \mathcal{B}(\mathbb{E}, \mathbb{F})$, the following conditions are equivalent*

1. *the range $A(B_{\mathbb{E}}(0,1))$ of the unit ball $B_{\mathbb{E}(0,1)}$ of $\mathbb{E}$ is relatively compact.*

2. *given any bounded sequence $(x_n)$ in $\mathbb{E}$, one can extract from the sequence $(Ax_n)$ a convergent sequence in $\mathbb{F}$.*

*Such an operator $A \in \mathcal{B}(\mathbb{E}, \mathbb{F})$ is called compact.*

**Proof 22** *The equivalence between the two properties follows from the equivalence of the corresponding characterisations of compactness.*

**Coexample 1** *The identity operator on $\mathbb{E}$ is compact if $\overline{B_{\mathbb{E}}(0,1)}$ is compact which implies that $\mathbb{E}$ is finite dimensional.*

Sobolev inclusions on closed manifolds (see e.g. [Ad2], [Gi]) give rise to compact operators.
Given a closed manifold $M$ and a vector bundle $E$ based on $M$, using a partition of the unity, one can define for any $s \in \mathbb{R}$, the $H^s$ Sobolev closure $\mathbb{E} := H^s(E)$ of the space $C^\infty(E)$ of smooth sections of $E$ (se e.g. [Gi]).

**Example 18** *For $t < s$, the inclusion $i : H^s(E) \to H^t(E)$ is a compact operator.*

An invertible operator $A : \mathbb{E} \to \mathbb{E}$ is not compact if $\mathbb{E}$ is infinite dimensional. Otherwise, one could extract from any bounded sequence $(u_n)$, a converging subsequence $(Au_{\phi(n)})$ so that $(u_{\phi(n)}) = (A^{-1}Au_{\phi(n)})$ would itself converge, contradicting the finite dimensionality of $\mathbb{E}$.

Finite rank operators (see e.g. [Bre]) form a subclass of compact operators.

**Definition 62** *A bounded operator $A$ in $\mathcal{B}(\mathbb{E}, \mathbb{E})$ has finite rank if its range is finite dimensional. The dimension of the range is called the rank of the operator.*

Finite rank projection operators are finite rank operators.

**Example 19** *Let $(e_k)$ be a complete orthonormal basis of a separable Hilbert space $\mathbb{H}$, the projection operator $P_k$ defined by $P_k e_j \equiv e_j$ if $j \leq k$, $P_k e_j \equiv 0$ otherwise, is a finite rank operator since its range has dimension $k$.*

Any finite rank operator is compact since the closure of the range $A(B(0,1))$ of the unit ball by a finite rank operator $A$ is compact as a closed and bounded subset of a finite dimensional space.

Let $\mathcal{K}(\mathbb{E}, \mathbb{F})$ denote the set of compact operators from $\mathbb{E}$ to $\mathbb{F}$. When $\mathbb{E} = \mathbb{F}$ we set $\mathcal{K}(\mathbb{E}) := \mathcal{K}(\mathbb{E}, \mathbb{E})$.

**Lemma 9** *$\mathcal{K}(\mathbb{E})$ is a two sided ideal in $\mathcal{B}(\mathbb{E})$.*

**Proof 23** *For if $A \in \mathcal{K}(\mathbb{E})$, $B \in \mathcal{B}(E)$ and a bounded sequence $(u_n)$ in $\mathbb{E}$, $(Bu_n)$ is also bounded. $A$ being compact we can extract from $(A\,Bu_n)$ a convergent subsequence $(A\,Bu_{\phi(n)})$ which shows that $AB$ is compact. Similarly we show that the product $B\,A$ is compact. Indeed, $A$ being compact, we can extract a subsequence $(u_{\phi(n)})$ of $(u_n)$ such that $(Au_{\phi(n)})$ converges and $B$ being bounded, $(B\,Au_{\phi(n)})$ therefore converges. This shows that $B\,A$ is compact.*

**Lemma 10** *$\mathcal{K}(\mathbb{E})$ is closed in $\mathcal{B}(\mathbb{E})$ and hence a Banach algebra for the operator norm $\| \cdot \|$.*

**Proof 24** *Let $A_n$ be a sequence of compact operators converging to $A$. To show the compactness of $\overline{A(B(0,1))}$ and hence of $A$, it is sufficient to show we can cover the image ball $A(B(0,1))$ by a finite number of balls with given radius. Let $\varepsilon > 0$ and $n \in I\!N$ such that $|||A_n - A||| < \frac{\varepsilon}{2}$. Since $A_n$ is compact, we can cover $A_n(B(0,1))$ by a finite number $N$ of balls with radius $\varepsilon/2$, $A_n(B(0,1)) \subset \bigcup_{i=1}^N B(h_i, \varepsilon/2)$. This induces a covering of $A(B(0,1))$ by a finite number of balls of radius $\varepsilon$ and ends the proof of the closedness of the set of compact operators.*

**Remark 14** *Since any finite rank operator is compact, and the set of compact operators being closed, the limit (in the operator norm topology) of a sequence of finite rank operators is also compact.*

**Example 20** *Any linear operator*

$$\begin{array}{rcl} A : \ell_2(I\!N) & \longrightarrow & \ell_2(I\!N) \\ (u_n) & \longmapsto & (\lambda_n u_n) \end{array}$$

*with* $(\lambda_n)$ *a sequence of complex numbers converging to zero, is compact. Indeed,* $A = \lim_{N\to\infty} P_N A$ *in the operator norm, where* $P_N$ *denotes the projection onto the* $N$*-dimensional space spanned by* $\{e_1, \cdots, e_N\}$ *with* $e_i$ *the sequence whose entries all vanish outside the* $i$*-th one. This convergence holds due to the convergence of the sequence* $(\lambda_n)$ *for*

$$|||A - P_N A||| = |||A(1-P_N)||| \le \sup_{\{n\ge N\}}|\lambda_n|\, |||(1-P_N)||| \le \sup_{\{n\ge N\}}|\lambda_n| \longrightarrow_{n\to\infty} 0.$$

**Lemma 11** *When* $\mathbb{E} = \mathbb{H}$ *is a Hilbert space,* $\mathcal{K}(\mathbb{H})$ *is a* $*$*-ideal i.e.* $A \in \mathcal{K}(\mathbb{H}) \Rightarrow A^* \in \mathcal{K}(\mathbb{H})$.

**Proof 25** *Indeed, let* $A$ *be compact and let us assume that* $A^*$ *is not compact. Then there is a sequence* $(u_n)$ *in the unit ball* $B(0,1)$ *such that* $\|A^*u_n - A^*u_m\| \ge \varepsilon > 0$ *for any* $n, m \in I\!N$*. Let* $v_n = A^*u_n$*, then*

$$< Av_n - Av_m, u_n - u_m >= \|A^*u_n - A^*u_m\|^2 \ge \varepsilon^2$$

*so that by the Cauchy-Schwartz inequality and using the fact that* $\|u_n\| \le 1$*, we get*

$$\varepsilon^2 \le \|Av_n - Av_m\|\,\|u_n - u_m\| \le 2\|Av_n - Av_m\|$$

*in which case* $(Av_n)$ *would not have a convergent subsequence. This would contradict the compactness of* $A$*.*

We sum up these properties in the following statement.

**Theorem 8** $\mathcal{K}(\mathbb{H})$ *is a* $C^*$*-algebra.*

## 7.2 Spectral decomposition for compact self-adjoint operators

We now investigate spectral properties of self-adjoint compact operators.

**Lemma 12** *Let* $\mathbb{H}$ *be a Hilbert space and* $A \in \mathcal{B}(\mathbb{H})$ *be a self-adjoint compact operator. Either* $|||A|||$ *or* $-|||A|||$ *is an eigenvalue of* $A$*.*

**Proof 26** *If* $A = 0$ *the statement is trivial, so that we can assume that* $A$ *is not identically zero. Since* $|||A||| = \sup_{\|u\|=1}|\langle u, Au\rangle|$ *there is a sequence* $(u_n)$ *in* $H$ *such that* $\|u_n\| = 1$ *and* $|\langle u_n, Au_n\rangle|$ *converges to* $|||A|||$*. One can extract a subsequence* $(u_{\phi(n)})$ *such that* $\langle u_{\phi(n)}, Au_{\phi(n)}\rangle$ *(it is real due to the self-adjointness), converges to some* $\lambda$ *in* $\{-|||A|||, |||A|||\}$*. Using the compactness of* $A$ *one can extract another subsequence* $(u_{\psi(n)})$ *such that* $\left(v_{\psi(n)}\right) := \left(A(u_{\psi(n)})\right)$ *converges. Relabeling the sequences we simply assume that* $(\langle u_n, v_n\rangle)$ *converges. Thus the expression*

$$\|Au_n - \lambda u_n\|^2 = \|Au_n\|^2 - 2\lambda\,\langle Au_n, u_n\rangle + \lambda^2 \le \lambda^2 - 2\lambda\,\langle Au_n, u_n\rangle + \lambda^2$$

*tends to zero as* $n$ *tends to infinity. Hence* $(u_n)$ *converges to* $u := \frac{1}{\lambda}\lim_{n\to\infty} Au_n$*. By the continuity of* $A$ *it follows that* $Au = \lambda u$*.*

We are now ready to give the *spectral theorem* which provides a spectral resolution for compact self-adjoint operators.

**Theorem 9** *A compact self-adjoint operator in a Hilbert space $\mathbb{H}$ has a compact and at most countable (possibly finite) purely discrete real spectrum, with at most one limit point zero. The sequence $(\lambda_n)$ of non-zero eigenvalues can be arranged so that $|\lambda_n| \geq |\lambda_{n+1}|$ tends to zero as n tends to infinity unless A has finite rank, in which case there is a finite set of eigenvalues. Moreover we have the following discrete spectral resolution:*

$$A\,u = \sum_{n=1}^{\infty} \lambda_n \,\langle u, u_n\rangle u_n \quad \forall u \in \mathbb{H},$$

*and the orthonormal sequence $(u_n)$ of eigenvectors associated with $(\lambda_n)$ verifies $H = \overline{\langle u_n, n \in I\!N\rangle} \oplus \mathrm{Ker}(A)$. This spectral resolution is finite if the operator has finite rank $N$ in which case we have $A\,u = \sum_{n=1}^{N} \lambda_n \,\langle u, u_n\rangle u_n \quad \forall u \in \mathbb{H}$.*

**Proof 27** *1. The compactness of the spectrum is a general property of bounded operators; indeed it is closed as the complement of an open set and bounded as a consequence of the boundedness of the operator. The fact that the eigenvalues are real is due to the self-adjointness.*

2. *The non-zero eigenvalues $\lambda_n$ and the eigenvectors $u_n$ are built recursively using the above lemma. Assuming that $A \neq 0$, let $\lambda_1 \in \{-|||A|||, |||A|||\}$ and $u_1$ an associated eigenvector as in the above proposition. Let $\mathbb{H}_1 = \langle u_1\rangle$ be the subspace spanned by $u_1$, then $A(\mathbb{H}_1) \subset \mathbb{H}_1$. We easily check that $A(\mathbb{H}_1^\perp) \subset \mathbb{H}_1^\perp$ and we can therefore restrict $A$ to $\mathbb{H}_1^\perp$: let $A_1$ denote this restriction. If $A_1$ is identically zero, the statement holds. Otherwise $A_1$ is a compact operator so that as above, from $A_1$ we build an operator $A_2$ defined as the restriction of $A_1$ to the orthogonal complement of the space $\mathbb{H}_2 := \langle u_1, u_2\rangle$ where $u_2 \in \mathbb{H}_1^\perp$ with norm 1 is such that $A_1 u_2 = A u_2 = \lambda_2 u_2$ for some $\lambda_2 \in \{-|||A_1|||, |||A_1|||\}$ . We continue this procedure until we reach an operator $A_i$ which vanishes identically, or otherwise we pursue indefinitely. This way, we build a sequence $(\lambda_n)$ of eigenvalues and orthonormal associated eigenvectors $(u_n)$ such that*

$$|\lambda_n| = |||A_n||| = \sup_{u \in \langle u_1, \cdots, u_n\rangle^\perp} \frac{\|Au\|}{\|u\|}.$$

3. *The sequence $(|\lambda_n|)$ is decreasing by construction. Let us show that it converges to zero as n tends to infinity; otherwise there would be a common positive lower bound $\varepsilon$ for all the $|\lambda_n|$, in which case $(v_n) := (u_n\, \lambda_n^{-1})$ would be a sequence bounded by $\varepsilon^{-1}$. The compactness of $A$ would then yield the existence of a subsequence $(v_{\phi(n)})$ such that $(u_{\phi(n)}) = (A\, v_{\phi(n)})$ is convergent, which is impossible since the $(u_n)$ form an orthonormal set. Indeed, were $u_{\phi(n)}$ to converge to some limit $\ell$ then we would have $\langle u_{\phi(n)}, u_{\phi(n+1)}\rangle = 0 \Longrightarrow \|\ell\|^2 = 0 \Longrightarrow \ell = 0$, which contradicts the fact that $u_{\phi(n)}$ has norm 1.*

4. *Since $\{\lambda_n, n \in I\!N\}$ is a subset of the spectrum $\sigma(A)$ which is closed, it follows that 0 lies in $\sigma(A)$ so that $\{\lambda_n, n \in I\!N\} \cup \{0\} \subset \sigma(A)$. We need to show that these two sets actually coincide.*
*Let $\mathbb{H}_\infty := \langle u_n, n \in I\!N\rangle$ be the space spanned by the vectors $u_n, n \in I\!N$. Then*

$$|||A_{|_{\mathbb{H}_\infty^\perp}}||| \leq |||A_{|_{\mathbb{H}_n^\perp}}||| = \sup_{u \in \langle u_1, \cdots, u_n\rangle^\perp} \frac{\|Au\|}{\|u\|} = |\lambda_n| \quad \forall n \in I\!N$$

*vanishes since* $\lambda_n$ *tends to zero as* $n$ *tends to infinity. Hence* $A_{|_{\mathbb{H}_\infty^\perp}} = 0$ *which implies that* $\mathbb{H} = \overline{\langle u_n, n \in I\!N \rangle} \oplus \mathrm{Ker}(A)$.
*Letting* $P_0$ *denote the orthogonal projection onto* $\mathrm{Ker}(A)$ *we write* $u = P_0 u + \sum_{n\in I\!N} \langle u, u_n \rangle u_n$, *which yields* $Au = \sum_{n\in I\!N} \lambda_n \langle u, u_n \rangle u_n$, *since* $A \circ P_0 = 0$.

5. *Using this decomposition we can show that* $\{\lambda_n, n \in I\!N\} \cup \{0\} = \sigma(A)$. *For this, we need to show that any* $\lambda$ *outside the set* $S := \{\lambda_n, n \in I\!N\} \cup \{0\}$ *does not belong to* $\sigma(A)$ *or in other words, that* $(A - \lambda I)^{-1}$ *exists and is bounded.*

6. *For this purpose, we consider the distance* $d$ *between* $\lambda$ *and the set* $S$, *which is positive since* $\sigma(A)$ *is closed and we have* $\lambda - \lambda_n \geq d$ *which in the limit yields* $|\lambda| \geq d$. *We observe that the operator* $A - \lambda I$ *is invertible since for* $u = \sum_{n\in I\!N} \langle u, u_n \rangle u_n + P_0 u \in H$ *we have* $(A - \lambda I)^{-1} u = \sum_{n\in I\!N} (\lambda_n - \lambda)^{-1} \langle u, u_n \rangle u_n - \lambda^{-1} P_0 u$. *Furthermore,*

$$\|(A - \lambda I)^{-1} u\|^2 = \sum_{n\in I\!N} \frac{|\langle u, u_n\rangle|^2}{|\lambda_n - \lambda|^2} + \frac{\|P_0 u\|^2}{|\lambda|^2} \leq \frac{\|u\|^2}{d^2}$$

*so that the operator* $(A - \lambda I)^{-1}$ *is bounded and* $\lambda \notin \sigma(A)$.

The spectral theorem extends to any compact operator.

**Corollary 4** *(see e.g. [Dr], [Ru]) Let* $\mathbb{H}_1$ *and* $\mathbb{H}_2$ *be two Hilbert spaces. A compact operator* $A : \mathbb{H}_1 \to \mathbb{H}_2$ *has a compact and at most countable spectrum, with at most one limit point zero. The sequence* $(\mu_n)$ *of non-zero eigenvectors of* $A^*A$ *is either finite or tends to zero as* $n$ *tends to infinity and we have the following discrete spectral resolution:*

$$A\,u = \sum_{n=1}^{\infty} \sqrt{\mu_n}\, \langle u, u_n \rangle v_n \quad \forall u \in \mathbb{H}_1,$$

*for some orthonormal subsets* $(u_n)$ *(consisting of eigenvectors of* $A^*A$*) in* $\mathbb{H}_1$ *and* $(v_n)$ *in* $\mathbb{H}_2$. *If* $A$ *has finite rank* $N$ *this sum is finite and we have* $Au = \sum_{n=1}^{N} \sqrt{\mu_n}\, \langle u, u_n \rangle v_n \quad \forall u \in \mathbb{H}_1$.
*The numbers* $(\lambda_n) := (\sqrt{\mu_n})$ *are called the* singular values *of* $A$.

**Proof 28** *The proof uses the spectral decomposition of the self-adjoint compact operator* $A^*A$:

$$A^*Au = \sum_{n=1}^{\infty} \mu_n \langle u, u_n \rangle u_n \quad \forall u \in H,$$

*where* $(\mu_n)$ *is the sequence of (non-negative) eigenvalues of* $A^*A$ *and* $(u_n)$ *the associated sequence of eigenvectors. Using this spectral resolution we define the square root* $\sqrt{A^*A}$ *by:*

$$\sqrt{A^*A}u := \sum_{n=1}^{\infty} \sqrt{\mu_n}\, \langle u, u_n \rangle u_n \quad \forall u \in H.$$

*Notice that* $(v_n) := \left(\frac{Au_n}{\sqrt{\mu_n}}\right)$ *is an orthonormal set in* $\mathbb{H}_2$; *we define the "phase operator"* $U$ *such that* $A = U\,\|A| = U\,\sqrt{A^*A}$ *by*

$$Uv := \sum_{n=1}^{\infty} \langle v, u_n \rangle\, v_n \quad \forall v \in \mathbb{H}_2. \tag{15}$$

*With this notation we have $(v_n) = (Uu_n)$. Since $Au = U\sqrt{A^*A}u \quad \forall u \in \mathbb{H}_1$, applying (15) to $v = \sqrt{A^*A}u$ yields*

$$Au = \sum_{n=1}^{\infty} \sqrt{\mu_n}\, \langle u, u_n \rangle\, v_n \quad \forall u \in \mathbb{H}_1.$$

**Corollary 5** *A bounded operator $A : \mathbb{H}_1 \to \mathbb{H}_2$ is compact whenever it is the limit of a sequence (possibly finite) of finite rank operators.*

**Proof 29** *We know that a limit (in the operator norm of bounded operators) of a sequence of finite rank operators is compact since the set of compact operators is closed in the set of bounded operators.*
*Conversely, it follows from Corollary 4 (and with the notation of the corollary) that any compact operator $A \in \mathcal{K}(\mathbb{H}_1, \mathbb{H}_2)$ can be written as the limit $A = \lim_{N\to\infty} A_N$ of a sequence of finite rank operators $A_N$ defined by*

$$A_N\, u = \sum_{n=1}^{N} \sqrt{\mu_n}\, \langle u, u_n \rangle\, v_n \quad \forall u \in \mathbb{H}_1.$$

The density of finite rank operators in the set of compact operators actually extends to a Banach setting.

**Theorem 10** *Let $\mathbb{E}$ and $\mathbb{F}$ be two Banach spaces. Any finite rank operator $A \in \mathcal{B}(\mathbb{E}, \mathbb{F})$ is compact and any compact operator in $A \in \mathcal{B}(\mathbb{E}, \mathbb{F})$ is a limit in the bounded operator norm, of finite rank operators.*

**Proof 30** *We give an alternative proof to the one using the discrete spectral resolution, which holds in the Banach setting. Let $A$ be a compact operator; the closure $\overline{A\,(B(0,1))}$ of the range $A\,(B(0,1))$ of the unit ball is compact. Given $\varepsilon > 0$, it can therefore be covered by a finite number of balls $B(h_i, \frac{\varepsilon}{2})$ centered at $h_i$ with given radius $\varepsilon$ in such a way that $\overline{A\,(B(0,1))} \subset \cup_{i=1}^{N} B(h_i, \frac{\varepsilon}{2})$. Let $\mathbb{F}$ denote the subspace generated by $h_i, i = 1, \cdots, N$ and let $P_{\mathbb{F}}$ denote the orthogonal projection onto $F$. Then $A_\varepsilon \equiv P_{\mathbb{F}} A$ has finite rank. Let us check that $|||A_\varepsilon - A||| \leq \varepsilon$. For $h$ in $B(0,1)$ there is some $i_0 \in \{1, \cdots, N\}$ such that $\|Ah - h_{i_0}\| < \frac{\varepsilon}{2}$. Since $\|P_F\| \leq 1$, this implies that $\|P_{\mathbb{F}} A\, h - P_{\mathbb{F}}\, h_{i_0}\| < \frac{\varepsilon}{2}$ and hence $\|P_{\mathbb{F}} A\, h - h_{i_0}\| < \frac{\varepsilon}{2}$. It follows that $\|P_{\mathbb{F}} A\, h - A\, h\| < \varepsilon$ for any $h \in B(0,1)$ and hence $|||A_\varepsilon - A||| < \varepsilon$, which shows that the compact operator $A$ could be approximated by finite rank ones $A_\varepsilon$.*

Here again, an example can be found on $l_2$ sequences.

**Example 21** *To a sequence $(\alpha_n)$ of real numbers converging to $0$, we can associate a compact operator*

$$\begin{array}{rcl} A : l_2 & \to & l_2 \\ (u_n) & \mapsto & (\alpha_n u_n). \end{array}$$

*It is compact as the limit in $\mathcal{B}(l_2)$ of operators of finite rank $A_k$ defined by*

$$A_k((u_n)) = (\alpha_0 u_0, \cdots, \alpha_k u_k, 0, \cdots, 0).$$

## 7.3 Fredholm operators

Useful references are [Gi], [LM], [N], [RS].

If $V$ is a finite dimensional space, a linear map $A : V \to V$ is either non-injective or it is bijective and the equation $A\,u = v, v \in V$ has unique solution. We want to generalise this "alternative" to operators of the type $K - \lambda I$ with $K$ compact by means of the concept of Fredholm operators.

**Definition 63** *Given two Banach spaces $\mathbb{E}$ and $\mathbb{F}$, an operator $A \in \mathcal{B}(\mathbb{E}, \mathbb{F})$ is Fredholm whenever it is invertible "up to a compact operator", i.e. whenever there are operators $B \in \mathcal{B}(\mathbb{F}, \mathbb{E})$ and $C \in \mathcal{B}(\mathbb{F}, \mathbb{E})$ such that $B\,A - I_{\mathbb{E}}$ et $A\,C - I_{\mathbb{F}}$ are compact.*

**Remark 15** *In Definition 63 one can equivalently require that $B\,A - I_{\mathbb{E}}$ and $A\,C - I_{\mathbb{F}}$ have finite rank.*

An invertible operator is clearly Fredholm. Any operator $I - A$ with $A$ compact, is also Fredholm.
Here is an instructive example of Fredholm operator.

**Example 22** *Composing the right shift and left shift operator $L : l^2 \to l^2$ introduced in Example 12 yields $RL((u_n)) = (0, u_1, u_2, \cdots, u_n, \ldots)$ so that $RL = I - \pi_0$ where $\pi_0$ is the projection onto the one-dimensional subspace $(u_0, 0, \cdots, 0, \cdots) \in l^2$ whereas $LR = I$. Since $\pi_0$ has finite rank and hence is compact, the operators $L$ and $R$ are Fredholm.*

If $A \in \mathcal{B}(\mathbb{E}, \mathbb{F})$ is Fredholm so is $A + K$ for any compact operator $K \in \mathcal{B}(\mathbb{E}, \mathbb{F})$. Indeed, the existence of operators $B \in \mathcal{B}(\mathbb{F}, \mathbb{E})$ and $C \in \mathcal{B}(\mathbb{F}, \mathbb{E})$ such that $A\,B - I$ and $C\,A - I$ are compact implies that $(A + K)\,B - I = A\,B - I + K\,B$ and $C(A + K) - I = CA - I + K$ are compact since compact operators form an ideal in the bounded operators.
If $\mathbb{E}$ and $\mathbb{F}$ are Hilbert spaces, since the adjoint of a compact operator is compact, the adjoint of a Fredholm operator is Fredholm.

**Proposition 20** *Let $\mathbb{H}_1$ and $\mathbb{H}_2$ be two Hilbert spaces and let $A \in \mathcal{B}(\mathbb{H}_1, \mathbb{H}_2)$. The following conditions are equivalent:*

*i) $A$ is Fredholm*

*ii) $KerA$ and $KerA^*$ are finite dimensional and $R(A)$ et $R(A^*)$ are closed.*

*iii) The kernels $KerA$ et $KerA^*$ are finite dimensional and*

$$\mathbb{H}_1 = Ker(A) \oplus R(A^*),$$

$$\mathbb{H}_2 = Ker(A^*) \oplus R(A)$$

*where the sums are orthogonal.*

**Remark 16** *In ii) it is sufficient to require that $KerA$ and $KerA^*$ are finite dimensional since the closedness of the ranges $R(A)$ and $R(A^*)$ then follows (see e.g. [Sh] Lemma 8.1 in [Sh] or Lemma 35.22 in [Dr]).*

**Proof 31** *(i) ⇒ (ii): We show that Ker(A) is finite dimensional by proving that its unit ball is compact. Let $(u_n)$ be a sequence in the unit sphere of Ker(A) so that $\|u_n\| = 1$. By assumption A admits a left inverse B modulo a compact so $I - BA$ is compact, and from $u_n = (I - B\,A)u_n$ and since, we can extract a convergent subsequence, which proves that the unit ball of Ker(A) is compact and hence that KerA is finite dimensional.*

*Let us now check that the range of A is closed under the assumption that A admits a left inverse B modulo a compact. Let $(u_n)$ be a sequence in $\mathbb{H}_1$ and let us assume that $v_n := A\,u_n \to v$. We want to show the existence of u in $\mathbb{H}_1$ such that $v = Au$. Without any restriction, we can also assume that $(u_n)$ lies in the orthogonal complement $\mathrm{Ker}^\perp A$ to KerA.*
*Let us first assume that $(u_n)$ is bounded. Since $u_n = B\,v_n + (I - B\,A)u_n$, $I - B\,A$ being compact, we can extract a convergent subsequence $(u_{\phi(n)})$ such that $(I - B\,A)(u_{\phi(n)})$ converges to some w. B being bounded, the sequence $(B\,v_n)$ converges to $Bv$ so that that $(u_{\phi(n)})$ converges to some $u := B\,v + w$. Thus $v = \lim_n v_{\phi(n)} = \lim_n Au_{\phi(n)} = Au$ lies in the range of A so that R(A) is closed.*
*If now the sequence $(u_n)$ is unbounded, then $\|u_n\|$ tends to $+\infty$ when $n \to +\infty$. Applying the result obtained in the bounded case to $u'_n \equiv \frac{u_n}{\|u_n\|}$ yields a subsequence $(u'_{\phi(n)}) \in Ker^\perp A$ that converges to $u'$ such that $\|u'\| = 1$ and $A\,u' = \lim_n \frac{Au_n}{\|u_n\|} = \lim_n \frac{v_n}{\|u_n\|} = 0$. Since A is closed KerA is also closed and $u' \in$ KerA which leads to a contradiction. Since the adjoint of a Fredholm operator is Fredholm, $KerA^*$, resp. $R(A^*)$ are finite dimensional, resp. closed.*

*(ii) ⇒ (iii): Given a closed operator A and densely defined on $\mathbb{H}_1$, we know that*

$$KerA^* + \overline{R(A)} = \mathbb{H}_2 \tag{16}$$

*and*

$$KerA^{**} + \overline{R(A^*)} = \mathbb{H}_1 \tag{17}$$

*Since R(A) and $R(A^*)$ are closed it follows that*

$$KerA^* + R(A) = \mathbb{H}_2$$

*and*

$$KerA + R(A^*) = \mathbb{H}_1.$$

*(iii) ⇒ (i): A is bijective from $Ker^\perp A = R(A^*)$ onto $R(A) = Ker^\perp(A^*)$, so we can find two operators C defined on R(A) and D defined on $R(A^*)$ such that $C\,A = I/Ker^\perp A$ and $A\,D = I/Ker^\perp A^*$. Let us denote by the same symbols C resp. the extension of C by 0 on $KerA^*$, resp. of D by 0 on KerA. They are bounded operators by the closed graph theorem and by construction we have $I - C\,A = \pi_{KerA}$ and $I - A\,D = \pi_{KerA^*}$ where $\pi_{\mathbb{H}_2}$ is the orthogonal projection on the vector space $\mathbb{H}_2$. Since these two projections have finite rank, it follows that A is Fredholm.*

**Example 23** *The left shift operator $L : \ell^2 \to \ell^2$ introduced in Example 12 has one dimensional kernel Ker $L = \{(u_n) \in \ell^2, \quad u_n = 0 \quad \forall n \neq 0\}$ and range $\ell^2$. Its adjoint $R = L^*$ (see Example 13) has kernel Ker $R = \{0\}$ and range $\{(u_n) \in \ell^2, \quad u_0 = 0\}$.*

We end this paragraph by the **Fredholm alternative** which generalises the finite dimensional alternative quoted at the beginning of the section. Let $K \in \mathcal{K}(\mathbb{H})$, then $A := K - \lambda I$ is a Fredholm operator. If moreover $K$ is self-adjoint, then either $\lambda$ is an eigenvalue of $K$ or otherwise Ker $(A) = \{0\}$ so that $R(A) = \mathbb{H}$ and the equation $Ku - \lambda u = v$ has a unique solution $u = (K - \lambda u)^{-1} v$ for any $v \in \mathbb{H}$. By the closed graph theorem the operator $(K - \lambda u)^{-1}$ is then bounded, in which case $\lambda$ lies in the resolvent $\rho(K) := \mathbb{C} - \sigma(K)$. This is another way to see that the spectrum of a self-adjoint compact operator is purely discrete.

## 7.4 The index of a Fredholm operator

Given a Fredholm operator $A : \mathbb{H}_1 \to \mathbb{H}_2$, the *index of $A$* is the positive integer given by

$$\begin{aligned} \mathrm{ind}(A) &:= \dim \mathrm{Ker}(A) - \dim \mathrm{Ker}(A^*) \\ &= \dim \mathrm{Ker}(A) - \mathrm{codim}\, \mathrm{R}(A) \end{aligned} \tag{18}$$

since the range of $A$ and the kernel of $A^*$ are topological complements in $\mathbb{H}_2$.
An invertible operator has vanishing index since both its kernel and cokernel have dimension zero. Since $\mathrm{ind}(A^*) = -\mathrm{ind}(A)$, a self-adjoint operator also has vanishing index.

**Remark 17** *A useful even if elementary observation is the implication*

$$\mathrm{ind}(A) \neq 0 \Longrightarrow (\mathrm{Ker}(A) \neq \{0\} \quad \text{or} \quad \mathrm{Ker}(A^*) \neq \{0\}) .$$

The index is additive on products.

**Proposition 21** *Given two Fredholm operators $A$ and $B$ then their product $A\,B$ is Fredholm and*

$$ind(A\,B) = ind(A) + ind(B).$$

**Proof 32** *(Partial) Let us show the index relation, leaving the proof of the Fredholm property of the product as an exercise. We write*

$$Ker(AB) = Ker(B) \oplus B^{-1}\left(R(B) \cap Ker(A)\right)|_{\mathrm{Ker}^{\perp}(B)}$$

$$\begin{aligned} Ker(B^*A^*) &= Ker(A^*) \oplus (A^*)^{-1}\left(R(A^*) \cap Ker(B^*)\right)_{\mathrm{Ker}^{\perp}(A^*)} \\ &= Ker(A^*) \oplus (A^*)^{-1}\left(Ker^{\perp}(A) \cap R^{\perp}(B)\right)_{\mathrm{Ker}^{\perp}(A^*)} . \end{aligned}$$

*Hence*

$$\begin{aligned} ind(A\,B) &= dim\ Ker(A\,B) - dim\ Ker(B^*A^*) \\ &= dim\ Ker(B) + dim\,(R(B) \cap Ker(A)) \\ &- dim\ Ker(A^*) - dim\left(Ker^{\perp}(A) \cap R^{\perp}(B)\right) \\ &= dim\ Ker(B) + dim\ Ker(A) \quad \text{since} \quad R^{\perp}(B) = \mathrm{Ker}(B^*) \\ &- dim\,(Ker(B^*) \cap Ker(A)) \\ &- dim\ Ker(A^*) - dim\left(Ker(B^*) \cap Ker^{\perp}(A)\right) \\ &= dim\ Ker\ (A) + dim\ Ker(B) - dim\ Ker(A^*) - dim\ Ker(B^*) \\ &= ind(A) + ind(B) \end{aligned}$$

A fundamental property of the index is that it is locally constant.

**Theorem 11** *Given two Hilbert spaces $\mathbb{H}_1$ and $\mathbb{H}_2$,*

1. *The set $\mathcal{F}(\mathbb{H}_1, \mathbb{H}_2)$ of Fredholm operators in $\mathcal{B}(\mathbb{H}_1, \mathbb{H}_2)$ is an open subset.*
2. *The index map $ind : \mathcal{F}(\mathbb{H}_1, \mathbb{H}_2) \to \mathbb{Z}$ is continuous and locally constant on $\mathcal{F}(\mathbb{H}_1, \mathbb{H}_2)$.*

**Proof 33** *Let $A : \mathbb{H}_1 \to \mathbb{H}_2$ be a fixed Fredholm operator then $\mathbb{H}_1 = \mathrm{Ker}(A) \oplus R(A^*)$ and $\mathbb{H}_2 = \mathrm{Ker}(A^*) \oplus R(A)$. Let us consider the spaces $\widetilde{\mathbb{H}_1} := Ker(A^*) \oplus \mathbb{H}_1$ and $\widetilde{\mathbb{H}_2} = Ker(A) \oplus \mathbb{H}_2$ together with the map*

$$\begin{array}{ccl} \mathrm{Hom}(\mathbb{H}_1, \mathbb{H}_2) & \longrightarrow & \mathrm{Hom}(\widetilde{\mathbb{H}_1}, \widetilde{\mathbb{H}_2}) \\ A & & \tilde{A}, \quad \tilde{A}(u, h) := \pi_{Ker(A)} h \oplus u \oplus Ah. \end{array}$$

*If $h = h_0 \oplus h' \in \mathrm{Ker}(A) \oplus R(A^\star)$ then*

$$\tilde{A}(u, h) = h_0 \oplus u \oplus Ah' \tag{19}$$

*so that $\tilde{A}$ lies in the set $Iso\left(\widetilde{\mathbb{H}_1}, \widetilde{\mathbb{H}_2}\right)$ of isomorphisms of the Hilbert spaces $\widetilde{\mathbb{H}_1}$ and $\widetilde{\mathbb{H}_2}$.*
*Using (19) we can express the Fredholm operator $A$ in terms of the invertible operator $\tilde{A}$. Let $i_1(A) : \mathbb{H}_1 \to \tilde{\mathbb{H}}_1$ be the natural inclusion and $\pi_1(A) : \widetilde{\mathbb{H}_1}, \to \mathbb{H}_1$, be the natural projection. We have $\pi_1(A) \circ i_1(A) = I_{\mathbb{H}_1}$ and $i_1(A) \circ \pi_1(A) = I_{\widetilde{\mathbb{H}_1},} - \pi_{\mathrm{Ker} A^*}$ so that $i_1(A)$ and $\pi_1(A)$ are Fredholm. The same holds for $i_2(A)$ and $\pi_2(A)$. It follows from (19) that*

$$A = \pi_2(A)\, \tilde{A}\, i_1(A).$$

*Given another operator $B \in \mathcal{B}(\mathbb{H}_1, \mathbb{H}_2)$ we set*

$$\tilde{B} := \pi_2(A)\, \tilde{B}\, i_1(A). \tag{20}$$

*By (19) we have $|||\tilde{B} - \tilde{A}||| = |||B - A|||$ so that a small perturbation $B$ of of $A$ in the space $\mathcal{B}(\mathbb{H}_1, \mathbb{H}_2)$ induces a small perturbation $\tilde{B}$ of $\tilde{A}$. Since $\tilde{A}$ is an isomorphism and since $Iso(\widetilde{\mathbb{H}_1}, \widetilde{\mathbb{H}_2})$ is open in $Hom(\widetilde{\mathbb{H}_1}, \widetilde{\mathbb{H}_2})$, so is $\tilde{B}$ an isomorphism. The operator $\tilde{B}$ being invertible and hence Fredholm, using the additivity of the index on products it follows from (20) and the Fredholm property of $\pi_2(A)$ and $i_1(A)$, that $B$ is also Fredholm; thus $\mathcal{F}(\mathbb{H}_1, \mathbb{H}_2)$ is open in $\mathcal{B}(\mathbb{H}_1 \mathbb{H}_2)$. Since $ind(\tilde{B}) = 0$, by (21) we have*

$$\begin{aligned} ind(B) &= ind(i_1(A)) + ind(\tilde{B}) + \mathrm{ind}(\pi_2(A)) \\ &= \mathrm{ind}(i_1(A)) + \mathrm{ind}(\pi_2(A)) \\ &= \dim(\mathrm{Ker}(A)) - \dim(\mathrm{Ker}(A^*)) \\ &= \mathrm{ind}(A), \end{aligned}$$

*which shows that the index is locally constant.*

The notion of index is illustrated by Example 23 at the beginning of this chapter.

**Example 24** *The operator $L$ has index $1$. It is easy to check that the index of $L^n$ is $n$ and the index of $R^n$ is $-n$, which shows that the index is surjective onto $\mathbb{Z}$.*

**Corollary 6** *Given an operator* $A \in \mathcal{F}(\mathbb{H}_1, \mathbb{H}_2)$ *and* $K \in \mathcal{K}(A, B)$ *then* $A + K \in \mathcal{K}(A, B)$ *and we have*

$$\operatorname{ind}(A + K) = \operatorname{ind}(A).$$

**Proof 34** *The fact that* $A + K$ *is Fredholm was observed previously. The family* $A_t := A + tK, t > 0$ *defines a continuous family in* $\mathcal{F}(\mathbb{H}_1, \mathbb{H}_2)$*; hence the map* $f(t) := \operatorname{ind}(A_t)$ *is locally constant and we have* $\operatorname{ind}(A + K) = f(1) = f(0) = \operatorname{ind}(A)$.

, *anoperator*A$\in \mathcal{B}(\mathbb{H})$ is self-adjoint if $A = A^*$.
Here is another characterisation (14) of the norm of a bounded operator when the operator is self-adjoint.

**Lemma 13** *Let $\mathbb{H}$ be a Hilbert space equipped with a scalar product $\langle\cdot,\cdot\rangle$ and $A \in \mathcal{B}(\mathbb{H})$ a self-adjoint operator then*

$$|||A||| = \sup_{\|u\|=\|v\|=1}|\langle u, Av\rangle| = \sup_{\|u\|=1}|\langle u, Au\rangle|.$$

**Proof 35** *Taking $v = u$ in (14) we infer that*

$$|||A||| = \sup_{\|u\|=\|v\|=1}|\langle u, Av\rangle| \geq \sup_{\|u\|=1}|\langle u, Au\rangle|.$$

*Conversely, let $C := \sup_{\|u\|=1}|\langle u, Au\rangle|$ and let us assume that $A$ is self-adjoint. We want to show that $\sup_{\|u\|=\|v\|=1}|\langle u, Av\rangle| \leq C$. From*

$$\langle u+v, A(u+v)\rangle - \langle u-v, A(u-v)\rangle = 4\,\mathrm{Re}\langle u, Av\rangle$$

*we infer that*

$$\|u\| = \|v\| = 1 \Longrightarrow |\mathrm{Re}\langle u, Av\rangle| \leq \frac{C}{4}\left(\|u+v\|^2 + \|u-v\|^2\right) \leq \frac{C}{2}\left(\|u\|^2 + \|v\|^2\right) = C.$$

*Choosing $\theta$ such that the image $R_\theta u$ of $u$ under the rotation of angle $\theta$ gives rise to a real number $\langle R_\theta u, Av\rangle$, and applying the above inequality to $R_\theta u$ instead of $u$ we get*

$$\|u\| = \|v\| = 1 \Longrightarrow |\langle u, Av\rangle| \leq C$$

*and consequently the identity*

$$|||A||| = \sup_{\|u\|=\|v\|=1}|\langle u, Av\rangle| = \sup_{\|u\|=1}|\langle u, Au\rangle|.$$

Here are a few useful spectral properties of self-adjoint operators.

**Lemma 14** *Let $\mathbb{H}$ be a Hilbert space and $A \in \mathcal{B}(\mathbb{H})$ be a self-adjoint operator. The following properties hold:*

1. *The eigenvalues of $A$ are real.*
2. *The eigenspaces of $A$ associated to different eigenvalues are orthogonal.*
3. *Any non-zero vector $u$ such that*

$$\frac{|\langle Au, u\rangle|}{\|u\|^2} = |||A|||$$

*is an eigenvector of $A$ with eigenvalue in $\{-|||A|||, |||A|||\}$.*

**Proof 36** 1. *Let $\lambda \in \mathbb{C}$ be an eigenvalue of $A$ and $u$ an associated eigenvector, then $< Au, u >= \lambda < u, u >=< u, Au >= \bar{\lambda} < u, u >$. But since $\|u\| \neq 0$, it follows that $\bar{\lambda} = \lambda$ and $\lambda \in \mathbb{R}$.*

2. *Given two eigenvalues $\lambda_i, i = 1, 2$, we have*

$$\lambda_1 \neq \lambda_2 \Rightarrow \mathrm{Ker}(A - \lambda_1 I) \perp \mathrm{Ker}(A - \lambda_2 I).$$

*Indeed, if $\lambda_1 \neq \lambda_2$ are two eigenvalues associated with the eigenvectors $u_1$ and $u_2$, then $< Au_1, u_2 >= \lambda_1 < u_1, u_2 >=< u_1, Au_2 >= \lambda_2 < u_1, u_2 >$, from which it follows that $< u_1, u_2 >= 0$.*

3. *By the Cauchy-Schwartz inequality, we have*

$$|||A||| = \frac{|\langle Au, u\rangle|}{\|u\|^2} \leq \frac{\|Au\| \, \|u\|}{\|u\|^2} \leq |||A|||$$

*and in particular*

$$|\langle Au, u\rangle| = \|Au\| \, \|u\|.$$

*This leads to the existence of a complex constant $\lambda$ such that $Au = \lambda u$ and $|\lambda| = |||A|||$. The fact that this eigenvalue of $A$ is real then follows from the first item so that $\lambda \in \{-|||A|||, |||A|||\}$.*

**Proposition 22** *Given a Hilbert space $\mathbb{H}$, the algebra $\mathcal{B}(\mathbb{H})$ equipped with the adjoint map $A \mapsto A^*$ is a $C^*$-algebra.*

**Proof 37** *We shall take for granted the fact that $\mathcal{B}(\mathbb{H})$ defines a Banach space for the topology induced by the operator norm.*

1. *The product of two operators $A, B$ in $\mathcal{B}(\mathbb{H})$ satisfies the inequality*

$$|||B\,A||| \leq |||A||| \, |||B|||.$$

2. *The product of two bounded operators is bounded as a consequence of the above inequality and $\mathcal{B}(\mathbb{H})$ is a unital algebra since it contains the identity operator.*

3. *Since*

$$\begin{aligned} |||A||| &= \sup_{\|u\|_1=\|v\|_2=1} |\langle Au, v\rangle_2| \\ &= \sup_{\|u\|_1=\|v\|_2=1} |\langle u, A^* v\rangle_2| \\ &= \sup_{\|u\|_1=\|v\|_2=1} |\langle A^* v, u\rangle_2| \\ &= |||A^*|||, \end{aligned}$$

*the map $A \longmapsto A^*$ preserves the boundedness and defines an involution on $\mathcal{B}(\mathbb{H})$.*

4. *Using Lemma **??**, we have*

$$\begin{aligned} |||A^* A||| &= \sup_{\|u\|=1} |\langle A^* Au, u\rangle| \\ &= \sup_{\|u\|=1} |\langle Au, Au\rangle| \\ &= |||A|||^2. \end{aligned}$$

The *Gelfand-Naimark* theorem says that every abstract $C^*$-algebra with identity is isometrically $*$-isomorphic to a $C^*$-algebra of operators. To prove that result, one uses the *Gelfand-Naimark-Segal* or *GNS* construction which produces a representation from a state. To a state $\rho$ on a $C^*$-algebra $\mathcal{A}$, i.e. a positive linear functional $\rho : \mathcal{A} \to \mathbb{C}$ ($\rho(a^*a \geq 0 \forall a \in \mathcal{A}$), one can associate a positive semi-definite bilinear form $\langle a, b\rangle = \rho(b^* a)$ with kernel $\mathcal{N}_\rho = \{a \in \mathcal{A}, \rho(a^* a) = 0\}$ which is a subvector space of $\mathcal{A}$ and a left ideal in $\mathcal{A}$. This bilinear form therefore induces a positive definite form $\langle \cdot, \cdot \rangle$ on $A/\mathcal{N}_\rho$ and hence a pre-Hilbert space structure on that quotient space, which by completion gives rise to a Hilbert space $\mathbb{H}_\rho$. The left regular representation

$$\begin{aligned} \pi_\rho : \mathcal{A} &\to \mathcal{B}(\mathbb{H}_\rho) \\ b &\mapsto (a \mapsto ba) \end{aligned}$$

is cyclic with cyclic vector $x_\rho := 1_{\mathcal{A}} + \mathcal{N}_\rho$ and $\rho(a) = \langle \pi_\rho(a) x_\rho, x_\rho \rangle$ for any $a \in \mathcal{A}$.

## 7.5 Closed graph theorem

Useful references are [Br], [Ru].

The operators one comes across in geometry or in physics usually are unbounded and only defined on a dense domain of the Banach space.

**Definition 64** *Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be two Banach space. The graph of an operator*

$$A : D(A) \subset E \to F$$

*defined on a domain $D(A)$ is the set:*

$$Gr(A) := \{(u, Au) \in E \times F, u \in D(A)\}.$$

It can be equipped with the (graph) norm

$$\|(u, v)\| := \|u\|_E + \|v\|_F.$$

Notice that whenever $A$ is invertible and its inverse bounded, then the graph of $A^{-1}$ is the symmetric of the graph of $A$ w.r.to the diagonal axis.

**Definition 65** *The operator $A$ is closed whenever its graph is closed for the graph norm.*

When $E$ and $F$ are separable, there is another characterisation for closed operators.

**Lemma 15** *An operator $A : D(A) \subset E \to F$ is* **closed** *if, given any sequence $(u_n)$ converging to $u \in E$ such that $Au_n$ converges in $F$, then the limit $u$ lies in the domain $D(A)$ and $Au_n \to Au$.*

**Proof 38** *The graph of $A$ is closed whenever for a sequence $(u_n, Au_n)$ in $\mathrm{Gr}(A)$ which converges in $E \times F$ to $(u, v)$ we have that $u \in D(A)$ and $v = Au$ i.e. $Au_n$ converges to $v = Au$, which corresponds to the characterisation in the lemma.*

*We shall henceforth assume that the Banach spaces under consideration are separable.*

**Lemma 16** *Any bounded linear operator defined on a closed domain is closed.*
*Furthermore, a closed linear operator $A : D(A) \subset \mathbb{E} \to \mathbb{F}$ defined on a dense domain $D(A)$ of $\mathbb{E}$ extends in an unique way to a bounded operator on $E$ whenever there is a constant $C > 0$ such that $\|Au\|_{\mathbb{F}} \leq C\|u\|_{\mathbb{E}}, \forall u \in D(A)$.*

**Example 25** *Let $\mathbb{F} = C([0, 1])$ equipped with the norm $\|f\|_\infty = \sup_{x\in[0,1]}|f(x)|$ and $\mathbb{E} = C^1([0, 1])$ equipped with the norm $\|f\|_{\infty,1} := \|f\|_\infty + \|f'\|_\infty$. The operator $A : f \mapsto f'$ is defined on $D(A) = C^\infty([0, 1]$ which is dense in $\mathbb{E}$ and we have $\|A(f)\|_\infty \leq \|f\|_{\infty,1}$ for any $f \in D(A)$. This yields back the well-known fact that $A$ extends to a bounded linear operator $A : \mathbb{E} \to \mathbb{F}$.*

**Proof 39**
- *Let $A$ be a bounded linear operator on a closed domain $D(A)$. A sequence $(u_n, Au_n)$ in $\mathrm{Gr}(A)$ which converges in $\mathbb{E} \times \mathbb{F}$, therefore converges to $(u, v) \in D(A) \times \mathbb{F}$. It follows from the continuity of $A$ that $v = \lim Au_n$ so that the sequence $(u_n, Au_n)$ converges in $\mathrm{Gr}(A)$.*

- *To prove the second one, all we need is to define the image of any element $u \in \mathbb{E}$ by an extension of $A$. Since $D(A)$ is dense in $\mathbb{E}$, $u$ can be seen as a limit $u = \lim_{n\to\infty} u_n$ of a sequence $(u_n)$ in $D(A)$. Since $(u_n)$ is convergent it is a Cauchy sequence, and hence so is the sequence $(Au_n)$ a Cauchy sequence so that it converges to some $v \in \mathbb{F}$ since $\mathbb{F}$ is complete. The operator $A$ being closed, this implies that $u$ lies in the domain $D(A)$ and $Au = v$. This extended operator (also denoted by $A$) is clearly a bounded operator.*
  *Moreover this extension does not depend on the choice of the sequence. For if $(u'_n)$ is another sequence tending to $u$, from the inequality $\|Au_n - Au'_n\| \leq C\|u_n - u'_n\|$, it follows that $Au'_n \to Au$.*

**Proposition 23 (Open mapping Theorem)** *A surjective bounded linear operator $A : \mathbb{E} \to \mathbb{F}$ between two Banach spaces is open i.e., it sends open subsets to open subsets.*
*Consequently, if it is invertible, its inverse is continuous.*

**Proof 40** *It relies on the following result which we take for granted. Under the assumptions of the proposition, the image $A(B_{\mathbb{E}}(0, r))$ of an open non void ball in $E$ contains a non void ball $B_{\mathbb{F}}(0, \varepsilon)$ in $\mathbb{F}$.*
*Assuming this, we want to show that the range $A(U)$ of an open subset $U$ in $\mathbb{E}$ is open i.e., that any $y = A(x)$ in $A(U)$ is the center of a ball $B_{\mathbb{F}}(y, \varepsilon)$ in $A(U)$.*
*The subset $U - x$ of $\mathbb{E}$ is open and contains $0$ so it contains an open non void ball $B_E(0, r)$. Thus, $A(U - x) = A(U) - y$ which contains $A(B_{\mathbb{E}}(0, r))$ contains a non void ball $B_{\mathbb{F}}(0, \varepsilon)$. This implies that $A(U)$ contains the non void open ball $B_{\mathbb{F}}(y, \varepsilon) = B_{\mathbb{F}}(0, \varepsilon) + y$.*

We have shown that a bounded linear operator is closed; the closed graph theorem provides a converse statement.

**Theorem 12 (Closed graph Theorem)** *Let $E$ and $F$ be two Banach spaces. A closed linear operator $A : E \to F$ with closed domain $D(A)$ is bounded. In particular, a closed linear operator with domain $E$ is bounded.*

**Proof 41**

- *Since $\mathbb{E}$ and $\mathbb{E}$ are complete, the cartesian product $\mathbb{E}\times\mathbb{E}$ is complete. Since $D(A)$ is closed, its graph $\mathrm{Gr}(A) =$ is closed in $\mathbb{E}\times\mathbb{E}$ and hence complete.*
- *The map $P : \mathrm{Gr}(A) \to D(A)$ defined as $P(u, v) = u$ is linear and bounded since $\|u\|_{\mathbb{E}} \leq \|(u, Au)\|_{\mathrm{Gr(A)}}$. It is a bijection with inverse $u \longmapsto (u, Au)$. The open mapping theorem implies that its inverse is continuous and hence $\|u\|_{\mathbb{E}} + \|Au\|_{\mathbb{F}} = \|P^{-1}(u)\|_{\mathrm{Gr(A)}} \leq C\,\|u\|_{\mathbb{E}}$ for some positive constant $C$ (chosen large enough). Thus $\|Au\|_{\mathbb{F}} \leq (C - 1)\,\|u\|_{\mathbb{E}}$ and $A$ is bounded.*

*In the following, we assume the operators are closed and defined on a dense domain $D(A)$.*

**Proposition 24** *The inverse of a bijective linear operator $A : D(A) \subset \mathbb{E} \to \mathbb{F}$ is a bounded operator $A^{-1} : \mathbb{F} \to \mathbb{E}$.*

**Proof 42** *Since the graph of $A$ is closed so is the graph of $A^{-1}$ and the result follows from the closed graph theorem.*

**Definition 66** *The resolvent of A is the set*

$$\begin{aligned}\rho(A) &:= \{\lambda \in \mathbb{C}, A-\lambda I \quad \textit{is bijective }\}\\ &= \{\lambda \in \mathbb{C}, A-\lambda I \quad \textit{is bijective and} \quad (A-\lambda I)^{-1} \in \mathcal{B}(H)\}.\end{aligned}$$

*The spectrum $\sigma(A)$ of $A$ is the complement of the resolvent:*

$$\sigma(A) := \mathbb{C}/\rho(A).$$

*The point spectrum $\sigma_p(A)$ is the set*

$$\sigma_p(A) := \{\lambda \in \mathbb{C}, Ker(A-\lambda I) \neq \{0\}\}.$$

In finite dimensions, $\sigma_p(A) = \sigma(A)$, which is not generally the case in infinite dimensions.

**Example 26** *Letting $H = l^2$ be the space of $l^2$ convergent sequences and let $A : l^2 \to l^2$ be the operator sending a sequence $(u_n)$ to a sequence $(v_n)$ with $v_0 = 0, v_i = u_{i-1}$. $0$ does not belong to $\sigma_p(A)$ but $0 \in \sigma(A)$.*

## 7.6 Adjoint of an (unbounded) operator

Useful references are [Br], [RS].

Let $\mathbb{H}_1$ and $\mathbb{H}_2$ be Hilbert spaces equipped with the Hermitian products $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$ respectively. We extend here the notion of adjoint of an operator to unbounded operators.

**Definition 67** *The adjoint of an operator $A : D(A) \subset \mathbb{H}_1 \to \mathbb{H}_2$ defined on a dense domain $D(A)$ is an operator $A^*$ defined on*

$$D(A^*) := \{v \in \mathbb{H}_2, \exists C(v) > 0, |< Au, v >_2| \leq C(v)\|u\|_1 \quad \forall u \in D(A)\}$$

*by $< Au, v >_2 = < u, A^* v >_1$ $\forall u \in D(A), v \in D(A^*)$.*

This defines $A^*$ in an unique way. For if $v_1^*$ et $v_2^*$ are elements in $\mathbb{H}_1$ such that $< Au, v >_2 = < u, v_i^* >_1, i = 1, 2$ for any $u \in D(A)$, then $0 = < u, v_1^* - v_2^* >_1$ for any $u \in D(A)$ dense in $E$, which implies $v_1^* = v_2^*$.

**Remark 18** *Provided $R(A) \subset D(A^*)$ we have $\mathrm{Ker}(A^*A) = \mathrm{Ker}(A)$. Indeed, clearly the inclusion $\mathrm{Ker}(A) \subset \mathrm{Ker}(A^*A)$ holds. Conversely we have*

$$A^*Au = 0 \Longrightarrow \langle A^*Au, u \rangle = 0 \Longrightarrow \|Au\| = 0 \Longrightarrow Au = 0,$$

*so that* $\mathrm{Ker}(A^*A) \subset \mathrm{Ker}(A)$.

Whenever $E = F = H$ is a Hilbert space equipped with the inner product $\langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_E = \langle \cdot, \cdot \rangle_F$, we say $A$ is *self-adjoint* if $A = A^*$, i.e

$$D(A) = D(A^*) \quad \text{and} \quad < Au, v > = < u, Av > \quad \forall u \in D(A).$$

The graph of the adjoint $A^*$ of an operator $A$ is given by

$$(Gr'(-A^*))^\perp = \overline{Gr(A)}$$

where "prime" means the symmetric set w.r.to the diagonal axis. This is an easy consequence of the fact that

$$< (u, Au), (-A^* v, v) >=< u, -A^* v > + < Au, v >= 0$$

for $u \in D(A)$, $v \in D(A^*)$.

**Proposition 25** *The domain $D(A^*)$ of the adjoint $A^*$ of a densely defined closed operator $A : D(A) \subset \mathbb{H}_1 \to \mathbb{H}_2$ is dense in $\mathbb{H}_1$, and the adjoint is also closed.*

**Proof 43** *Let us assume that the domain of $A^*$ is not dense in $\mathbb{H}_2$. Then there exists a non zero vector $u \in \mathbb{H}_2$ such that $< u, v >_2= 0$ for any $v \in D(A^*)$. Using the closedness of $A$, this would imply that*

$$\begin{aligned} < u, v >_2= 0 \quad &\Leftrightarrow \quad < u, v >_2 + < 0, -A^* v >_1= 0 \\ &\Leftrightarrow \quad (0, u) \in (Gr'(-A^*))^\perp = Gr(A) \end{aligned}$$

*and hence $A(0) = u \neq 0$ which contradicts the linearity of $A$.*
*Let us check that $A^*$ is closed. Let $(v_n, A^* v_n)$ be a Cauchy sequence in $D(A^*) \times \mathbb{H}_1$ converging to $(y, x) \in \mathbb{H}_2 \times \mathbb{H}_1$. For any $u \in D(A)$, we have*

$$< Au, y >_2= \lim_{n\to\infty} < Au, v_n >_2= \lim_{n\to\infty} < u, A^* v_n >_1=< u, x >_1$$

*so that $| < Au, y >_2 | \leq \|x\|_1 \cdot \|u\|_1$ for any $v \in D(A)$, which implies $y \in D(A^*)$ and $A^* y = x$.*

Let $A : D(A) \subset H \to H$. When the space $\mathbb{H}_p(A)$ spanned by the eigenvectors of $A$ coincides with the total Hilbert space $H$, letting $\{e_n, n \in \mathbb{N}\}$ be an orthonormal basis of eigenvectors of $H$, the operator has the following *discrete resolution*:

$$Au = \sum_n \lambda_n < u, e_n > \quad \forall u \in D(A).$$

Given a map $f : \sigma_p(A) \subset \mathbb{C} \to \mathbb{C}$, using this spectral representation of $A$ we can define the map $f(A)$ on the domain:

$$D(f(A)) := \{u \in H, \sum_n f(\lambda_n)^2 < u, e_n >^2 < \infty\}$$

by

$$f(A)u = \sum_n f(\lambda_n) < u, e_n > e_n \quad \text{for} \quad u \in D(f(A)).$$

**Example 27** *When $A$ has a discrete resolution with purely discrete spectrum and positive eigenvalues outside a discrete set of eigenvalues, the function $f(x) = e^{-tx}$ defines a bounded operator $e^{-tA}$ called the heat-operator associated to $A$. Heat-operators will be investigated more thoroughly in the sequel.*

Let $A : D(A) \subset \mathbb{H}_1 \to \mathbb{H}_2$ be a closed operator defined on a dense domain $D(A)$ of a Hilbert space $(\mathbb{H}_1, \langle \cdot, \cdot \rangle_1)$ with values in $(\mathbb{H}_2, \langle \cdot, \cdot \rangle_2)$, then the domain $D(A^*)$ is dense in $\mathbb{H}_2$ and we can define the adjoint $A^{**}$ of $A^*$ which coincides with $A$.

Let us first check that $A \subset A^{**}$. Given $x \in D(A)$, $| < x, A^* y > | = | < Ax, y > | \leq$

$\|Ax\|\|y\|$ for any $y \in D(A^*)$ and hence $x \in D(A^{**})$. Since $< x, A^*y >=< Ax, y >$ the operators $A$ and $A^{**}$ coincide on $D(A)$. Since $A^{**}$ is closed as the adjoint of a closed operator, so is its graph and we have $Gr((A^*)^*) = (Gr'(-A^*))^\perp = Gr(A)$, for $Gr(A)$ is closed. This ends the proof of the identity $A^{**} = A$.

A useful property for $A : D(A) \subset \mathbb{H}_1 \to \mathbb{H}_2$ is

$$R^\perp(A) = \mathrm{Ker}(A^*). \tag{21}$$

Indeed, we have $R^\perp(A) \subset \mathrm{Ker}A^*$. For if $< y, Ax >= 0 \;\forall x \in D(A)$ then $y \in D(A^*)$ and $< A^*y, x >= 0 \;\forall x \in D(A)$. But $D(A)$ is dense in $\mathbb{H}$ so it follows that $y \in \mathrm{Ker}A^*$. Let us now check the other inclusion. Given $x \in \mathrm{Ker}A^*$, we have $< A^*x, y >= 0$ for any $y \in D(A)$ and hence $< x, A^{**}y >=< x, Ay >= 0$ for any $y \in D(A)$ which shows that $x \in R(A)^\perp$.
It follows from (21) that

$$\mathrm{Ker}(A^*) \oplus \overline{R(A)} = \mathbb{H}_2,$$

which applied to $A^*$ yields:

$$R^\perp(A^*) = \mathrm{Ker}(A)$$

and hence

$$\mathrm{Ker}(A) \oplus \overline{R(A^*)} = \mathbb{H}_1.$$

The non-zero eigenvalues $\lambda_n$ and the eigenvectors $u_n$ are built recursively using the previous proposition. Assuming that $A \neq 0$, let $\lambda_1 \in \{-|||A|||, |||A|||\}$ and $u_1$ an associated eigenvector as in the above proposition. Let $\mathbb{H}_1 = \langle u_1 \rangle$ be the subspace spanned by $u_1$, then $A(\mathbb{H}_1) \subset \mathbb{H}_1$. We easily check that $A(\mathbb{H}_1^\perp) \subset \mathbb{H}_1^\perp$ and we can therefore restrict $A$ to $\mathbb{H}_1^\perp$: let $A_1$ denote this restriction. If $A_1$ is identically zero, the statement holds. Otherwise $A_1$ is a compact operator so that as above, from $A_1$ we build an operator $A_2$ defined as the restriction of $A_1$ to the orthogonal complement of the space $\mathbb{H}_2 := \langle u_1, u_2 \rangle$ where $u_2 \in \mathbb{H}_1^\perp$ with norm 1 is such that $A_1 u_2 = A u_2 = \lambda_2 u_2$ for some $\lambda_2 \in \{-|||A_1|||, |||A_1|||\}$ . We continue this procedure until we reach an operator $A_i$ which vanishes identically, or otherwise we pursue indefinitely. This way, we build a sequence $(\lambda_n)$ of eigenvalues and orthonormal associated eigenvectors $(u_n)$ such that defined on the subspace $h^2 := \{(u_n)_{n\in\mathbb{N}}, \quad \sum_{n=0}^\infty n^2 |u_n|^2 < \infty\}$ of $l^2$-sequences.

# 8 Differential and pseudo-differential operators

## 8.1 Differential operators

Useful references are [BGV], [F], [H], [LM], [R].

We need some notations:
For a multi index $\alpha := (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$, let us set $|\alpha| := \sum_{k=1}^n \alpha_k$ and for $\xi \in \mathbb{R}^n, \xi^\alpha := \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$. We also set $D_x^\alpha := (-i)^{|\alpha|} \frac{\partial^{|\alpha|}}{\partial x^\alpha}$ with $\frac{\partial^{|\alpha|}}{\partial x^\alpha} := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$.
With these notations we have $\mathcal{F}(D^\alpha f)(\xi) = \xi^\alpha \hat{f}(\xi)$ for any Schwartz function $f \in \mathcal{S}(\mathbb{R}^n)$ where $\mathcal{F}$ also denoted by $\hat{}$ denotes the Fourier transform $\hat{f}(\xi) := (2\pi)^{-\frac{n}{2}} \int e^{-i\langle x, \xi \rangle} f(x) dx$.
In what follows, $K := \mathbb{R}$ or $K := \mathbb{C}$.

A *differential operator* of non-negative integer order $m$ on an open subset $U$ of $I\!R^n$ is a linear map

$$A : C^k(U, K^p) \to C^{k-m}(U, K^q)$$

of the form

$$A = \sum_{|\alpha| \leq m} a_\alpha(x) D_x^\alpha \tag{22}$$

where $a_\alpha(x)$ is a $(q,p)$ matrix of smooth $K$-valued functions with $a_\alpha \neq 0$ for some $\alpha$ and such that $|\alpha| := \sum_{i=1}^n \alpha_i = m$ (i.e. $A$ differentiates $m$ times).

**Example 28** *The Laplacian* $\Delta := -\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ *on* $U := I\!R^n$ *provides an example of differential operator of order* 2.

A change of coordinates $\tilde{x} = \tilde{x}(x)$ on $U$ gives for any $j \in \{1, \cdots, n\}$:

$$\frac{\partial}{\partial x_j} = \sum_{k=1}^n \frac{\partial \tilde{x}_k}{\partial x_j} \frac{\partial}{\partial \tilde{x}_k}.$$

As a consequence, in the new coordinates, the operator $A$ reads

$$A = \sum_{|\alpha| \leq m} \tilde{a}_\alpha(\tilde{x}) \tilde{D}_x^\alpha$$

for some other $(q,p)$ matrix $\tilde{a}_\alpha$ of smooth -valued functions on $U$, which shows that the operator can be described in a similar way in these new coordinates. If we have $a_\alpha(x) = 0$ for all $|\alpha| = m$, then $\tilde{a}_\beta(\tilde{x}) = 0$ for all $|\beta| = m$ so that the order is conserved under a change of coordinates.

Given a smooth manifold $M$ of dimension $n$, it therefore makes sense to define a differential operator $A : C^\infty(M, I\!R) \to C^\infty(M, I\!R)$ of order $m$ as a linear operator which has the above description (22) in any local chart of $M$ since another local chart would give the same type of local description via a change of coordinates.

Let $\mathrm{GL}_r()$ be the group of invertible $K$-valued $r \times r$ matrices. Given two maps $\tau_1 : U \to \mathrm{GL}_p()$ and $\tau_2 : U \to \mathrm{GL}_q(K)$, $\tau_2 A \tau_1$ defines another differential operator or order $m$ on $U$ since

$$\tau_2 A \tau_1 = \sum_{|\alpha| \leq m} \tau_2 a_\alpha(x) \tau_1 \frac{\partial^{|\alpha|}}{\partial x^\alpha},$$

is of the same type as (22). Letting $\pi_E : E \to M$ and $\pi_F : F \to M$ be two vector bundles based on $M$ of rank $p, q$ respectively, this shows that the shape of the operator described in (22) is invariant under a change of trivialization $\tau_E : U \to \mathrm{GL}_p()$ on $E$ and $\tau_F : U \to \mathrm{GL}_q()$ on $F$.

It therefore makes sense to set the following

**Definition 68** *A differential operator of order* $m$ *on a smooth manifold* $M$ *is a linear map* $A : C^\infty(E) \to C^\infty(F)$, *where* $E, F$ *are two vector bundles based on* $M$ *of rank* $p, q$ *over* $K$ *respectively, such that each point* $x$ *of* $M$ *has a neighborhood* $U$ *with coordinates* $(x_1, \cdots, x_n)$ *over which there is a local trivialization* $E_{|_U} \simeq U \times^p$ *and* $F_{|_U} \simeq U \times^q$ *in which the operator* $A$ *reads*

$$A = \sum_{|\alpha| \leq m} a_\alpha(x) D_x^\alpha.$$

**Example 29 (The Laplace-Beltrami operator)** *Let $(M, g)$ be a Riemannian manifold with Riemannian metric $g$. Here we take $E = F = M \times \mathbb{C}$. In a local chart, the metric reads $g(x) = g_{ij}(x)dx_i dx_j$; let $(g^{ij})$ denotes the inverse matrix of $(g_{ij})$ and detg the determinant of $(g_{ij})$. The Laplace-Beltrami operator is defined by*

$$\begin{aligned}\Delta_g &:= -\frac{1}{\sqrt{detg}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \sqrt{detg}\, g^{ij} \frac{\partial}{\partial x_j} \\ &= -\sum_{i,j=1}^{n} g^{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \textit{terms of lower order.}\end{aligned}$$

Setting $g_{ij} = \delta_{ij}$, the Kronecker symbol of $\{i, j\}$ yields back the local expression of the Laplacian on $\mathbb{R}^n$ introduced in Example 28.

The Fourier transform description of differential operators, called the *momentum space* description, is often useful in physics. Given a point $x \in M$ and a local trivialization on a subset $U$ containing $x$, we identify $T_x^* M$ with $\mathbb{R}^n$, $x$ with a point in $\mathbb{R}^n$, and write a local section $u$ of the trivialized bundle $E$ over $U$:

$$u(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{i\langle x, \xi\rangle} \hat{u}(\xi) d\xi$$

where $\langle x, \xi\rangle$ is the inner product on $\mathbb{R}^n$ induced by the Riemannian metric at point $x \in X$. Given a differential operator $A : C^\infty(E) \to C^\infty(F)$ and $u \in C^\infty(E)$ with compact support in some trivialising neighborhood $U$ of a point $x \in M$, the Fourier transform of $Au$ reads:

$$Au(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{i\langle x, \xi\rangle} \sigma(A)(x, \xi)\, \hat{u}(\xi) d\xi = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{i\langle x-y, \xi\rangle} \sigma(x, \xi)\, u(y)\, d\xi\, dy,$$

where

$$\sigma(A)(x, \xi) := \sum_{|\alpha| \leq m} a_\alpha(x) \xi^\alpha$$

is the *total symbol* of $A$, which is only defined locally in a neighborhood of the point $x$. Using Fourier transforms, one can check that the symbol of the product of two differential operators $A : C^\infty(F) \to C^\infty(G)$ and $B : C^\infty(E) \to C^\infty(F)$, where $E$, $F$ and $G$ are three vector bundles over a closed manifold $M$ is given by the star-product of the two symbols $\sigma(A)$ and $\sigma(B)$:

$$\sigma(A) \star \sigma(B) = \sum_\gamma \frac{(-i)^{|\gamma|}}{\gamma!} \partial_\xi^\gamma \sigma(A)(x, \xi) \cdot \partial_x^\gamma \sigma(B)(x, \xi), \tag{23}$$

where $\cdot$ stands for the multiplication of matrices. This is a finite sum since $\sigma(A)$ is polynomial in $\xi$.
In contrast the leading part of the symbol of an operator is globally defined. Indeed, let $x$ and $\tilde{x} = \tilde{x}(x)$ be two systems of coordinates on $U$. The Schwartz property by which one can exchange partial differentiations yields for $|\alpha| = m$:

$$\frac{\partial^{|\alpha|}}{\partial x^\alpha} = \sum_{|\beta|=m} \left[\frac{\partial \tilde{x}}{\partial x}\right]_\beta^\alpha \frac{\partial^{|\beta|}}{\partial \tilde{x}^\beta}$$

where $\left[\frac{\partial \tilde{x}}{\partial x}\right]$ is the symmetrization of the $m$-th tensor product of the matrix $\left(\frac{\partial \tilde{x}}{\partial x}\right)$. Hence for any $|\alpha| = m$, the matrices $a_\alpha(x)$ transforms to:

$$\tilde{a}_\alpha(\tilde{x}) = \sum_{|\beta|=m} a_\beta(x) \left[\frac{\partial \tilde{x}}{\partial x}\right]_\beta^\alpha.$$

Thus the expression $\sum_{|\alpha|=m} a_\alpha(x) \frac{\partial^{|\alpha|}}{\partial x^\alpha}$ extends to a section of $\otimes_{sym}^m TM \otimes Hom(E, F)$. Identifying the symmetrised tensor product $S^m(V) := \otimes_{sym}^m V$ for some finite dimensional vector space $V$ (via the evaluation map) with the space $S^m(V^*)$ of homogeneous polynomials of degree $m$ on $V^*$, we identify this expression with the *leading symbol* of $A$:

$$\sigma_L(A)(x,\xi) := \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha.$$

**Example 30** *The leading symbol of the Laplace-Beltrami operator reads* $\sigma_L(\Delta_g)(x,\xi) = \sum_{i,j=1}^n g^{ij}\xi_i\xi_j := \|\xi\|^2$.

For fixed $x \in M$, it is a homogeneous polynomial of degree $m$ in the variable $\xi \in T_x^*M$. It follows from (23) that the leading symbol behaves multiplicatively:

$$\sigma_L(AB) = \sigma_L(A)\sigma_L(B) \tag{24}$$

for two differential operators $A : C^\infty(F) \to C^\infty(G)$ and $B : C^\infty(E) \to C^\infty(F)$, where $E$, $F$ and $G$ are three vector bundles over a closed manifold $M$. Whenever the manifold $M$ is Riemannian and the bundle $E$ is Hermitian, for a differential operator $A : C^\infty(E) \to C^\infty(E)$ we have

$$\sigma_L(A^*) = \sigma_L(A)^*.$$

In particular, $\sigma_L(A^*A)(x,\xi) = (\sigma_L(A)(x,\xi))^* \ (\sigma_L(A)(x,\xi))$ is a non-negative matrix for any $(x,\xi) \in T_x^*M$.

Given an invertible differential operator, its inverse is not differential any longer hence the need to enlarge the class of differential operators to pseudo-differential ones.

## 8.2 Pseudo-differential operators on manifolds

Useful references are [LM], [Sh], [T].

In what follows we shall only briefly sketch the definitions and properties which can be of use to us later on.

A *pseudo-differential* operator on a closed smooth manifold $M$ is a linear operator $A : C^\infty(E) \to C^\infty(F)$ where $E, F$ are two vector bundles based on $M$ of rank $p, q$ respectively, such that each point $x$ of $M$ has a neighborhood $U$ with coordinates $(x_1, \cdots, x_n)$ over which there are local trivialisations $E_{|_U} \simeq U \times \mathbb{C}^p$ and $F_{|_U} \simeq U \times \mathbb{C}^q$ in which for any section $u$ of $E$ with compact support in $U$, we have:

$$\widehat{Au}(\xi) = \sigma(x,\xi)\hat{u}(x)$$

for some $p \times q$-matrix valued function $\sigma(x,\xi)$ (called the *symbol of A*) obeying the following growth condition. There is a real constant $r$ such that for any multiindices $\alpha, \beta$ there is a constant $C_{\alpha,\beta}$ satisfying the following requirement:

$$|D_x^\alpha D_\xi^\beta \sigma(x,\xi)| \leq C_{\alpha,\beta}(1+\|\xi\|)^{r-|\beta|} \quad \forall (x,\xi) \in T^*X.$$

Let $Sym_{p,q}^r$ denote the set of functions satisfying this requirement (for convenience, we shall often drop the subscripts $(p,q)$).

**Remark 19** *A pseudo-differential operator* $A : C^\infty(E) \to C^\infty(F)$ *has a distribution kernel* $K(x,y)$ *with values in* $F_x \otimes E_y^*$ *where* $F_x$*, resp.* $E_y$ *is the fibre of* $F$ *above* $x$*, resp. of* $E$ *above* $y$*:*

$$Au(x) = \int_M K(x,y)u(y)\,dy \quad \forall u \in C^\infty(E)$$

*which is smooth outside the diagonal. Its singularities are concentrated on the diagonal* $M \times M$*.*

The product formula (23) for differential operators extends to **properly supported** pseudo-differential operators.

**Definition 69** *Given two open subsets* $U, V$ *of* $\mathbb{R}^n$*,*

1. *a distribution* $K \in \mathcal{D}'(U \times V)$ *is* **properly supported** *if the two canonical projection maps* $U \times V \supset \mathrm{Supp}(K) \to U$ *and* $U \times V \supset \mathrm{Supp}(K) \to V$ *mapping* $(x,y)$ *to* $x$ *and* $y$ *respectively are proper maps (i.e., the preimage of a compact set is compact);*

2. *an operator* $A: \mathcal{D}(U) \to C^\infty(U)$ *is* **properly supported** *if its Schwartz kernel is.*

**Example 31** *Differential operators are properly supported since their kernel has support contained in the diagonal.*

**Remark 20** *[Sh, Proposition 3.1] A properly supported operator maps* $\mathcal{D}(U)$ *to* $\mathcal{D}(U)$ *as a consequence of (??). It moreover extends to a a continuous map* $\mathcal{E}'(U) \longrightarrow \mathcal{E}'(U)$*.*

**Example 32** *If* $A$ *is a pseudodifferential operator on* $U$*, then for any* $\chi \in \mathcal{D}(U)$ *and* $\tilde{\chi} \in \mathcal{D}^\infty(U)$ *the operator* $\chi\, A\, \tilde{\chi}$ *is properly supported and its kernel* $\chi(x)\, K(x,y)\, \tilde{\chi}(y)$ *where* $K$ *is the kernel of* $A$*, has compact support in* $U \times U$*.*

**Example 33** *If* $A$ *is a pseudodifferential operator on* $U$*, then for any* $\chi \in \mathcal{D}(U)$ *and* $\tilde{\chi} \in \mathcal{D}^\infty(U)$ *the operator* $\chi\, A\, \tilde{\chi}$ *is properly supported and its kernel* $\chi(x)\, K(x,y)\, \tilde{\chi}(y)$ *where* $K$ *is the kernel of* $A$*, has compact support in* $U \times U$*.*

The formula for the symbol of a composition of properly supported pseudodifferential operators, is derived from that proved for differential operators by replacing the equality sign by an asymptotic expansion of the following type. For symbols $\sigma$ and $\sigma_j, j \in \mathbb{N}$ of decreasing order $m_j$, we set

$$\sigma \sim \sum_{j=1}^\infty \sigma_j,$$

if for each integer $m$ there exists some integer $K$ for which $\sigma-\sum_{j=1}^{K}\sigma_j \in Sym^{-m} \quad \forall k \geq K$. The symbol of the product of two pseudo-differential operators $A : C^\infty(F) \to C^\infty(G)$ and $B : C^\infty(E) \to C^\infty(F)$, where $E$, $F$ and $G$ are three vector bundles over a closed manifold $M$ is given by the star-product of the two symbols $\sigma(A)$ and $\sigma(B)$:

$$\sigma(A) \star \sigma(B) \sim \sum_\gamma \frac{(-i)^{|\gamma|}}{\gamma!} \partial_\xi^\gamma \sigma(A)(x,\xi) \cdot \partial_x^\gamma \sigma(B)(x,\xi). \tag{25}$$

We single out a subclass of pseudo-differential operators, namely those whose symbol is poly-homogeneous.

**Definition 70** *A symbol $\sigma$ on an open subset $U$ of $\mathbb{R}^n$ is classical or poly-homogeneous of order $a \in \mathbb{C}$* [3] *whenever*

$$\sigma \sim \sum_{j=1}^\infty \sigma_{a-j}$$

*with $\sigma_{a-j}$ positively homogeneous of degree $a-j$ in $\xi$ i.e.,*

$$\sigma_{a-j}(x,t\xi) = t^{a-j}\sigma_{a-j}(x,\xi) \quad \forall t>0 \quad \forall (x,\xi) \in T_x^*U.$$

The polyhomogeneous property is preserved under a change of local trivialisation so that we can set the following definition.

**Definition 71** *A pseudo-differential operator whose symbol in a given chart is classical, is called a classical pseudo-differential operator.*

**Example 34**

1. *Differential operators or order $m$ are classical pseudo-differential operators of order $m$.*
2. *The inverse of a differential operator of order $m$ is a classical pseudo-differential operator of order $-m$.*

The leading symbol of a classical pseudo-differential operator of order $a$ is given by

$$\sigma_L(x,\xi) := \sigma_a(x,\xi) \quad \forall (x,\xi) \in T_x^*M.$$

It is globally defined and multiplicative as a consequence of (25) so that formula (24) extends to pseudo-differential operators.
A pseudo-differential operator $A$ whose order is smaller than any negative integer is called a *smoothing* operator.

**Example 35**

1. *A finite rank pseudo-differential operator is smoothing.*
2. *A pseudo-differential operator whose kernel is smooth is a smoothing operator.*

Let $M$ be a closed Riemannian manifold and $E$ be a Hermitian vector bundle over $M$ and let as before $H^s(E)$ denote the $H^s$-Sobolev closure of the space $C^\infty(E)$ of smooth sections of some vector bundle $E$ over $M$. We have the following fundamental result.

**Proposition 26** *Let $E \to M$ and $F \to M$ be two vector bundles over a closed manifold $M$. A pseudo-differential operator $A : C^\infty(E) \to C^\infty(F)$ of real order $a$ extends to a bounded operator*

$$A : H^s(E) \to H^{s-a}(F).$$

[3] The symbol $\sigma$ lies in $Sym^r$ where $r$ is the real part of $a$.

**Proof 44** *(see e.g. [Gi, Lemma 1.3.5 ], [LM, Proposition 3.2, chapter III ]).*
*We need to prove that*

$$|\langle Au, v\rangle| \leq C\, \|u\|_s\, \|v\|_{s-a} \quad \forall u \in H^s(E), v \in H^{s-a}(F).$$

*Using a partition of unity on the base manifold $M$ and localising the operator via a local trivialisation, one can reduce the proof to the case of a pseudo-differential operator $A : C_c^\infty(\mathbb{R}^n) \to C_c^\infty(\mathbb{R}^n)$ whose symbol $\sigma(x,\xi)$ has compact support in $x$ where $n$ is the dimension of $M$. We first observe that*

$$\widehat{Au}(\eta) = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} e^{i\langle x, \xi-\eta\rangle} \sigma(x,\xi)\, \hat{u}(\xi) d\xi\right) dx = \int_{\mathbb{R}^n} \tau(\eta-\xi,\xi)\, \hat{u}(\xi)\, d\xi$$

*where we have set $\tau(\eta,\xi) := \int_{\mathbb{R}^n} e^{i\langle x,\eta\rangle}\sigma(x,\xi)dx$. Using the Plancherel formula, for $u, v$ in $C_c^\infty(\mathbb{R}^n)$ we write*

$$\begin{aligned} \langle Au, v\rangle &= \int_{\mathbb{R}^n} \widehat{Au}(\eta)\hat{v}(\eta)d\eta \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \tau(\eta-\xi,\xi)\, \hat{u}(\xi)\hat{v}(\eta) d\xi d\eta. \end{aligned}$$

*We set*

$$\Psi(\xi,\eta) := \tau(\eta-\xi,\xi)(1+|\xi|)^{-s}(1+|\eta|)^{s-a},$$

*$U(\xi) := \hat{u}(\xi)(1+|\xi|)^s$ and $V(\eta) = \hat{v}(\eta)(1+|\eta|)^{a-s}$. The Cauchy-Schwartz inequality yields*

$$\begin{aligned} |\langle Au, v\rangle| &\leq \int\int |\Psi(\xi,\eta)\, U(\xi) V(\eta)| d\xi d\eta \\ &\leq \left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} |\Psi(\xi,\eta)| d\eta\right) |U|^2(\xi) d\xi\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} |\Psi(\xi,\eta)| d\xi\right) |V|^2(\eta) d\eta\right)^{\frac{1}{2}} \\ &\leq C\|u\|_s\, \|v\|_{s-a}, \end{aligned}$$

*for some positive constant $C$, provided $\int_{\mathbb{R}^n} |\Psi(\xi,\eta)|d\eta$ and $\int_{\mathbb{R}^n} |\Psi(\xi,\eta)|d\xi$ are finite. Since $|\sigma(x,\xi)| \leq C'\|\xi\|^a$ for some constant $C'$ and since the symbol $\sigma$ has compact support in $x$, for any multiple index $\gamma$ we have*

$$|\eta^\gamma\, \tau(\eta,\xi)| \leq \int_{\mathbb{R}^n} |D_x^\gamma \sigma(x,\xi)| dx \leq C'_\gamma (1+\|\xi\|)^a$$

*for some constant $C'_\gamma$. Thus for any positive integer $k$, there is a positive constant $C_k$ such that $|\tau(\eta,\xi)| \leq C_k\, (1+\|\xi\|)^a\, (1+\|\eta\|)^{-k}$ from which it follows that*

$$|\Psi(\eta,\xi)| \leq C_k (1+\|\eta-\xi\|)^{-k}\, (1+\|\xi\|)^{a-s}\, (1+\|\eta\|)^{s-a}.$$

*The triangle inequality yields $(1+\|\xi\|)^r \leq (1+\|\eta\|)^r\, (1+\|\eta-\xi\|)^{|r|}$ for any real number $r$. When applied to $r = a-s$ this gives*

$$|\Psi(\eta,\xi)| \leq C_k (1+\|\eta-\xi\|)^{|a-s|-k}$$

*which is $L^1$ in each of the variables for large enough $k$.*

**Corollary 7** *A pseudo-differential operator $A : C^\infty(E) \to C^\infty(F)$ of negative order extends to a compact operator*

$$A : H^s(E) \to H^s(F)$$

*for any $s \in \mathbb{R}$. In particular, any smoothing operator is compact.*

**Proof 45** *The assertion follows from the compactness of the inclusion $i : H^t(F) \to H^s(F)$ for $t > s$. Indeed, $t := s - a$ with $a$ the order of $A$ is larger than $s$, hence $A : H^s(E) \to H^{s-a}(F)$ composed with the inclusion $i$ is compact as the composition of a bounded and a compact operator, which yields the compactness of $A : H^s(E) \to H^s(F)$. The rest of the proposition easily follows.*

This has straightforward consequences which yield a hint towards more general properties. Coupled with the spectral theorem for compact operators Corollary 7 yields a discrete spectral decomposition for a class of pseudo-differential operators, namely inverses of differential operators.

**Corollary 8** *Let $E$ be a finite rank Hermitian vector bundle over a closed Riemannian manifold $M$. An invertible pseudo-differential operator with positive order $A : C^\infty(E) \to C^\infty(E)$, which is essentially self-adjoint:*

$$\langle Au, v \rangle = \langle u, Av \rangle \quad \forall u, v \in C^\infty(E),$$

*has an infinite set of (non-zero) real eigenvalues $\{\lambda_n, n \in \mathbb{N}\}$ such that $|\lambda_n|$ tends to infinity as $n$ tends to infinity. It has the following discrete spectral resolution:*

$$Au = \sum_{n=1}^{\infty} \lambda_n \langle u_n, u \rangle u_n \quad \forall u \in C^\infty(E)., \tag{26}$$

*with a smooth orthonormal set $\{u_n, n \in \mathbb{N}\}$ of eigenvectors.*

**Remark 21** *Invertibility is not necessary; ellipticity is sufficient. However, we only give the proof in the invertible case, which is rather straightforward.*

**Proof 46** *Let $a$ be the positive order of $A$. By Corollary 7 its inverse, which has negative order $-a$ is a compact operator $B := A^{-1} \in \mathcal{K}(L^2(E))$ (here we have set $s = 0$). It is self-adjoint as the bounded inverse of an essentially self-adjoint operator. By Theorem 9 the sequence $(\mu_n)$ of eigenvalues (which do not vanish in view of the invertibility) of $B$ tends to zero as $n$ goes to infinity and the corresponding orthonormal sequence of eigenvectors $(u_n) \in L^2(E)$ satisfies*

$$A^{-1}v = \sum_{n=1}^{\infty} \mu_n \langle u_n, v \rangle u_n \quad \forall v \in L^2(E). \tag{27}$$

*Recall that $A^{-1} : H^s(E) \to H^{s+a}(E)$ is bounded for any real number $s$. Applied to $s = 0$ this yields that $u_n$ lies in $H^a(E)$ since $A^{-1}u_n = \mu_n u_n$. When iteratively applied to $s = ka$ with $k \in \mathbb{N}$, this procedure yields that $u_n \in H^{ka}(E)$ for any positive integer $k$ so that $u_n$ finally lies in $C^\infty(E)$. Equation (27) applied to $v = Au$ yields $u = \sum_{n=1}^{\infty} \langle v, u_n \rangle u_n$ for any $u$ in $C^\infty(E)$ and hence, by continuity of $A$ on $C^\infty(E)$*

$$Au = \sum_{n=1}^{\infty} \lambda_n \langle v, u_n \rangle u_n \quad \forall u \in C^\infty(E).$$

*Hence the discrete spectral resolution of $A$.*

## 8.3 Elliptic operators, their index and associated heat-operators

The "discrete spectral resolution" proven above when assuming invertibility of the whole operator also holds under the weaker assumption that the leading symbol be invertible. As we shall see, the leading symbol governs many properties of the operator.

**Definition 72** *Let $E$ and $F$ be two finite rank bundles over a closed manifold $M$. A pseudo-differential operator $A : C^\infty(E) \to C^\infty(F)$, whose leading symbol $\sigma_L(A)(x,\xi)$ is an invertible matrix for any $\xi \neq 0$ and any $x \in M$ is called elliptic.*

**Example 36**

1. *The Laplace-Beltrami operator $\Delta_g$ on a Riemannian manifold $(M,g)$ is elliptic for its leading symbol is $\sigma_L(\Delta_g)(x,\xi) = \|\xi\|^2$ is invertible for any $\xi \neq 0$.*
2. *Any invertible pseudo-differential operator is elliptic for if $A$ is invertible so is $\sigma_L(A)$ by the multiplicativity property (24) of the leading symbol.*

From the properties of the leading symbol, it follows that if $A$ is elliptic then so is $A^*$, and hence so is $A^*A$.

**Remark 22** *In fact the injectivity of the leading symbol of $A$ is enough to have the ellipticity of $A^*A$ since $\sigma_L(A)(x,\xi)$ injective implies that $\sigma_L(A^*)(x,\xi) = (\sigma_L(A)(x,\xi))^*$ is onto and hence that $\sigma_L(A^*A) = (\sigma_L(A))^* \, (\sigma_L(A))$ is bijective.*

There are at least two ways to build a pseudo-differential operator whose leading symbol is a given homogeneous symbol $p \in C^\infty(T^*M \otimes \mathrm{Hom}(E,F))$; both use an inverse Fourier transform. The first one uses a partition of unity patching up localised pseudo-differential operators whose Fourier transform is $p$ (see [Gi] par. 1.3.3), the second one uses a Riemannian metric on the base manifold and a connection on the bundle building the associated parallel transport into the inverse Fourier transform (see [LM] formula (3.19)) of $p$. Let us denote by $\mathrm{Op}(\sigma)$ the corresponding pseudo-differential operator, where we have left out the explicit dependence on the various choices, whether a partition of unity or a metric and connection; different choices lead to operators which differ by a smoothing operator.
Such a construction yields an "inverse up to a smoothing operator" $\mathrm{Op}(\sigma_L(A)^{-1})$ (called parametrix) of an elliptic operator $A$ under the mere invertibility of the leading symbol, which is clearly a weaker assumption than the invertibility of the operators since the latter implies the former:

**Proposition 27** *([LM] Theorem 4.3, [Gi] Lemma 1.4.5) Let $E$ and $F$ be two Hermitian vector bundles over a closed Riemannian manifold $M$. Any elliptic pseudo-differential operator $A : C^\infty(E) \to C^\infty(F)$ has a parametrix, i.e. an inverse map up to a smoothing (and hence compact) operator. In other words, there is a pseudo-differential operator $B : C^\infty(F) \to C^\infty(E)$ such that $AB$ and $BA$ differ from the identity map on $C^\infty(E)$, resp. $C^\infty(F)$ by a smoothing operator.*
*If it is of real order $a$, it extends to a Fredholm operator (denoted here by the same letter $A$):*

$$A : H^s(E) \to H^{s-a}(F).$$

*More precisely,*

- *$Ker\, A$ and $Ker A^*$ are finite dimensional vector subspaces of $C^\infty(E)$ and $C^\infty(F)$ respectively.*

- *Decomposition theorem*

$$H^s(E) = \text{Ker}(A) \oplus \text{R}(A^*) \quad \forall s \in I\!R,$$

$$H^s(F) = \text{Ker}(A^*) \oplus \text{R}(A) \quad \forall s \in I\!R,$$

*where the sums are orthogonal w.r.to the $H^s$-inner product,*

$$C^\infty(E) = \text{Ker}(A) \oplus \text{R}(A^*),$$

$$C^\infty(F) = \text{Ker}(A^*) \oplus \text{R}(A)$$

*where the sums are orthogonal w.r.to the $L^2$-inner product.*

In view of property (18) we can define the *index* of an elliptic pseudo-differential operator $A : C^\infty(E) \to C^\infty(F)$, where $E \to M$ and $F \to M$ are two finite rank Hermitian vector bundles over a closed closed manifold $M$:

$$\text{ind}(A) = \text{dimKer}(A) - \text{dimR}(A). \tag{28}$$

Given two operators $A_0$ and $A_1$ with the same leading symbol as $A$, their convex sum $A(\varepsilon) := \varepsilon A_1 + (1 - \varepsilon)A_0$ has the same leading symbol for any $\varepsilon \in [0, 1]$ and defines a continuous one-parameter family of elliptic operators. Since the index is locally constant, $\text{ind}(A_0) = \text{ind}(A_1)$ so that the index only depends on the leading symbol.

Similar arguments to the one used in Corollary 8 further yield a discrete spectral resolution, under the weaker ellipticity assumption. The inverstibility of the operator assumed in Corollary 8 is replaced hre by the inverstibility of its leading symbol.

**Theorem 13** *[Gi] (Lemma 1.6.3 and Lemma 1.12.6) Let $E \to M$ be a Hermitian vector bundle over a closed Riemannian manifold $M$. Any essentially self-adjoint elliptic differential operator $A : C^\infty(E) \to C^\infty(F)$ of positive order* $\text{ord}(A)$ *admits a discrete spectral resolution*

$$Au = \sum_{n=1}^{\infty} \lambda_n \langle u, u_n \rangle u_n \quad \forall u \in C^\infty(E),$$

*where $(\lambda_n)$ is the sequence of eigenvalues, which tends to infinity as $n$ tends to infinity, and $(u_n)$ is an orthonormal set in $C^\infty(E)$ for the $L^2$ product induced by the Riemannian metric on $M$ and the Hermitian product on $E$.*
*Ordering the eigenvalues such that $|\lambda_1| \leq |\lambda_2| \leq \cdots$ we have*

$$|\lambda_n| \sim Cn^{\frac{\text{ord}(A)}{\dim M}}.$$

Let $E \to M$ be a Hermitian vector bundle over a closed Riemannian manifold $M$. A (formally) self-adjoint elliptic differential operator $A : C^\infty(E) \to C^\infty(E)$ with positive leading symbol satisfies the following condition ([Gi] Lemma 1.6.4)

$$\exists C > 0, \quad \langle Au, u \rangle \geq -C|u|^2 \quad \forall u \in C^\infty(E).$$

Thus, the positivity of the leading symbol implies the existence of a lower bound of the whole operator, and hence of its spectrum.

Since the spectrum is purely discrete, it only contains a finite number of non-positive eigenvalues, in which case we have the following asymptotic behaviour:

$$\lambda_n \sim C n^{\frac{\mathrm{ord}(A)}{\dim M}}.$$

The heat equation, associated with a (formally) self-adjoint elliptic differential operator $A : C^\infty(E) \to C^\infty(F)$ of positive order $\mathrm{ord}(A)$ with positive leading symbol is a system of equations:

$$\begin{aligned} (\partial_t + A)u(x,t) &= 0 \quad \text{(evolution equation)} \\ \lim_{t\to 0} u(x,t) &= u(x) \quad \text{(initial condition)}. \end{aligned} \tag{29}$$

**Remark 23** *When $A = \Delta_g$ is the laplace-Beltrami operator on a Riemannian manifold, this equation allegedly governs the diffusion of heat from an initial distribution $u(x) = u(0,x)$ on the manifold under consideration.*

The spectral resolution of a self-adjoint differential operator given by Theorem 13 yields the existence and uniqueness of the solution.

**Proposition 28** *Let $E$ and $F$ be two Hermitian vector bundles over a closed Riemannian manifold $M$. The heat-equation (29) associated with a (formally) self-adjoint elliptic differential operator $A : C^\infty(E) \to C^\infty(F)$ of positive order $\mathrm{ord}(A)$ with positive leading symbol admits a unique solution:*

$$\left(e^{-tA}u\right)(x,t) = \sum_{n=1}^{\infty} e^{-t\lambda_n} \langle u, u_n\rangle\, u_n = \int_M K_t(x,y)\, u(y)\, dy$$

*where $Au = \sum_{n=1}^{\infty} \lambda_n \langle u_n, u\rangle\, u_n \quad \forall u \in C^\infty(E)$ is the discrete resolution of $A$ and where*

$$K_t(x,y) := \sum_{n=1}^{\infty} e^{-t\lambda_n}\, u_n(x) \otimes u_n^*(y) \in F_x \otimes E_y^*,$$

*is a smooth kernel function for $t > 0$, called the heat-kernel of $A$.*

**Proof 47** *We refer the reader to [Gi] Lemma 1.6.5; we nevertheless observe that since the eigenvalues are positive up to a finite number, the smoothness of the heat-kernel follows from the convergence of $\sum_{n=1}^{\infty} \lambda_n^k\, e^{-\lambda_n}$ for any non-negative integer $k$.*

**Example 37** *The Laplacian $\Delta_{S^1}$ on the unit circle $S^1$ associated with the canonical metric on $S^1$ induced from that on $\mathbb{R}$ and acting on*

$$C^\infty(S^1) = \{f \in C^\infty([0,2\pi]), \quad f^{(k)}(0) = f^{(k)}(2\pi) \quad \forall n \in \mathbb{Z}_{\geq 0}\}$$

*has purely discrete spectrum $\{n^2, n \in \mathbb{Z}\}$ and the solution to the associated heat-operator equation reads:*

$$u(t,\theta) = \sum_{n\in\mathbb{Z}} e^{-n^2 t} \langle u, e^{i\,n\theta}\rangle\, e^{in\theta}$$

*with the scalar product given by $\langle u, v\rangle := \frac{1}{2\pi}\int_0^{2\pi} u(\theta)\, \overline{v(\theta)}\, d\theta$.*

The heat-operator associated with an essentially self-adjoint elliptic differential operator $A$ has a well-defined trace

$$\mathrm{tr}\left(e^{-tA}\right) = \sum_{n=1}^{\infty} e^{-t\lambda_n}$$

with the above notations. In particular the trace of the heat operator $e^{-t\Delta_g}$ associated with the Laplace-Beltrami on a manifold equipped with a Riemannian metric $g$ played an important historical role insofar that its asymptotic expansion as $t$ tends to zero contains information on the manifold.

**Example 38** *In the case of the Laplacian on the circle we have [R] Theorem 1.12, a result which dates back to Jacobi (ca. 1870):*

$$\mathrm{tr}\left(e^{-t\Delta_{S^1}}\right) = \sum_{n\in\mathbb{Z}} e^{-n^2 t} \sim_{t\to 0} \frac{2\pi}{\sqrt{4\pi t}},$$

*where the numerator* $2\pi$ *is to be interpreted as the length of the unit circle.*

This one-dimensional example generalises to the Laplace-Beltrami operator $\Delta_g$ on a closed $n$-dimensional Riemannian manifold $(M, g)$ for which we have

$$\mathrm{tr}\left(e^{-t\Delta_g}\right) \sim_{t\to 0} \mathrm{Vol}(M)\, t^{-\frac{n}{2}}. \tag{30}$$

The issue as to how much geometric information on the manifold one can get from the spectral information contained in this asymptotic expansion has been nicely formulated in the title "*Can you hear the shape of a drum?*" of a famous article by Kac [Kac].

**Remark 24** *(see e.g. [R] Corollary 3.25) In particular, isospectral closed Riemannian manifolds i.e., such that the associated Laplace-Beltrami operators have the same eigenvalues counted with multiplicity, have the same dimension* $n$ *and the same volume* $\mathrm{Vol}(M)$.

## 8.4 From heat-kernel expansions to zeta functions

The *Mellin transform* relates complex powers to exponentials. The Gamma function

$$\Gamma(s) = \int_0^\infty t^{s-1}\, e^{-t}\, dt \quad ,$$

is holomorphic on the half-plane $\mathrm{Re}(s) > 0$ and extends to a meromorphic function on the plane with simple poles at negative integers. A change of variable $t := \lambda\, u$ yields the Mellin transform

$$\lambda^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1}\, e^{-t\lambda}\, dt \quad \forall \lambda > 0. \tag{31}$$

With the same notations as (26) we define the complex power $(A')^{-s}$ of the restriction $A'$ of the operator $A$ to the orthogonal subspace to its (finite dimensional) kernel, via its spectral resolution by:

$$(A')^{-s}\, u = \sum_{n=1, \lambda_n \neq 0}^{\infty} \lambda_n^{-s}\, \langle u_n, u\rangle\, u_n \quad \forall u \in C^\infty(E), \tag{32}$$

where for negative eigenvalues, we use some determination of the logarithmic which we omit to explicit here. Since the eigenvalues $\lambda_n$ of $A$ behave asymptotically as $\lambda_n \sim C n^{\frac{a}{n}}$ the complex power $(A')^{-s}$ of the operator $\zeta$-function has a well-defined trace on the half-plane $\mathrm{Re}(s) > \frac{n}{a}$ called the $\zeta$*-function* associated with $A$.

$$\zeta_A(s) := \mathrm{tr}\left((A')^{-s}\right) := \sum_{n\in I\!N, \lambda_n\neq 0} \lambda_n^{-s}. \tag{33}$$

Replacing $e^{-t\lambda}$ in (31) by $\mathrm{tr}\left(e^{-tA'}\right) = \sum_{n\in I\!N,\lambda_n\neq 0} e^{-t\lambda_n}$ we get

$$\zeta_A(s) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\,\mathrm{tr}\left(e^{-tA'}\right)\, dt. \tag{34}$$

The asymptotic behaviour (30) of the heat-operator for the Laplace-Beltrami operator generalises to any self-adjoint elliptic differential operator with positive leading symbol $A$ acting on a space of smooth sections of a hermitian vector bundle $E$ over a closed Riemannian manifold $M$. Indeed, there are some constants $\alpha_j = \int_M \alpha_j(x)\, dx, j \in Z\!\!\!Z_{\geq 0}$ such that (see [Gi] Lemma 1.8.2)

$$\mathrm{tr}\left(e^{-tA}\right) = \sum_{j=0}^{J} \alpha_j\, t^{\frac{j-n}{a}} + o(t^{\frac{J-n}{a}}), \quad \forall J \in I\!N, \tag{35}$$

where $n$ is the dimension of the manifold $M$ and $a$ the order of $A$, which we assume to be positive.

**Proposition 29** *Let $A$ be a self-adjoint elliptic differential operator with positive leading symbol $A$ acting on a space of smooth sections of a hermitian vector bundle $E$ over a closed Riemannian manifold $M$. Its $\zeta$-function –which is holomorphic on the half-plane* $\mathrm{Re}(s) > \frac{n}{a}$*– extends to a meromorphic function on the plane with simple poles which is holomorphic at zero and we have*

1. $\Gamma(s)\,\zeta_A(s) = \sum_{j<J} \frac{\alpha_j}{s-\frac{n+j}{a}} - \frac{\mathrm{dimKer}(A)}{s} + R_J(s)$, *where $R_J$ is a holomorphic map on* $\mathrm{Re}(s) > \frac{n-J}{a}$,
2. $$\zeta_A(0) = \mathrm{fp}_{t=0}\mathrm{tr}\left(e^{-tA}\right) = \alpha_n. \tag{36}$$

**Proof 48** *We split $\Gamma(s)\,\zeta_A(s)$ in a holomorphic part $\int_1^\infty t^{s-1}\,\mathrm{tr}\left(e^{-tA'}\right)\, dt$ and a meromorphic part $\int_0^1 t^{s-1}\,\mathrm{tr}\left(e^{-tA'}\right)\, dt$ in which we insert the asymptotic expansion of $\mathrm{tr}\left(e^{-tA'}\right) = \mathrm{tr}\left(e^{-tA}\right) - \mathrm{dimKer}(A)$ induced by (35). The result then follows from the asymptotic expansion $\Gamma(s) \sim_0 s^{-1}$.*

# 9 A brief incursion into index theory

Let $f : E \to F$ be a linear map between two finite dimensional vector spaces $E$ and $F$. Then

$$\dim\mathrm{Ker}(f) - \dim\mathrm{Coker}(f) = \dim(E) - \dim(F)$$

since $f$ induces an isomorphism under any complementary subspace of $\mathrm{Ker}(f)$ to a $\mathrm{Im}(f)$. Although the dimensions of the kernel and the range of $f$ depend on $f$, this formula shows that their difference does not.

If now $A : C^\infty(E) \to C^\infty(F)$ is an elliptic operator acting on smooth sections of a Hermitian vector bundle $E$ over a closed Riemannian manifold $M$ to smooth sections of a Hermitian vector bundle $F$ over $M$, by Proposition 27 it has finite dimensional kernel and cokernel so that

$$\mathrm{ind}(A) = \dim\mathrm{Ker}(A) - \dim\mathrm{Coker}(A)$$

is well-defined in spite of the infinite dimensionality of the source and the target spaces of $A$. Both the the dimensions of the kernel and the cokernel of $A$ depend on $A$ but by Theorem 11, their difference is insensitive to "small" (i.e. compact) variations of $A$. The index problem for elliptic differential operators was posed by Israel Gel'fand [Ge] in 1960; in view of the homotopy invariance of the index he asked for a formula involving topological invariants.

## 9.1 From the Gauss-Bonnet to the Atiyah-Singer theorem

The answer to Gelfands question was announced by M.F. Atiyah and I.M. Singer (1963) and later proved together with various generalisations in a series of papers by the two authors.

Let $M$ be an even dimensional spin manifold and let $E = S \otimes W$ be the tensor product of the spinor bundle $S$ of $M$ and some exterior bundle $W$. Since $S = S^+ \oplus S^-$ is $\mathbb{Z}_2$-graded, so is $E = E^+ \oplus E^-$. Let $D^+ : C^\infty(E^+) \to C^\infty(E^-)$ be the corresponding Dirac operator. The *Atiyah-Singer index theorem* (see e.g. [BGV], [LM]) expresses the index of $D^+$ in terms of topological invariants, namely as the integral over the base manifold of some Chern-Weil forms, namely the $\hat{A}$-genus on $M$ and the Chern character on $W$ described at the end of chapter 2:

$$\mathrm{ind}(D^+) = \frac{1}{(2i\pi)^{\frac{n}{2}}} \int_M \hat{A}(\nabla) \wedge \mathrm{ch}(\nabla^W).$$

There have since then been various proofs of the Atiyah-Singer theorem, among which one by E. Getzler (1985) which is a purely analytic proof (based on ideas of E. Witten and Alvarez-Gaume) involving the asymptotic expansion of the heat-kernel of the square of the Dirac operator combined with a clever algebraic calculation of the top order of the asymptotic expansion since then called "Getzler rescaling" [BGV]. A proof of the Atiyah-Singer index theorem using K-theory combined with the heat-kernel asymptotic expansion can be found in [Gi].

An important ingredient in the various proofs is the Mac-Kean and Singer formula. Given a graded hermitian vector bundle $E = E^+ \oplus E^-$ on a closed Riemannian manifold $M$. An elliptic operator $D^+ : C^\infty(E^+) \to C^\infty(E^-)$ together with its (formal) adjoint $D^-$ give rise to a self-adjoint operator $D = \begin{pmatrix} 0 & \Delta^- \\ D^+ & 0 \end{pmatrix}$, whose square reads $\Delta = \Delta^+ \oplus \Delta^- := D^- D^+ \oplus D^+ D^-$. The operator $D$ being self-adjoint and elliptic has a discrete resolution and it is easy to check that

$$\mathrm{Spec}(\Delta^+) \setminus \{0\} = \mathrm{Spec}(\Delta^+) \setminus \{0\}. \tag{37}$$

On the other hand, from the asymptotic behaviour of the eigenvalues $\{\lambda_n^+, n \in I\!N\}$ of $\Delta^+$ and of the eigenvalues $\{\lambda_n^-, n \in I\!N\}$ of $\Delta^-$, we know that for any positive $\varepsilon$

$$\begin{aligned} \mathrm{str}\left(e^{-\varepsilon\Delta}\right) &:= \mathrm{tr}\left(e^{-\varepsilon\Delta^+}\right) - \mathrm{tr}\left(e^{-\varepsilon\Delta^-}\right) \\ &:= \sum_{n\in I\!N} e^{-\varepsilon\,\lambda_n^+} - \sum_{n\in I\!N} e^{-\varepsilon\,\lambda_n^-} \end{aligned}$$

By (37) the terms involving non-zero eigenvalues cancel and we have the *Mac-Kean Singer equation:*

$$\mathrm{str}\left(e^{-\varepsilon\Delta}\right) = \mathrm{ind}(D^+). \tag{38}$$

Combined with (39) this yields

$$\mathrm{ind}(D^+) = \mathrm{fp}_{\varepsilon=0}\mathrm{str}\left(e^{-\varepsilon\,\Delta}\right) = s\zeta_\Delta(0), \tag{39}$$

where $s\zeta_\Delta := \zeta_{\Delta^+} - \zeta_{\Delta^-}$.

Index theorems go back to Gauss. Let $S$ be an oriented smooth surface embedded in $I\!R^3$. We consider the corresponding *Gauss map*

$$\begin{aligned} N : S &\to S^2 \\ x &\mapsto N_x \end{aligned}$$

that takes a point on the surface to the unit normal vector $N_x$ at point $x$ pointing outwards. The end point of the vector $N_x$, which is identified with the vector, lies on the 2-dimensional unit sphere $S^2$. Since $\|N_x\|^2 = 1$, the tangent vector $T_x N_x$ at the end point $N_x$ is orthogonal to $N_x$ so that the tangent spaces $T_{N_x} S^2$ and $T_x S$ are isomorphic. We can therefore consider the tangent map $T_x N : T_x S \to T_{N(x)} S^2$ to the Gauss map as an endomorphism of $T_x S$. The *Gaussian curvature* $K(x)$ at a point $x$ in $S$ – which measures the deformation of an infinitesimal area on $S$ around $x$ under the Gauss map (see e.g. [R] Chapter 2, paragraph 2.1) – corresponds to its determinant:

$$K(x) := \det(N_x). \tag{40}$$

**Example 39** *The curvature of the standard* 2*-sphere of radius* $r$ *is given by* $r^{-2}$.

As shown by Gauss in his famous Theorema Egregium, in spite of its apparent dependence on the embedding of $S$ in $I\!R^d$, the Gaussian curvature $K$ can be defined intrinsically in terms of the Christoffel symbols. As the following theorem shows, the Gaussian curvature has an influence on the topology of the surfacec.

**Theorem 14 (Gauss-Bonnet theorem)** *For a closed oriented surface* $S$ *of genus* $p$ *in* $I\!R^3$

$$\chi(M) = \int_S K\, dA \tag{41}$$

*where* $\chi(S) := 2 - 2p$ *is the Euler characteristic of* $S$ *and* $dA$ *the infinitesimal area element induced by the metric on* $I\!R^3$.

In view of higher dimensional generalisations, it is useful to recognise (49) as an index theorem; for this we interpret the topological invariant given by the Euler chracteristic in terms of the index of a Dirac-type operator of forms. We shall then very modestly state two index theorems for Dirac-type operators on forms, leaving out the proofs which would lead us out of the scope of these notes.

### 9.2 Generalised Laplacians and generalised Dirac operators

Generalised Laplacians and generalised Dirac operators are useful examples of elliptic operators.

**Definition 73** *A generalised Laplacian on a vector bundle $E$ over a closed Riemannian manifold $M$ is a second order differential operator $A$ such that $\sigma_L(A)(x,\xi) = \|\xi\|^2$, where $\|\cdot\|$ is the norm induced by the metric.*

**Example 40** *The Laplace-Beltrami operator on a Riemannian manifold is a generalised Laplacian.*

A generalised Laplacian $A$ is elliptic since $\sigma_L(A)(x,\xi) = \|\xi\|^2$ is invertible whenever $\xi \neq 0$.

Let $E \to M$ be a vector bundle based on a Riemannian manifold $M$ and let $E$ be equipped with a connection $\nabla^E$. The Levi-Civita connection $\nabla$ on $M$ yields a connection $\nabla^{T^*M}$ on $T^*M$ which, when combined with $\nabla^E$, yields a connection $\nabla^{T^*M\otimes E} = \nabla^{T^*M} \otimes 1 + 1 \otimes \nabla^E$ on $T^*M \otimes E$. Composed with $\nabla^E$, this yields an operator $\nabla^{T^*M\otimes E}\nabla^E : C^\infty(E) \to C^\infty(T^*M \otimes T^*M \otimes E)$. Using the metric on $C^\infty(TM \otimes TM)$, by contraction, one can build its trace to obtain a second-order differential operator

$$\Delta^E := -\mathrm{tr}(\nabla^{T^*M\otimes E}\nabla^E),$$

which defines a generalised Laplacian on $C^\infty(E)$.

**Example 41** *When $E := M \times K$, and $\nabla^E = \nabla$, the Levi-Civita connection on $M$, it yields back the Laplace-Beltrami operator and we have:*

$$\Delta_g = -\mathrm{div} \circ \nabla = \nabla^*\nabla,$$

*where* div *denotes the divergence defined by:*

$$-\langle \mathrm{div} U, f\rangle = \langle U, \nabla f\rangle \quad \forall f \in C^\infty(M), U \in C^\infty(TM),$$

*and where $\nabla^*$ stands for the adjoint (in the operator sense) of the connection $\nabla$. Indeed, in local coordinates the divergence reads* $\mathrm{div} U = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x_j}(U^j\sqrt{g})$, *which combined with the local formula $(\nabla f)^j = \sum_{i,j=1}^n g^{ij}\partial_i f$ yields the first identity.*

**Example 42** *More generally, take $E = \Lambda T^*M$ equipped with the connection $\nabla^E$ induced by the Levi-Civita connection on $M$, then*

$$\Delta^{\Lambda T^*M} = -tr(\nabla^{T^*M\otimes\Lambda T^*M}\nabla^{\Lambda T^*M}) = \nabla^*\nabla$$

*where we have set for short $\nabla = \nabla^{\Lambda T^*M}$ and $\nabla^*$ its adjoint.*

The operator $\Delta^E$ is non-negative in the following sense:

$$\langle \Delta^E u, u\rangle = \langle \nabla^E u, \nabla^E v\rangle \geq 0 \quad \forall u, v \in C^\infty(E).$$

Following Dirac, we now look for a differential operator $D^E$ whose square is a Laplacian $\Delta^E$. When $E$ is the trivial bundle $E = \mathbb{R}^n \times \mathbb{C}$, $\Delta^E$ is the ordinary Laplacian $\Delta = -\sum_{i=1}^n \frac{\partial^2}{\partial^2 x_i}$ on $\mathbb{R}^n$. Looking for an operator $D = \sum_{i=1}^n c_i \frac{\partial}{\partial x_i}$ such that $D^2 = \Delta$ leads to the Clifford relations of section 3.5, namely:

$$c_i c_j + c_j c_i = -2\delta_{ij}.$$

$D = \sum_{i=1}^n c_i \frac{\partial}{\partial x_i}$ then yields a first order differential operator which provides a square root of the Laplacian.
This extends to manifolds and bundles up to the fact that one does not generally get an exact identification of the square of the Dirac operator with the canonical Laplacian anymore due to extra terms involving the geometry of the manifold and the bundle. However such an identification still holds "up to lower order operators" i.e., on the level of leading symbols, hence the following definition.

**Definition 74** *A generalised Dirac operator is a first order differential operator $D$ whose square is a generalised Laplacian i.e., whose leading symbol $\sigma(D)(x,\xi)$ satisfies*

$$\sigma_L(D^2)(x,\xi) = \|\xi\|^2 \quad \forall (x,\xi) \in T_x^* M.$$

**Remark 25** *Locally, a solution of the equation $\left(\sigma(D)(x,\xi)\right)^2 = \|\xi\|^2 \quad \forall (x,\xi) \in T_x^* M$ is given by $\sigma(D)(x,\xi) = \sum_{i=1}^n c_i\, \xi_i$ where the $c_i$ satisfy the Clifford relations.*

Extending Dirac's construction to build a "square root" of a Laplacian $\Delta^E$ "up to lower order terms" acting on sections of a vector bundle $E$, requires the use of a Clifford connection. We use here the notations of section 3.5.
Let $M$ be a Riemannian manifold and let $E \to M$ be a Clifford module based on $M$. A connection $\nabla^E$ on $E$ is called a *Clifford connection* if it commutes with the Clifford multiplication on $E$:

$$[\nabla^E, c(a)] = c(\nabla a) \quad \forall a \in C^\infty(C(M)),$$

where $C(M)$ is the bundle of Clifford algebras on $M$. Here $\nabla$ is the Levi-Civita connection on $M$.

**Example 43**

- **The exterior power of the cotangent bundle:** *Let $E = \Lambda T^* M$ be equipped with the connection $\nabla^E$ induced by the Levi-Civita connection on $M$. Then $\nabla^E$ is a Clifford connection for the Clifford action $c(v) = \varepsilon(v) - i(v) \quad \forall v \in C^\infty(TM)$ on $\Lambda T^* M$ seen as a Clifford module.*

- **Spinor bundles:** *Let $E = S \otimes W$ where $S$ is the spinor bundle on $M$. The Clifford module acts on $E$ via a Clifford multiplication $c$. Since $Spin(V)$ is a finite covering of $SO(V)$, we can lift the Levi-Civita connection on $SO(TM)$ to a connection on the spinor bundle $S$. Combined with a connection $\nabla^W$ on $W$, it yields a Clifford connection $\nabla^{S\otimes W}$ on $S \otimes W$.*

- **Spin$^c$ bundles:** *Let $M$ be a Spin$^c$-manifold, so that the orthonormal frame bundle $SO(TM) \to M$ lifts to some Spin$^c(V)$-bundle where $V$ is the model* *space for $M$. Since $Spin^c(V) \to SO(V)$ is not a finite covering, the Levi-Civita connection on $M$ does not automatically lift to a connection on $Spin^c(V)$. We need additional information, namely a connection on the $SO(V) \times S^1$-bundle obtained from the quotient of $Spin^c(V)$ by $\pm$, which lifts to a connection on $Spin^c(V)$. This connection $\tilde{\nabla}$ is obtained from combining the Levi-Civita connection on $M$ with a connection on the $U(1)$ bundle obtained from $Spin^c(V)$ by dividing out by $Spin(V)$.*

Using a Clifford connection $\nabla^E$ we consider a first order differential operator

$$D^E := \sum_{i=1}^{n} c(e^i)\nabla^E_{e_i}$$

called a *Dirac operator.*

**Example 44** • **A Dirac operator associated to the de Rham operator:** *By Proposition 10 $c := \varepsilon - i$ defines a Clifford multiplication on $\Omega(M) = C^\infty(\Lambda T^*M)$ and by means of the Levi-Civita connection $\nabla$, the operator $d + d^*$ can be interpreted as a Dirac operator acting on sections of the Clifford module $\Lambda T^*M$ (compare with (6):*

$$D^{T^*M} := \sum_{i=1}^{n} c(e^i)\nabla^E_{e^i} = d + d^*. \tag{42}$$

- **The twisted Dirac operator:** *When $M$ is a spin manifold, $S$ the spinor bundle and $W$ an exterior vector bundle based on $M$ with connection $\nabla^W$, the operator*

$$D^{S\otimes W} := \sum_{i=1}^{n} c(e^i)\nabla^{S\otimes W}_{e_i}$$

*is called a twisted Dirac operator. In the absence of exterior bundle $W$, i.e. when $E = S$, it is often denoted by $D$ and simply called the Dirac operator on $M$. When the dimension of $M$ is odd, $D^{S\otimes W}$ is an essentially self-adjoint Dirac operator (on the adequate domain), when the dimension is even, $E = E^+ \oplus E^- = S^+ \otimes W \oplus S^- \otimes W$ is $\mathbb{Z}_2$-graded and $D$ is odd for this grading, i.e. $D^{S\otimes W} = \begin{pmatrix} 0 & \left(D^{S\otimes W}\right)^- \\ \left(D^{S\otimes W}\right)^+ & 0 \end{pmatrix}$.*

- **Dirac operators for Spin$^c$-structures:** *To a connection $\tilde{\nabla}$ on a $Spin^c$-bundle, there is also an associated Dirac operator :*

$$\tilde{D} = \sum_{i=1}^{n} c(e_i)\tilde{\nabla}_{e_i}.$$

## 9.3 The Bochner-Weitzenböck and Lichnerowicz formulae

In general, the square of a Dirac operator $D^E$ only coincides with the Laplacian $\Delta^E$ up to a zero-order differential operator as we shall see from the Lichnerowicz and Bochner-Weitzenböck formulae below.
The square of $D^{\Lambda T^*M}$ gives rise to the *Hodge Laplacian*

$$\Delta = d^*d + dd^*$$

acting on forms on a smooth manifold $M$. Here $d$ is the exterior differential on forms and $d^*$ its adjoint for the natural $L^2$-product on form.s This operator relates to the *Bochner Laplacian* $\Delta^{\Lambda T^*M}$ by a *Bochner-Weizenböck* relation:

**Proposition 30**

$$(d+d^*)^2 = \Delta^{\Lambda T^*M} + \sum_{i<j} c(dx_i)c(dx_j)\Omega(e_i, e_j)$$

*where* $\Omega(u,v) := [\nabla_u, \nabla_v] - \nabla_{[u,v]}$ *is the curvature tensor on* $\Lambda T^*M$ *equipped with the connection induced by the Levi-Civita connection.*

**Proof 49** *Since the operators involved in the equality to be proven are differential operators and since the curvature operator is tensorial, the proof can be carried out choosing an orthornormal tangent frames* $(e_1(x), \cdots, e_n(x))$ *at a given point* $x$ *and does not depend on the way we extend it to a field of orthonormal frames in a neighborhood of* $x$. *We choose to extend it to a field of orthonormal frames* $(e_1, \cdots, e_n)$ *such that* $(\nabla e_j)_x = 0$ *at point* $x \in M$.

$$\begin{aligned}
(d+d^*)^2\alpha &= \sum_{i,j=1}^{n} c(dx_i)\nabla_i(c(dx_j)\nabla_j\alpha) \\
&= \sum_{i,j=1}^{n} c(dx_i)c(dx_j)\nabla_i(\nabla_j\alpha) \\
&= -\nabla_i \sum_{i=1}^{n} \nabla_i\nabla_j\alpha + \sum_{i<j}^{n} c(dx_i)c(dx_j)\left(\nabla_i\nabla_j - \nabla_j\nabla_i\right)\alpha \\
&= \Delta^{\Lambda T^*M}\alpha - \sum_{i<j} c(dx_i)c(dx_j)\Omega(e_i, e_j)\alpha.
\end{aligned}$$

In order to relate the square of the twisted Dirac operator $D^{S\otimes W}$ on a spin manifold with $\Delta^{S\otimes W}$ we need the notion of twisting curvature. Let us set $E = S \otimes W$ equipped with the Clifford connection $\nabla^E := \nabla^S \otimes 1 \oplus 1 \otimes \nabla^W$ which combines the spin-connection $\nabla^S$ induced by the Riemannian connection on $M$ and the connection $\nabla^W$ on the exterior bundle.

We first observe that the curvature $\Omega^E$ of $\nabla^E$ decomposes under the isomorphism $\mathrm{End}(E) \simeq C(M) \otimes End_{C(M)}(E)$ as follows

$$\left(\nabla^E\right)^2 = R + F^{E/S}$$

where $R$ is a $C(M)$- valued 2-form $M$ given by the formula

$$R(e_i, e_j) = \frac{1}{4} \sum_{k,l=1}^{n} \langle \Omega(e_i, e_j)e_k, e_l) c(e^k)c(e^l)\rangle$$

where $\Omega$ is the Riemannian curvature on $M$ and $e_i, i = 1, \cdots, n$ is an orthonormal frame of the tangent bundle $TM$ and $e^i, i = 1, \cdots, n$ the dual frame. The remaining two form $F^{E/S}$ is called the *twisting curvature*, which in this case coincides with the curvature $\Omega^W$ on the exterior bundle $W$.

The square of a Dirac operator $D^E$ differs from the Laplacian $\Delta^E$ by a term involving the scalar curvature of $M$ and the twisting curvature of the Clifford connection $\nabla^E$ as can be seen from the *Lichnerowicz formula* (see e.g. Theorem 3.52 of [BGV]) or equivalently the general Bochner identity (see Theorem 8.2 of [LM]), which relates the square $(D_E)^2$ of the twisted Dirac operator $D^E := \sum_{i=1}^n c(e_i)\nabla^E_{e_i}$ with the Laplace-Beltrami operator

$$\begin{aligned}\Delta^E &= \operatorname{tr}\left(\nabla^{T^*M\otimes E}\nabla^E\right)\\ &= -\sum_{i=1}^n \left(\nabla^{T^*M\otimes E}\nabla^E\right)_{e_i,e_i}\\ &= -\sum_{i=1}^n \left(\nabla^E_{e_i}\nabla^E_{e_i} - \nabla^E_{\nabla^E_{e_i}e_i}\right) \qquad (43)\end{aligned}$$

associated with the superconnection $\nabla^E$ on $E$, where $\nabla^{T^*M\otimes E}$ is the connection induced on the tensor product bundle $T^*M\otimes E$ by the Levi-Civita connection on $M$ and the connection $\nabla^E$ on $E$. Here $\{e_i, i=1,\cdots,n\}$ is a local orthonormal tangent frame.

**Proposition 31**

$$\left(D^E\right)^2 = \Delta^E + R^E = \Delta^E + R^W + \frac{s}{4}, \qquad (44)$$

*where $r_M$ stands for the scalar curvature on $M$ and*

$$R^E := \sum_{i<j} c(e_i)\, c(e_j)\, \left(\nabla^E\right)^2(e_i,e_j); \quad R^W := \sum_{i<j} c(e_i)\, c(e_j)\, \left(\nabla^W\right)^2(e_i,e_j). \qquad (45)$$

*In particular, for a flat auxillary bundle we have:*

$$D_W^2 = \Delta_M + \frac{s}{4},$$

*where $\Delta_M$ is the Laplace-Beltrami operator on the Riemannian manifold $M$.*

**Proof 50** *We choose a local orthonormal tangent frame $\{e_i, i=1,\cdots,n\}$ at point $x\in M$ such that $\left(\nabla^E_{e_i}\right)_x = 0$ for all $i\in\{1,\cdots,n\}$. Since $D^E = \sum_{i=1}^n c(e_i)\,\nabla^E_{e_i}$, at that point $x$ we have:*

$$\begin{aligned}\left(D^E\right)^2 &= \sum_{i,j=1}^n c(e_i)\,\nabla^E_{e_i}\, c(e_j)\,\nabla^E_{e_j}\\ &= \sum_{i,j=1}^n c(e_i)\, c(e_j)\, \left[\left(\nabla^E\right)^2(e_i,e_j) + \nabla^E_{\nabla_{e_i}e_j}\right]\\ &= -\sum_{i=1}^n \left(\nabla^E\right)^2(e_i,e_i) + \sum_{i<j} c(e_i)\, c(e_j)\, \left[\left(\nabla^E\right)^2(e_i,e_j) - \left(\nabla^E\right)^2(e_j,e_i)\right]\\ &= \Delta^E + \sum_{i<j} c(e_i)\, c(e_j)\, \left(\nabla^E\right)^2(e_i,e_j)\\ &= \Delta^E + R^E.\end{aligned}$$

*The curvature term* $\left(\nabla^E\right)^2 \in \Omega^2(\mathrm{End}(E))$ *decomposes as* $\left(\nabla^E\right)^2 = \left(\nabla^S\right)^2 \otimes 1 + 1 \otimes \left(\nabla^W\right)^2$ *so that* $R^E = \sum_{i<j} c(e_i)\, c(e_j)\, \left(\nabla^S\right)^2 (e_i, e_j) + R^W$. *A careful computation (see e.g. the proof of Theorem 3.52 in [BGV]) shows that* $\sum_{i<j} c(e_i)\, c(e_j)\, \left(\nabla^S\right)^2 (e_i, e_j) = \frac{s}{4}$.

## 9.4 The Hodge de Rham decomposition theorem

From the Bochner-Weitzenböck formula, we know that the Hodge Laplacian $\Delta = (d + d^*)^2$ differs from the Laplacian $\Delta^{\Lambda T^* M}$ by a zeroth-order differential operator. They therefore have the same leading symbol; since we know that $\Delta^{\Lambda T^* M}$ is elliptic, so is $\Delta$. As a consequence of Proposition 27 applied to the restriction $\Delta_p$ of $\Delta$ to $p$-forms, the space of *p-harmonic forms* given by:

$$\mathcal{H}_p(M) := \{\alpha \in \Omega^p(M), \Delta_p \alpha = 0\}$$

is finite dimensional. Its dimension is called the *p-th Betti number* and is denoted by $\beta_p(M)$. Since $\mathrm{Ker}(\Delta_p) = \mathrm{Ker}\left(d_{|\Omega^p(M)}\right)$ we have

$$\beta_p(M) = \dim(H^p(M)).$$

On the other hand, the decomposition theorem for Fredholm operators yields a *Hodge decomposition theorem*:

$$\Omega^p(M) = \mathcal{H}_p(M) \oplus R(d + d^*)_{|\Omega^p(M)} = \mathcal{H}_p(M) \oplus R\left(d_{|\Omega^{p-1}(M)}\right) \oplus R\left(d^*_{|\Omega^{p+1}(M)}\right),$$

the direct sums corresponding to orthogonal sums w.r.to the inner product on forms. As a consequence, we have:

$$\beta_p(M) = \dim\left(\mathcal{H}_p(M)\right).$$

The Hodge star isomorphism between $p$-forms and $n-p$-forms yields the isomorphism:

$$H^p(M) \simeq H^{n-p}(M)$$

and hence

$$\beta_p(M) = \beta_{n-p}(M).$$

The *Euler characteristic* which is given by the alternating sum of the Betti numbers defines a topological invariant of the manifold:

$$\xi(M) = \sum_{p=0}^{n} \beta_p(M). \tag{46}$$

There is another possible interpretation of the Euler characteristic as the index of the zero section in the tangent bundle $TM$. Let $f : M \to N$ be a smooth map from a closed oriented smooth $m$-dimensional manifold $M$ to another closed oriented smooth $n$-submanifold $N$ of an oriented manifold $W$ of dimension $m + n$ such that $f$ is transverse to $N$. A point $x \in f^{-1}(N)$ has positive or negative type according to whether the composition:

$$M_x \to W_{f(x)} \to W_{f(x)}/N_{f(x)}$$

preserves or reverses orientation. Here the first map is the tangent map $T_x f$ to $f$ at point $x$. Accordingly we set $i_x(f,N) = 1$ or $i_x(f,N) = -1$. The *intersection number* of $(f,N)$ is the integer:

$$i(f,N) := \sum_{x \in f^{-1}(N)} i_x(f,N).$$

It is invariant under homotopies of the map $f$.

Now if $s_0 : M \to TM$ is the zero section of the tangent bundle of $M$, we have:

$$\xi(M) = i(s_0, M).$$

As a consequence, since any section $s$ of the tangent bundle is homotopic to the zero section by the map $(t,x) \mapsto ts(x)$, if the tangent bundle to $M$ has a section which is nowhere zero then $\xi(M) = 0$. Since $\xi(S^{2n}) = 2$, every vector field on $S^{2n}$ vanishes somewhere. In other words, a "hairy ball cannot be combed".

We shall also need so split the space of harmonic forms as the sum

$$\mathcal{H}^p(M) = \mathcal{H}^{p,sd}(M) \oplus \mathcal{H}^{p,asd}(M)$$

of the space of *self-dual harmonic p-forms*

$$\mathcal{H}^{p,sd}(M) := \{\alpha \in \mathcal{H}^p(M), \star\alpha = \alpha\}$$

and of the space of *anti self-dual harmonic p-forms*

$$\mathcal{H}^{p,asd}(M) := \{\alpha \in \mathcal{H}^p(M), \star\alpha = -\alpha\}$$

which are both trivially finite dimensional since the space of harmonic $p$-forms is. We set

$$\beta_p^+(M) := \dim\left(\mathcal{H}^{p,sd}(M)\right), \quad \beta_p^-(M) := \dim\left(\mathcal{H}^{p,asd}(M)\right).$$

Clearly we have:

$$\beta_p(M) = \beta_p^+(M) + \beta_p^-(M).$$

## 9.5 Three index theorems on forms

To give a flavour of index theory, we state here (without proof) three index theorems of forms.
We saw that the operator $D = d + d^*$ gives rise to a Dirac operator $D$ acting on forms on a closed manifold $M$. Two different gradings on $\Omega(M)$ give rise to two different *chiral Dirac operators.*

## 9.6 The Chern-Gauss-Bonnet index theorem on forms

Let us first equip $\Omega(M)$ with the $\mathbb{Z}_2$-grading given by the parity of the form

$$\Omega(M) = \Omega^{ev}(M) \oplus \Omega^{odd}(M)$$

where $\Omega^{ev}(M) = \mathrm{Ker}(I - P)$ is the algebra of forms of even degree and $\Omega^{odd} = \mathrm{Ker}(I + P)$ the space of forms of odd degree. Here $P$ denotes the parity operator which is 1 on even forms and $-1$ on odd forms. The index of the Dirac operator $D_P^+ := (d + d^*)_{|_{\mathrm{Ker}(I-P)}}$ can be expressed in terms of the Euler characteristic defined in (46):

**Lemma 17**
$$\mathrm{ind}(D_P^+) = \chi(M).$$

*Proof.*
$$\begin{aligned}
\mathrm{ind}(D_P^+) &= \mathrm{dimKer}(D_P^+) - \mathrm{dimKer}(D_P^-) \\
&= \mathrm{dimKer}((1-P)D) - \mathrm{dimKer}((1+P)D) \\
&= \sum_p \mathrm{dimKer}(D_{|_{\Omega^{2p}(M)}}) - \sum_p \mathrm{dimKer}(D_{|_{\Omega^{2p+1}(M)}}) \\
&= \sum_{p=0}^{n} (-1)^p \mathrm{dimKer}(D_{|_{\Omega^p(M)}}) \\
&= \sum_{p=0}^{n} (-1)^p \mathrm{dim}\mathcal{H}^p(M) \\
&= \sum_{p=0}^{n} (-1)^p \beta_p(M) \\
&= \chi(M).
\end{aligned}$$

The Chern-Gauss-Bonnet index theorem (which we do not prove here) provides a local expression of the index:

**Theorem 15**
$$\mathrm{ind}(D_P^+) = \chi(M) = (2\pi)^{-\frac{n}{2}} \int_M e(\nabla)$$
*where $\nabla$ is the Levi-Civita connection on $M$, $e(\nabla)$ the Euler class of $M$ equipped with $\nabla$ introduced in (9).*

## 9.7 The Hirzebruch signature index theorem

Let us now introduce another $\mathbb{Z}_2$-grading on $\Omega(M)$ using the *chirality operator* defined on $p$ forms by:
$$\Gamma = (-1)^{pn+\frac{p(p-1)}{2}+l} i^{k(n)} \star$$
where $\star$ is the Hodge star defined in formula (3). We have set $k(n) := \frac{n}{2}$ if $n$ is even and $k(n) := \frac{n+1}{2}$ if $n$ is odd. Since $\Gamma^2 = I$ , the space of forms splits:
$$\Omega(M) = \Omega^+(M) \oplus \Omega^-(M)$$
where we have set $\Omega^+(M) = \mathrm{Ker}(\Gamma - I)$ and $\Omega^-(M) := \mathrm{Ker}(\Gamma + I)$. If the dimension $n$ is even, then $D = d + d^*$ anti-commutes with $\Gamma$, i.e. $\Gamma D = -D\Gamma$, so that the operator $D_\Gamma^+ := (d + d^*)_{|_{\mathrm{Ker}(I-\Gamma)}}$ acts from the space $\Omega^+(M)$ to the space $\Omega^-(M)$.

We henceforth specialise to the case $n = 2k = 4l$ is a multiple of 4, for which $\Gamma$ coincides with the Hodge star operator on $k$-forms. In particular we have:
$$\mathcal{H}^{k,sd}(M) = \{\alpha \in \Omega^k(M), \Gamma\alpha = \alpha\}$$
and
$$\mathcal{H}^{k,asd}(M) = \{\alpha \in \Omega^k(M), \Gamma\alpha = -\alpha\}$$
that are finite dimensional spaces with dimensions $\beta_k^+(M)$ and $\beta_k^-(M)$ respectively.

**Proposition 32** *In dimension $n = 2k = 4l$, the bilinear form*

$$\begin{aligned} \sigma : \Omega^k(M) \times \Omega^k(M) &\to I\!R \\ (\alpha, \beta) \quad \to \quad & \sigma(\alpha,\beta) := \int_M \alpha \wedge \beta \end{aligned}$$

*induces a non degenerate symmetric bilinear form on $H^k(M)$. Its signature, called the signature of $M$, is given by*

$$\sigma(M) := \text{sign}(\sigma) = \beta_k^+(M) - \beta_k^-(M).$$

**Proof 51** *We saw previously that $\int_M \alpha \wedge \beta = (-1)^k \int_M \beta \wedge \alpha$ on $k$-forms so that if $k$ is even, it yields a symmetric bilinear form. Given two closed forms $\alpha$ and $\beta$, $\sigma(\alpha, \beta)$ only depends on the cohomology class of $\alpha$ and $\beta$; indeed*

$$\begin{aligned} \int_M (\alpha + d\gamma) \wedge \beta &= \int_M \alpha \wedge \beta + \int_M d\gamma \wedge \beta \\ &= \int_M \alpha \wedge \beta + \int_M d(\gamma \wedge \beta) - \int_M \gamma \wedge d\beta \\ &= \int_M \alpha \wedge \beta, \end{aligned}$$

*where we have used Stokes' theorem to set the middle integral to zero and the fact that $\beta$ is closed to set the last integral to zero.*

*The form $\sigma$ is non degenerate; indeed, let us assume that $\int_M \alpha \wedge \beta = 0$ for any closed $k$-form $\beta$ and let us show that $\alpha = 0$. Pick a harmonic $k$-form $\alpha$ as representative of a cohomology class in $H^k(M)$. Then*

$$(d + d^*)\alpha = 0 \Rightarrow d\alpha = d^*\alpha = 0 \Rightarrow d(\star\alpha) = 0.$$

*Hence $\star\alpha$ is also a closed $k$-form and we can take $\beta = \star\alpha$. This yields $\alpha \wedge \star\alpha = \|\alpha\|^2 = 0$ so that $\alpha = 0$ which ends the proof of the non degeneracy.*

*Let us now compute this bilinear form in different cases. For $\alpha \in \mathcal{H}^{k,sd}(M)$, $\beta \in \mathcal{H}^{k,sd}(M)$, we have*

$$\sigma(\alpha,\beta) = \int_M \alpha \wedge \beta = \int_M \alpha \wedge \star\beta = \langle \alpha, \beta \rangle,$$

*if $\alpha \in \mathcal{H}^{k,asd}(M)$, $\beta \in \mathcal{H}^{k,asd}(M)$*

$$\sigma(\alpha,\beta) = \int_M \alpha \wedge \beta = \int_M \alpha \wedge \star\beta = -\langle \alpha, \beta \rangle$$

*and if $\alpha \in \mathcal{H}^{k,sd}(M)$, $\beta \in \mathcal{H}^{k,asd}(M)$*

$$\sigma(\alpha,\beta) = \int_M \star\alpha \wedge \beta = (-1)^k \int_M \beta \wedge \star\alpha = \langle \alpha, \beta \rangle = \int_M \star\alpha \wedge \star\beta = -\sigma(\alpha,\beta) = 0.$$

*As a consequence, $\sigma$ is diagonal on $\mathcal{H}^{k,sd}(M) \oplus \mathcal{H}^{k,asd}(M)$ with eigenvalues $+1, -1$ with multiplicity $\beta_k^+(M)$ and $\beta_k^-(M)$. The bilinear form $\sigma$ therefore has signature*

$$\text{sign}(\sigma) = \text{dim}\mathcal{H}^{k,sd}(M) - \text{dim}\mathcal{H}^{k,asd}(M) = \beta_k^+(M) - \beta_k^-(M).$$

The following Lemma relates the index of $D^+_\Gamma$ to the signature of the manifold:

**Lemma 18**

$$\mathrm{ind}(D^+_\Gamma) = \sigma(M).$$

**Proof 52** *We first observe that if $\alpha_i, i = 1, \cdots, \beta_j(M)$ is an orthonormal basis of $\mathcal{H}^j(M)$ with $j < k$ then $\alpha_i^+ = \alpha_i + \star\alpha_i, \alpha_i^- = \alpha_i - \star\alpha_i$, $i = 1, \cdots, \beta_j(M)$ yield an orthonormal basis of $\mathcal{H}^j \oplus \mathcal{H}^{n-j}$, where $\star$ is the Hodge star operator. Since the $\alpha_i^+$ are self-dual and the $\alpha_i^-$ are anti self-dual, they yield an orthonormal basis respectively of $\mathcal{H}^{j,sd}(M) \oplus \mathcal{H}^{n-j,sd}(M)$ and $\mathcal{H}^{j,asd}(M) \oplus \mathcal{H}^{n-j,asd}(M)$. As a consequence,*

$$\begin{aligned} & \beta_j^+(M) + \beta_{n-j}^+(M) = \dim\left(\mathcal{H}^{j,sd}(M) \oplus \mathcal{H}^{n-j,sd}(M)\right) \\ = \; & \dim\left(\mathcal{H}^{j,asd}(M) \oplus \mathcal{H}^{n-j,asd}(M)\right) = \beta_j^-(M) + \beta_{n-j}^-(M). \end{aligned}$$

*Hence we have,*

$$\begin{aligned} \mathrm{ind}(D^+_\Gamma) &= \mathrm{dimKer}(D^+_\Gamma) - \mathrm{dimKer}(D^-_\Gamma) \\ &= \mathrm{dimKer}(D_{|_{\Omega^{sd}(M)}}) - \mathrm{dimKer}(D_{|_{\Omega^{asd}(M)}}) \\ &= \sum_{j=0}^n \dim\left(\mathcal{H}^{j,sd}(M)\right) - \sum_{j=0}^n \dim\left(\mathcal{H}^{j,asd}(M)\right) \\ &= \sum_{j=0}^n \beta_j^+(M) - \sum_{j=0}^n \beta_j^-(M) \\ &= \sum_{j=0}^{k-1}\left(\beta_j^+(M) + \beta_{n-j}^+(M)\right) - \sum_{j=0}^{k-1}\left(\beta_j^-(M) + \beta_{n-j}^-(M)\right) + \beta_k^+(M) - b_k^-(M) \\ &= \beta_k^+(M) - b_k^-(M) \\ &= \sigma(M). \end{aligned}$$

The Hirzebruch signature index theorem (which we do not prove here) gives a local expression of the index as an integral of the $L$-genus introduced in (10):

**Theorem 16** *Let $M$ be an oriented closed $n = 4l$ Riemannian dimensional manifold.*

$$\mathrm{ind}(D^+_\Gamma) = \frac{(-1)^l}{\pi^{2l}} \int_M L(\nabla)$$

*where as before, $\nabla$ is the Levi-Civita connection.*

## 9.8 The Riemann-Roch theorem

Let $M$ be a Hermitian almost-complex manifold. Following a similar procedure as for the de Rham operator, we first recognise $\overline{\partial} + \overline{\partial}^*$ as a Dirac operator on complex forms.

The Clifford map $c = \varepsilon - \iota$ on the vector bundle of differential forms described above induces a Clifford action on the vector bundle of anti-holomorphic differential forms $\Lambda\left(T^{0,1}M\right)^*$ in the following way:

$$\alpha = \alpha^{1,0} + \alpha^{0,1} \in \Omega^1(M) \Longrightarrow c(\alpha) = \sqrt{2}\left(\varepsilon(\alpha^{1,0}) - \iota(\alpha^{0,1})\right).$$

Since $\overline{\alpha^{1,0}} = \alpha^{0,1}$, the operator $c(\alpha)$ is skew-adjoint

$$(c(\alpha))^* = \varepsilon^*(\overline{\alpha}^{1,0}) - \iota^*(\overline{\alpha}^{0,1}) = \iota(\alpha^{0,1}) - \varepsilon(\alpha^{1,0}) = -c(\alpha).$$

If the manifold is Kähler, we saw that $\nabla J = 0$, hence $\nabla$ preserves the holomorphic and anti-holomorphic tangent bundles. Since the Levi-Civita $\nabla$ commutes with $\varepsilon - \iota$, it therefore follows that $\nabla$ commutes with the above Clifford connection.
We further saw that $d = \varepsilon \circ \nabla$, hence for a local frame $(e^i)_{i=1,\ldots,n}$ of $\left(T^{1,0}M\right)^*$ dual to the tangent frame $(e_i)_{i=1,\cdots,n}$ we have

$$d = \varepsilon(e^i)\nabla_{e_i} + \varepsilon(\overline{e}^i)\nabla_{\overline{e}_i} \Longrightarrow \overline{\partial} = \varepsilon(\overline{e}^i)\nabla_{\overline{e}_i}.$$

A computation of the adjoint shows that

$$\overline{\partial}^* = -\iota(\overline{e}^i)\nabla_{\overline{e}_i}.$$

Hence

$$\overline{\partial} + \overline{\partial}^* = (\varepsilon - \iota)\,(\overline{e}^i)\nabla_{\overline{e}_i} = c \circ \nabla$$

corresponds to a Dirac operator on complex forms.
Similarly to the Lichnerowicz formula, the Bochner-Kodaira formula relates $(\overline{\partial}+\overline{\partial}^*)^2$ to the generalized Laplacian $\Delta^{0,\bullet} := -\sum_i \left(\nabla_{Z_i}\nabla_{\overline{Z}_i} - \nabla_{Z_i}\overline{Z}_i\right)$ up to curvature terms.
In particular, the operator $\overline{\partial} + \overline{\partial}^*$ seen as a differential operator acting on anti-holomorphic differential forms is elliptic.
We equip $\Lambda\left(T^{0,1}M\right)^*$ with the $\mathbb{Z}^2$-grading induced by the natural $\mathbb{Z}$-grading on forms so as to split $D = \overline{\partial}+\overline{\partial}^*$ into two odd differential operators $D^+$ and $D^-$ acting respectively on even and odd anti-holomorphic differential forms. Since $D$ is elliptic, the operator $D^+$ has a well-defined index which coincides with the alternating sum of the Hodge numbers:

$$\mathrm{ind}\left(D^+\right) = \sum_i (-1)^i \dim_{\mathbb{C}} H^{0,i}(M) = \sum_i (-1)^i h^{0,i}$$

called the *arithmetic genus* of $M$. The Riemann-Roch theorem (which do not prove here) states that

$$\mathrm{ind}\left(D^+\right) = \frac{1}{(2i\pi)^{n/2}} \int_M \mathrm{Td}(M).$$

# 10 Configuration and moduli spaces

This chapter offers an illustration of how the various tools from geometry and operator theory presented in the previous chapters can come into play in quantum field theory. A short description of spaces of (inequivalent) configurations arising in Yang-Mills, Seiberg-Witten and string theory is given here.

## 10.1 The geometric setting

A classical field theory with symmetries typically leads to the following geometric setting. A gauge group $\mathcal{G}$ (a group of symmetries) acts on an (infinite dimensional) space of configurations $X$, and one is interested in the *moduli space* of inequivalent configurations $\mathcal{M} := X/\mathcal{G}$.

The space of inequivalent configurations can play a role to study solutions of the classical field equations –namely the Euler-Lagrange equations minimizing the classical action– (Yang-Mills equation in Yang-Mills theory and the Seiberg -Witten equations in Seiberg-Witten theory) or to investigate the quantized theory from a path integration point of view (in string theory). In both cases the non-compactness of the moduli space can come into the way; Seiberg-Witten theory offers the advantage over Yang-Mills theory that the moduli space of classical solutions is compact and Seiberg-Witten invariants are built up from integrals on the moduli space of inequivalent solutions to the Seiberg-Witten equations. In the path integral approach to quantization, when the moduli space is a finite dimensional manifold (for string theory, the Teichmüller space of inequivalent conformal structures on a Riemann surfaces is a smooth finite dimensional manifold), one can reduce path integrals on infinite dimensional configuration spaces to ordinary integrals on the moduli space.

If the action is not free, the moduli space might not be a Hausdorff space; to cure this problem, one can either reduce the group of gauge transformations or reduce the configuration space in order to get a free action and a manifold structure on the quotient space. For this reason, in Yang-Mills and Seiberg-Witten theory one restricts to irreducible configurations, whereas in string theory, one restricts the gauge group, considering only the connected component of the identity of the group of diffeomorphisms of a surface.

Recall from the (slice) theorem of section 3.2, that given a Hilbert Lie group $\mathcal{G}$, acting $L^2$-isometrically on a smooth Hilbert manifold $X$ via a smooth, free and proper action, provided the tangent map $\tau_x := D_e\theta_x$ (see notations of section 3.2) has a closed range, then the quotient space $X/\mathcal{G}$ is a smooth Hilbert manifold. In practice, one typically comes across the following situation:

- $\mathcal{G}$ is modelled on some space $H^{s+k}(E)$ of Sobolev sections of some vector bundle $E$ based on a closed manifold $M$ of dimension $n$ ($k > 0$, $s$ is usually chosen large enough $s > \frac{n}{2}$ so that Sobolev sections are continuous),
- $X$ is modelled on some space $H^s(F)$ of Sobolev sections of another vector bundle $F$ based on $M$,
- for $x \in X$, $\tau_x : C^\infty(E) \to C^\infty(F)$ is a differential operator of order $k > 0$ with injective symbol.

The last proposition in section 4.8 then tells us that if $\tau_x$ were elliptic, it would extend to a Fredholm operator $\tau_x : H^{s+k}(E) \to H^s(F)$ and hence have a closed range (see section 3). Yet the mere injectivity of the symbol of $\tau_x$ which implies that $\tau_x^* \tau_x$ is elliptic, which is sufficient since it yields the closedness of the range of $\tau_x$ and as a consequence, the $L^2$- orthogonal splitting:

$$R(\tau_x) \oplus \operatorname{Ker}(\tau_x^*) = C^\infty(F).$$

**Remark 26** *The stronger assumption that $\tau_x$ be elliptic would require that its leading symbol be an isomorphism for non vanishing $\xi$, but this in turn implies that $E$ and $F$ should have the same rank, which is not always the case in practice. However, under the weaker requirement that the leading symbol be injective for non vanishing $\xi$ This splitting actually holds in the $H^s$ topology.*

We have:

**Theorem 17** *Let $\mathcal{G}$ be a Hilbert Lie group modelled on some space $H^{s+k}(E)$ of Sobolev sections of some Hermitian vector bundle $E$ based on a closed manifold $M$ of dimension $n$ (with $s > \frac{n}{2}$, $k > 0$) acting on a Hilbert manifold $X$ modelled on some space $H^s(F)$ of Sobolev sections of another Hermitian vector bundle $F$ based on $M$. We assume that the action is $L^2$-isometric for the $L^2$ Riemannian metric on $X$ built from inner products on the model space obtained by integrating along $M$ the inner products on the fibres. If the action is free, proper and smooth and if moreover, for any $x \in X$, $\tau_x : C^\infty(E) \to C^\infty(F)$ is a differential operator of order $k > 0$ with injective symbol, then the moduli space $X/\mathcal{G}$ is a smooth manifold.*

In applications, an additional difficulty occurs; because one restricts oneself to some Sobolev setting, the differential operator $\tau_x$ may have non smooth coefficients lying in some Sobolev space so that one then needs to adapt the classical results on differential operators with smooth coefficients (notice that a differential operator of order $a$ with $H^k$-coefficients takes smooth sections to $H^{k-a}$ sections unlike an operator with smooth coefficients which takes smooth sections to smooth sections). We shall elude this difficulty here, referring the reader to [KRo, Theorem 3.1.10] for a discussion on this point.

## 10.2 Inverse limit of Hilbert manifolds

We want to extend the slice theorem beyond Hilbert spaces, to spaces of smooth sections $C^\infty(M, E)$ of some bundle $E$ on a closed manifold $M$, which are Fréchet spaces and intersections $C^\infty(M, E) = \cap_{k \in \mathbb{N}} H^k(M, E)$ of Hilbert spaces $H^k(M, E)$ of $H^k$ sections of $E$. The group $\operatorname{Diff}(M)$, should it be equipped with a suitable Lie group structure, its Lie algebra is expected to be the space $C^\infty(M, TM)$ of smooth of the tangent bundle $TM$. This calls for the set up of inverse limits of Hilbert manifolds.

If $\{X_n, n \in \mathbb{N}\}$ is a countable family of topological spaces with continuous inclusions $X_{n-1} \hookrightarrow X_n$ then the intersection $X := \bigcap_n X_n$ can be endowed with the projective topology which corresponds to the weakest topology on $X$ that makes the inclusions $X \hookrightarrow X_n$ continuous. We denote the resulting topological space, called the *inverse limit* of the $X_n$ by $(X; X_n, n \in \mathbb{N})$.
If for every $n \in \mathbb{N}$, the space $X_n$ is a linear Hilbert space and the inclusion maps are linear, the resulting inverse limit $(X; X_n, n \in \mathbb{N})$ is a linear space called an inverse limit of Hilbert spaces or an *I.L.H. vector space* for short.

**Definition 75** *A $C^k$- I.L.H. manifold (resp. $C^\infty$- I.L.H. manifold) modelled on an I.L.H. linear space $(E; E_n, n \in I\!N)$ is an I.L.H. topological space $(X; X_n, n \in I\!N)$ such that*

- *$X_n$ is a $C^k$-Banach manifold (resp. $C^\infty$-Banach manifold) modelled on $\mathbb{E}_n$,*
- *For each $x \in X$, for any $n \in I\!N$, there is an open neighborhood $U_n(x)$ of $x$ in $X_n$ and homeomorphisms:*

$$\Phi_n : U_n(x) \to V_n \subset E_n$$

*which yield $C^k$ (resp. $C^\infty$) coordinate systems around $x \in X_n$ and satisfy:*

$$U_{n+1}(x) \subset U_n(x) \quad \textit{and} \quad \Phi_{n|_{U_{n+1}(x)}} = \Phi_{n+1},$$

- *$U(x) := \bigcap_n U_n(x)$ is an open neighborhood of $x$ in $(X; X_n, n \in I\!N)$.*

We have included the last condition in the definition of an I.L.H. manifold which makes it a strong I.L.H. manifold according to the usual convention, so that I.L.H. manifolds considered here are in fact *strong I.L.H. manifolds.*

**Definition 76** *A map $\phi : X \to Y$ between two $C^k$-I.L.H. manifolds is $C^k$-I.L.H. differentiable if it is the inductive limit of $C^k$-differentiable maps $\phi_n : X_{m(n)} \to Y_n$ for some $m(n)$ such that $\phi_{n|_{X_{m(n+1)}}} = \phi_{m(n+1)}$. It is smooth if it is $C^k$ for all $k \in I\!N$.*

**Remark 27** *There are examples for which one can choose $m(n) = n$ (e.g. the multiplication in the Weyl group, see below), but allowing $m(n) \neq n$ is necessary if we want to put an I.L.H. Lie group structure on the group of diffeomorphisms we need in the context of string theory.*

**Definition 77** *An I.L.H. topological group is called an I.L.H. Lie group if it is a smooth I.L.H. manifold with the group operations given by smooth I.L.H. maps.*

**Example 45** *The group of smooth diffeomorphisms on a closed manifold can be equipped with an I.L.H. Lie group structure [O], [AM] even though the group of diffeomorphisms of a fixed Sobolev class is not a Banach Lie group (due to the lack of smoothness of the left multiplication), hence the relevence of the concept of I.L.H. space.*

## 10.3 A slice theorem in the Fréchet setting

The results of this paragraph are based on [KRo]. Let $P$, $B$ be smooth I.L.H. manifolds, $\pi : P \to B$ a smooth I.L.H. map and $G$ an I.L.H. Lie group. Then $(P, B, G, \pi)$ is an I.L.H. principal bundle if and only if the transition maps are smooth I.L.H. maps.

We are now ready to state an extension of the above slice theorem to an I.L.H. manifold modelled on a space of smooth sections. The notion of properness extends to the I.L.H. setting in a straightforward way.

**Theorem 18** *Let $G$ be an I.L.H. Lie group acting transitively on the right on a smooth I.L.H. manifold $X$:*

$$\begin{array}{rcl} \Theta : \mathcal{G} \times X & \to & X \\ (g, x) & \mapsto & x \cdot g. \end{array}$$

*Let us assume that the I.L.H. manifolds $X$, resp. $G$ are modelled on the I.L.H. spaces $C^\infty(M,E) = \cap_{k\in\mathbb{N}} H^k(M,E)$, resp. $C^\infty(M,F) = \cap_{k\in\mathbb{N}} H^k(M,F)$ where $E \to M$ and $F \to M$ are two Hermitian vector bundles based on some closed Riemannian manifold $M$. The manifold $X$ is equipped with a (weak) $L^2$ Riemannian structure built from inner products on their model spaces obtained by integrating along $M$ the inner product on the fibres.*
*Under the following assumptions:*

- *The action of $G$ on $X$ is smooth I.L.H., $L^2$-isometric, free and proper,*
- *Setting $\theta_x := \Theta(\cdot, x)$ for any $x \in X$, the map:*

$$\begin{array}{rcl} \tau_x : \mathfrak{g} & \to & T_x X \\ u & \mapsto & D_e\theta_x(u) \end{array}$$

  *is an injective differential operator with injective symbol (so that $\tau_x^* \tau_x$ is elliptic),*

*then the quotient is a smooth I.L.H. manifold equipped with the induced $L^2$-structure and the canonical projection $\pi : X \to X/\mathcal{G}$ yields an I.L.H. principal fibre bundle.*

**Remark 28** *This can be compared with [H, Theorem 2.5.1] established in the tame Fréchet setup. Tame Fréchet spaces are special classes of graded Fréchet spaces, where by graded we mean a Fréchet space equipped with a fixed collection of seminorms $\{\|\cdot\|_n, n \in \mathbb{N}\}$ increasing in strength, $|\cdot\|_0 \leq |\cdot\|_1 \leq \cdots \leq |\cdot\|_n \leq \cdots$ and which induce the locally convex topology on the space. The space $C^\infty(M,E)$ of smooth sections of a vector bundle $E \to M$ over a closed manifold is a tame Fréchet space.*

Let us now turn to three examples in quantum field theory; Yang-Mills, Seiberg-Witten and string theory which use the above theorem.

## 10.4 Configurations in Yang-Mills gauge theory

Useful references are [CT], [FU], [KRo], [MM], [N], [Ts].

$G$ denotes a fixed compact connected Lie group.

Let $P$ be be a smooth principal bundle based on a closed manifold $M$ with structure group $G$. Let $adP := P \times_G Lie(G)$ be the the vector bundle based on $M$ with typical fibre given by the Lie algebra $Lie(G)$ of $G$ associated to the adjoint action of $G$ on $Lie(G)$. Let us set $E := adP$ and $F := T^*M \otimes adP$, a vector bundle the sections of which are 1-forms on $M$ with values in $adP$.

*The space of configurations:* Let $\mathcal{C}(P)$ (resp. $\mathcal{C}^s(P)$) denote the space of smooth (resp. $H^s$) connections on $P$. Since a connection on $P$ is a $G$-equivariant $Lie(G)$-valued one form $\omega$ on $P$ such that $\omega(\tilde{X}) = X \quad \forall X \in Lie(G)$ ($\tilde{X}$ is the canonical vector field generated by $X$), two connections differ by a horizontal one form on $P$ and hence an $adP$ valued one form on $M$. $\mathcal{C}(P)$ (resp. $\mathcal{C}^s(P)$) is an affine I.L.H. (resp. Hilbert) space with tangent vector space $C^\infty(F)$ (resp. $H^s(E)$).

*The gauge group:* Let $E_G := P \times_G G$ where $G$ acts on itself by the adjoint action, then the set $C^\infty(E_G)$ (resp. $H^s(E_G)$) of smooth (resp. $H^s$-Sobolev) sections of $E_G$ is an I.L.H. (resp. Hilbert) Lie group modelled on $C^\infty(E)$ (resp. $H^s(E)$). It corresponds to the group of automorphisms of $P$ that cover the identity map on $M$.

*The space of irreducible configurations:* A connection $A$ on $P$ induces a covariant derivation $\nabla^A$ on $adP$ from which one can define a differential operator of order 1:

$$\begin{aligned} d_A : \Omega^0(M,E) = C^\infty(E) &\to \Omega^1(M,E) = C^\infty(F) \\ \sigma &\mapsto \big(X \to \nabla^A_X(\sigma)\big) \end{aligned}$$

which extends to the exterior differential $d_A : \Omega^*(M,E) \to \Omega^{*+1}(M,E)$. The operator $d_A$ is generally not injective so that we need to restrict ourselves to *irreducible* connections, namely those for which $d_A$ is one to one. Notice that when $A$ is reducible, an element $u \in \mathrm{Ker} d_A$ generates gauge transformations $g_t := e^{tu}$ that leaves $A$ fixed. The space $\bar{\mathcal{C}}(P)$ (resp. $\bar{\mathcal{C}}^s(P)$) of irreducible smooth (resp. $H^s$) connections on $P$) is also an I.L.H. (resp. Hilbert) manifold modelled on $C^\infty(F)$ (resp. $H^s(F)$).

*An $L^2$-structure on the configuration space:* Since $G$ is compact, its Lie algebra $Lie(G)$ can be equipped with a positive definite inner product which is invariant under the adjoint action. The bundle $E = adP$ thus inherits an inner product which, combined with the Riemannian metric on $M$ yields an inner product on $F = T^*M \otimes adP$. Hence, the configuration space $\mathcal{C}(P)$ which is modelled on $C^\infty(F)$ can be equipped with an $L^2$-metric obtained by integrating along $M$ the inner product on $F$. This metric is invariant under the action $\Theta$.

*The gauge group action:* The I.L.H. Lie group $C^\infty(E_G)$ acts smoothly on the I.L.H. space $\mathcal{C}(P)$ of smooth connections on $P$:

$$\begin{aligned} \Theta : C^\infty(E_G) \times \mathcal{C}(P) &\to \mathcal{C}(P) \\ (g, A) &\mapsto A \cdot g := A + g^{-1} d_A g. \end{aligned}$$

The action $\Theta$ is $L^2$-isometric and it induces a smooth, free and proper action on the I.L.H. space of *irreducible* configurations:

$$\begin{aligned} \bar{\Theta} : C^\infty(E_G)/\mathcal{Z} \times \bar{\mathcal{C}}(P) &\to \bar{\mathcal{C}}(P) \\ (g, A) &\mapsto a \cdot g := A + g^{-1} d_A g. \end{aligned}$$

Here $\mathcal{Z}$ is the center of $C^\infty(E_G)$ and corresponds to $C^\infty(P \times_G Z(G))$ where $Z(G)$ is the center of $G$.

*The tangent operator $\tau_x$:* It is the tangent operator at identity to $\theta_A := \Theta(\cdot, A)$ and therefore corresponds to the first order differential operator $d_A : C^\infty(E) \to C^\infty(F)$ which has injective symbol.

*The moduli space of inequivalent connections:* Applying the slice theorem to the I.L.H. gauge group quotiented by its center $\mathcal{G} := C^\infty(E_G)/\mathcal{Z}$ acting on the I.L.H. manifolds of irreducible configurations $X := \bar{\mathcal{C}}(P)$ shows that the moduli space $X/\mathcal{G}$ of inequivalent irreducible connections on $P$ is a smooth I.L.H. manifold.

## 10.5 Configurations in Seiberg-Witten theory

Classical references are [Ma], [Mo].

The setting is similar in spirit to the Yang-Mills setting. Here $M$ is a closed 4-dimensional Spin$^c$ manifold and $\tilde{P}$ the lift of the orthonormal frame bundle $SO(TM)$ to a principal Spin$^c$ bundle based on $M$. Let $E := M \times I\!R$ and $F := T^*M \otimes I\!R \oplus S^+(\tilde{P})$ where $S^+(\tilde{P})$ is the spinor bundle associated to the Spin$^c$ structure on $M$.

*The space of configurations:* Let $\mathcal{C}(\mathcal{L})$ (resp. $\mathcal{C}^s(\mathcal{L})$) be the I.L.H. (resp. Hilbert) space of $U(1)$ smooth (resp. $H^s$ ) connections on $\mathcal{L}$, the determinant line bundle $\mathcal{L}$ associated to $\tilde{P}$. The space of smooth (resp. $H^s$) configurations is given by:

$$\mathcal{C}(\tilde{P}) := \mathcal{C}(\mathcal{L}) \times C^\infty(S^+(\tilde{P}))$$

resp.

$$\mathcal{C}^s(\tilde{P}) := \mathcal{C}^s(\mathcal{L}) \times H^s(S^+(\tilde{P})).$$

It is a smooth I.L.H. (resp. Hilbert) manifold.

*The gauge group:* The group of smooth (resp. $H^s$) automorphisms of $\tilde{P}$ that cover the identity on the frame bundle of $M$ is an I.L.H. (resp. Hilbert) Lie group which coincides with the group of smooth (resp. $H^s$) maps $C^\infty(M, S^1)$ (resp. $H^s(M, S^1)$).

*The gauge group action:* An element $g$ of $C^\infty(M, S^1)$ induces a bundle map $\mathrm{det}g$ on the determinant bundle $\mathcal{L}$ and a bundle map $S^+(g)$ on the spinor bundle $S^+(\tilde{P})$. It acts on the space of configurations by:

$$\begin{array}{rcl}\Theta : C^\infty(M, S^1) \times \mathcal{C}(\tilde{P}) & \to & \mathcal{C}(\tilde{P}) \\ (g, (A, \psi)) & \mapsto & (A, \psi) \cdot g := (\mathrm{det}g^* A, S^+(g^{-1})\psi).\end{array}$$

*Irreducible configurations:* Let $\bar{\mathcal{C}}(\tilde{P}) := \{(A, \psi) \in \mathcal{C}(\tilde{P}), \psi \neq 0\}$ (resp. $(\bar{\mathcal{C}}^s(\tilde{P}) := \{(A, \psi) \in \mathcal{C}^s(\tilde{P}), \psi \neq 0\})$ denote the space of irreducible configurations. It is a submanifold of $\mathcal{C}(\tilde{P})$ (resp. $\bar{\mathcal{C}}^s(\tilde{P})$) as an open subset of that I.L.H. (Hilbert ) manifold. The above action is free when restricted to irreducible configurations:

$$\begin{array}{rcl}\bar{\Theta} : C^\infty(M, S^1) \times \bar{\mathcal{C}}(\tilde{P}) & \to & \bar{\mathcal{C}}(\tilde{P}) \\ (g, (A, \psi)) & \mapsto & (A, \psi) \cdot g := (\mathrm{det}g^* A, S^+(g^{-1})\psi).\end{array}$$

The action $\Theta_A$ is smooth, free and proper.

*An $L^2$-isometric action:* The Riemannian metric on $M$ induces a Hermitian product on the spinor bundle $S^+(\tilde{P})$ and hence one on the bundle $F$. Integrating this inner product on $M$, yields an $L^2$-metric on $\bar{\mathcal{C}}^s(\tilde{P})$ which is invariant under the $\Theta$ action.

*The tangent operator $\tau_x$:* The tangent operator at $Id$ to the map $\theta_{(A,\psi)} := \tilde{\Theta}(\cdot, (A, \psi))$ is the first order differential operator

$$\begin{array}{rcl}\tau_{(A,\psi)} : C^\infty(E) & \to & C^\infty(F) \\ f & \mapsto & (2df, -f \cdot \psi)\end{array}$$

which has injective symbol.

*The moduli space of irreducible configurations:* We can apply the slice theorem to the gauge group $\mathcal{G} := C^\infty(M, S^1)$, the space of irreducible configurations $X := \bar{\mathcal{C}}(\tilde{P})$ and conclude that the moduli space $X/\mathcal{G}$ of inequivalent irreducible configurations is a smooth I.L.H. manifold.

## 10.6 The Teichmüller space in string theory

The geometric setting for string theory is also that of Teichmüller theory. Useful references are [AJPS], [D], [DHP], [Tr] and many references therein.

Here $M$ is a closed Riemann surface of genus $p$ (which we shall assume here is larger than 1), $E := \mathbb{R} \oplus TM$ and $F := T^*M \otimes_s T^*M$ the symmetrized product of the cotangent bundle, where $\otimes_s$ denotes the symmetrized tensor product.
In what follows, $s$ will be assumed large enough for the sections of the different bundles to be continuous.

*The configuration space:* Let $\mathcal{M}(M) := \{g \in C^\infty(T^*M \otimes_s T^*M), \det g > 0\}$ (resp. $\mathcal{M}^s(M) := \{g \in H^s(T^*M \otimes_s T^*M), \det g > 0\}$) be the space of smooth (resp. $H^s$) Riemannian metrics on $M$; it is an I.L.H. (resp. Hilbert) manifold modelled on $C^\infty(T^*M \otimes_s T^*M)$ (resp. $H^s(T^*M \otimes_s T^*M)$).

*The gauge group:* Let $\mathcal{W}(M) := \{e^\phi, \phi \in C^\infty(M, \mathbb{R})\}$ (resp. $\mathcal{W}^s(M) := \{e^\phi, \phi \in H^s(M, \mathbb{R})\}$) be the group of smooth (resp. $H^s$) *Weyl transformations*, $\mathcal{D}(M) := \{f \in C^\infty(M, M), f^{-1} \in C^\infty(M, M)\}$ (resp. $\mathcal{D}^s(M) := \{f \in H^s(M, M), f^{-1} \in H^s(M, M)\}$) the group of smooth (resp. $H^s$) diffeomorphisms of $M$. $\mathcal{W}(M)$, $\mathcal{D}(M)$ are I.L.H. Lie groups modelled on $C^\infty(M, \mathbb{R})$, $C^\infty(TM)$ respectively. For fixed large enough $s$, $\mathcal{W}^s(M)$ is a Hilbert Lie group, but $\mathcal{D}^s(M)$ is not; it is a Hilbert manifold modelled on $H^s(TM)$ which is only a topological group. The I.L.H. setting is therefore useful to put a Lie group structure on diffeomorphisms.

Let $\mathcal{D}_0(M)$ denote the connected component of the identity map in $\mathcal{D}(M)$ and let

$$\mathcal{G} := \mathcal{D}_0(M) \odot \mathcal{W}(M)$$

where $\odot$ stands for the semi-direct product, namely the product with a twisted product law $(f, \phi) \odot (f', \phi') := (f \circ f', \phi \circ f' + \phi)$.

*Group actions:* The Weyl group $\mathcal{W}(M)$ acts on the configuration space $\mathcal{M}(M)$ by pointwise multiplication:

$$\begin{array}{rcl} \mathcal{W}(M) \times \mathcal{M}(M) & \to & \mathcal{M}(M) \\ (\phi, g) & \mapsto & e^\phi g \end{array}$$

and this I.L.H. action is smooth, free and proper. The set

$$Conf(M) := \{[g] := \{e^\phi \cdot g, e^\phi \in \mathcal{W}(M)\}, \quad g \in \mathcal{M}(M)\}$$

is an I.L.H. manifold, the manifold of *conformal structures.* It is diffeomorphic to the I.L.H. manifold

$$\mathcal{J}(M) := \{J \in C^\infty(TM \otimes T^*M),\ J \quad \text{preserves orientation} \quad \text{and} \quad J^2 = -I\}$$

of smooth almost complex structures on $M$ .

Because the genus is assumed to be larger than 1, in each conformal class $[g]$ of $g \in \mathcal{M}^s(M)$, there is an smooth metric with curvature $-1$. Let us set $\mathcal{M}_{-1}(M) := \{g \in \mathcal{M}(M), s_g = -1\}$ where $s_g$ is the scalar curvature of $g$.
There is a diffeomorphism of I.L.H. manifolds [Tr]:

$$\mathcal{M}_{-1}(M) \simeq \mathcal{J}(M) \simeq Conf(M).$$

The gauge group $\mathcal{G}$ of interest to us acts on $\mathcal{M}(M)$ by:

$$\begin{array}{rcl}\mathcal{D}_0(M) \odot \mathcal{W}(M) \times \mathcal{M}(M) & \rightarrow & \mathcal{M}(M) \\ ((f,\phi), g) & \mapsto & f^* e^{\phi} g\end{array}$$

and the action is smooth, free and proper (using here again the fact that the manifold has genus larger than 1).

*An $L^2$-metric on the configuration space:* In order to understand the quotient space, we use a splitting of the tangent space to the manifold of metrics. It is isomorphic to the space of covariant two tensors and splits into pure trace and traceless two covariant tensors:

$$T_g\mathcal{M}(M) \simeq C^\infty(T^*M \otimes_s T^*M) = C^\infty(M, I\!R) \cdot g \oplus C^\infty_{0,g}(T^*M \otimes_s T^*M)$$

where $C^\infty_{0,g}(T^*M \otimes_s T^*M) := \{h \in C^\infty(T^*M \otimes_s T^*M), \quad tr_g(h) := g^{ab}h_{ab} = 0\}$. This splitting is orthogonal w.r.to the inner product induced by the metric $g$ on $M$ on the space of smooth covariant two tensors. This inner product on $T_g\mathcal{M}(M)$ induces an $L^2$-metric on $\mathcal{M}(M)$ which is only invariant under $\mathcal{D}(M)$ but not under $\mathcal{W}(M)$. As we saw above, the action of the Weyl group on $\mathcal{M}(M)$ being a straightforward pointwise multiplication, the fact that it is not $L^2$- isometric is not a major obstruction to apply the slice theorem when taking the quotient. It is however a serious obstacle from the path integration point of view and this non invariance of the metric under Weyl transformations is a source of *conformal anomaly*.

## 10.7 Conformal covariant operators

We view the Laplace-Beltrami operator $\Delta_g$ associated with a Riemannian metric $g$ as an example of a more general class of conformally covariant operators.
Given a vector bundle $E$ over a closed $n$-dimensional manifold $M$, let us consider maps

$$\begin{array}{rcl}\mathcal{M}\mathrm{et}(M) & \rightarrow & Cl(M,E) \\ g & \mapsto & A_g,\end{array}$$

where $\mathcal{M}\mathrm{et}(M)$ denotes the space of Riemannian metrics on $M$.

**Definition 78** *The operator $A_g \in Cl(M,E)$ associated to a Riemannian metric $g$ is* **conformally covariant** *of* bidegree $(a,b)$ *if the pointwise scaling of the metric $\bar{g} = e^{2f}g$, for $f \in C^\infty(M)$ yields*

$$A_{\bar{g}} = e^{-bf} A_g e^{af} = e^{(a-b)f} A'_g, \quad \text{for } A'_g := e^{-af} A_g e^{af}, \tag{47}$$

*for constants $a, b \in I\!R$.*

**Remark 29** *If $A_g$ is of bidegree $(a,b)$, then its (formal)adjoint $A_g^*$ is of bidegree $(-b,-a)$.*

Here are some known conformally covariant differential operators; more details are in Chang [**?**].

**Differential operators of order 1.** (Hitchin [Hit]) For $M$ spin, the Dirac operator $D_g := \gamma^i \cdot \nabla_i^g$ is a conformally covariant operator of bidegree $\left(\frac{n-1}{2}, \frac{n+1}{2}\right)$.

**Differential operators of order 2.** If $\dim(M) = 2$, the Laplace-Beltrami operator $\Delta_g$ is conformally covariant of bidegree $(0,2)$. It is well known that in dimension two

$$R_{\bar{g}} = e^{-2f}\left(R_g + 2\Delta_g f\right), \tag{48}$$

where $R_g$ is the scalar curvature, and by the Gauss-Bonnet theorem (compare with Theorem 14)

$$\int_M R_g \, dA_g = 2\pi\chi(M), \tag{49}$$

with the Euler characteristic $\chi(M)$ a topological and hence a conformal invariant. Here $dA_g$ is the area element (intrinsically) defined by the metric $g$.

On a Riemannian manifold of dimension $n$, the Yamabe operator, also called the conformal Laplacian,

$$L_g := \Delta_g + c_n \, R_g,$$

is a conformally covariant operator of bidegree $\left(\frac{n-2}{2}, \frac{n+2}{2}\right)$, where $c_n := \frac{n-2}{4(n-1)}$.

For $u \in C^\infty(TM)$,

$$P_g u := \nabla_g u - \frac{1}{2}\mathrm{tr}_g(\nabla_g u) \cdot g,$$

corresponds to the traceless part of $\nabla_g u$. The operator $P_g^* P_g$ is a generalised Laplacian on vector fields, which is conformally covariant of bidegree $(-2,2)$.

## 10.8 Conformal anomalies

This subsection based on [**?**] and [AJPS] derives the conformal anomaly of the zeta determinant of a conformally covariant operator.

Let $M$ be a closed Riemannian manifold and $\mathcal{M}\mathrm{et}(M)$ denote the space of Riemannian metrics on $M$. The space $\mathcal{M}\mathrm{et}(M)$ is trivially a Fréchet manifold as the open cone of positive definite symmetric (covariant) two-tensors inside the Fréchet space

$$C^\infty(T^*M \otimes_s T^*M) := \{h \in C^\infty(T^*M \otimes T^*M) : h_{ab} = h_{ba}\}$$

of all smooth symmetric two-tensors. The Weyl group $\mathcal{W}(M) := \{e^f : f \in C^\infty(M)\}$ acts smoothly on $\mathcal{M}\mathrm{et}(M)$ by Weyl transformations

$$W(g,f) = \bar{g} := e^{2f} g,$$

and given a reference metric $g$ in $\mathcal{M}\mathrm{et}(M)$, a functional $\mathcal{F} : \mathcal{M}\mathrm{et}(M) \to \mathbb{C}$ induces a map

$$\begin{aligned} \mathcal{F}_g = \mathcal{F} \circ W(g,\cdot) : C^\infty(M) &\to \mathbb{C}, \\ f &\mapsto \mathcal{F}(e^{2f} g). \end{aligned}$$

**Definition 79** *A functional $\mathcal{F}$ on $\mathcal{M}\mathrm{et}(M)$ is conformally invariant for a reference metric $g$ if $\mathcal{F}_g$ is constant on a conformal class, i.e.*

$$\mathcal{F}(e^{2f}g) = \mathcal{F}(g) \quad \forall f \in C^\infty(M).$$

*A functional $\mathcal{F}$ on $\mathcal{M}\mathrm{et}(M)$ is conformally invariant if it is conformally invariant for all reference metrics. A functional $\mathcal{F} : \mathcal{M}\mathrm{et}(M) \times M \to \mathbb{C}$ is called a pointwise conformal covariant of weight $w$ if*

$$\mathcal{F}(e^{2f}g, x) = w \cdot f(x)\mathcal{F}(g,x) \quad \forall f \in C^\infty(M), \quad \forall x \in M.$$

A functional $\mathcal{F} : \mathcal{M}\mathrm{et}(M) \to \mathbb{C}$ which is Fréchet differentiable has a differential

$$d\mathcal{F}(g) : T_g\mathrm{Met(M)} = C^\infty(T^*M \otimes_s T^*M) \to \mathbb{C},$$

$$d\mathcal{F}(g).h := \frac{d}{dt}\Big|_{t=0} \frac{\mathcal{F}(g+th) - \mathcal{F}(g)}{t}.$$

For such a functional $\mathcal{F}$, the differentiability of the Weyl map implies that the composition $\mathcal{F}_g : C^\infty(M) \to \mathbb{C}$ is differentiable at 0 with differential $d\mathcal{F}_g(0) : T_0C^\infty(M) = C^\infty(M) \to \mathbb{C}$.

**Definition 80** *The* **conformal anomaly** *for the reference metric $g$ of a differentiable functional $\mathcal{F}$ on $\mathcal{M}\mathrm{et}(M)$ is $d\mathcal{F}_g(0)$. In physics notation, the conformal anomaly in the direction $f \in C^\infty(M)$ is*

$$\begin{aligned} \delta_f\mathcal{F}_g &:= d\mathcal{F}_g(0).f = d\mathcal{F}(g).2f\,g \\ &= \lim_{t=0} \frac{\mathcal{F}(g+2tfg) - \mathcal{F}(g)}{t} = \frac{d}{dt}\Big|_{t=0} \mathcal{F}(e^{2tf}g). \end{aligned}$$

**Remark 30** *$\mathcal{F}$ is conformally invariant if and only if $d\mathcal{F}_g(0).f = 0$ for all $g \in \mathcal{M}\mathrm{et}(M), f \in C^\infty(M)$.*

For a fixed Riemannian metric $g = (g_{ab})$, we equip $C^\infty(M)$ with the $L^2$ metric

$$(f, \tilde{f})_g = \int_M f(x)\tilde{f}(x)\mathrm{dvol}_g(x).$$

We define an $L^2$ metric on $\mathcal{M}\mathrm{et}(M)$ by

$$\langle h, k\rangle_g := \int_M g^{ac}(x)g^{bd}(x)\, h_{ab}(x)\, k_{cd}(x)\, \mathrm{dvol}_g(x) = \int_M h^{cd}(x)\, k_{cd}(x)\, \mathrm{dvol}_g(x) \quad (50)$$

with $(g^{ab}) = (g_{ab})^{-1}$ and $h^{ab}(x) := g^{ac}(x)g^{bd}(x)\, h_{cd}(x)$. The $L^2$ metric induces a weak $L^2$-topology on $\mathcal{M}\mathrm{et}(M)$, and $L^2(T^*M\otimes_s T^*M)$, the $L^2$-closure of $C^\infty(T^*M\otimes_s T^*M)$ with respect to $\langle\ ,\ \rangle_g$, is independent of the choice of $g$ modulo Hilbert space isomorphism. The choice of a reference metric yields the inner product (50) on the tangent space $T_g\mathcal{M}\mathrm{et}(M) = C^\infty(T^*M \otimes_s T^*M)$, giving the weak $L^2$ Riemannian metric on $\mathcal{M}\mathrm{et}(M)$, and forming the completion of each tangent space.

The various inner products are related as follows:

**Lemma 19** *For $g$ in $\mathcal{M}\mathrm{et}(M), h$ in $C^\infty(T^*M\otimes_s T^*M)$ and $f$ in $C^\infty(M)$, show that*

$$\langle h, f\,g\rangle_g = (\mathrm{tr}_g(h), f)_g$$

*where we have set: $\mathrm{tr}_g(h) := h^b_b = g^{ab}h_{ab}$.*

■

**Definition 81** *If the differential* $d\mathcal{F}(g) : C^\infty(T^*M \otimes_s T^*M) \to \mathbb{C}$ *extends to a continuous functional* $\overline{d\mathcal{F}(g)} : L^2(T^*M \otimes_s T^*M) \to \mathbb{C}$*, then by Riesz's lemma there is a unique two-tensor* $T_g(\mathcal{F})$ *with*

$$\overline{d\mathcal{F}(g)} \cdot h = \langle h, T_g(\mathcal{F})\rangle_g, \quad \forall h \in L^2(T^*M \otimes_s T^*M).$$

$T_g(\mathcal{F})$ *is precisely the* $L^2$ *gradient of* $\mathcal{F}$ *at* $g$*.*

**Proposition 33** *Let* $\mathcal{F}$ *be a functional on* $\mathcal{M}\mathrm{et}(M)$ *which is differentiable at the metric* $g$ *and whose differential* $d\mathcal{F}(g)$ *extends to a continuous functional* $\overline{d\mathcal{F}(g)} : L^2(T^*M \otimes_s T^*M) \to \mathbb{C}$*. Then the differential* $d\mathcal{F}_g(0)$ *also extends to a continuous functional* $\overline{d\mathcal{F}_g(0)} : L^2(M) \to \mathbb{C}$*. Identifying the conformal anomaly at* $g$ *with a function in* $L^2(M)$*, we have*

$$\overline{d\mathcal{F}_g(0)} = 2\,\mathrm{tr}_g\left(T_g(\mathcal{F})\right).$$

*In particular, the functional* $\mathcal{F}$ *is conformally invariant iff* $\mathrm{tr}_g\left(T_g(\mathcal{F})\right) = 0$ *for all metrics* $g$*.*

**Proof 53** *The differential* $d(\mathcal{F}_g)_0$ *extends to a continuous functional because*

$$d\mathcal{F}_g(0)\cdot f = d\mathcal{F}(g)(2fg) \Longrightarrow \overline{d\mathcal{F}_g(0)}\cdot f = \overline{d\mathcal{F}(g)}(2fg)$$

*By Exercise* **??**,

$$\overline{d\mathcal{F}_g(0)}\cdot f = \overline{d\mathcal{F}(g)}\cdot(2fg) = \langle T_g(\mathcal{F}), 2f\,g\rangle_g = 2\left(\mathrm{tr}_g(T_g(\mathcal{F})), f\right)_g,$$

*as desired.*

**Definition 82** *Under the assumptions of the Proposition, the function*

$$x \mapsto \delta_x \mathcal{F}_g := 2\mathrm{tr}_g\left(T_g(\mathcal{F})\right)(x)$$

*is called the* **local anomaly** *of the functional* $\mathcal{F}$ *at the reference metric* $g$*.*

This is taken from bosonic string theory.
Let $(M,g)$ be a closed Riemann surface and $X : M \to \mathbb{R}^d$ a smooth map. Let

$$\Delta_g := -\frac{1}{\sqrt{\det g}}\,\partial_i \sqrt{\det g}\, g_{ij}\partial_j$$

(where as before $\det g$ stands for the determinant of the metric matrix $(g_{ij})$ and $(g^{ij})$ its inverse) denote the Laplace-Beltrami operator on $M$. The classical **Polyakov action** [?] (see also [AJPS] and references therein for a review) for bosonic string reads

$$\mathcal{A}(X,g) := \langle \Delta_g X, X\rangle_g. \tag{51}$$

1. Show that it yields a conformal invariant action depending on $X$.

2. For any $h \in C^\infty(M, T^*M \otimes_s T^*M)$, show that

$$d\mathcal{A}(X,\cdot)(g)\cdot h = \langle h, T_g(x)\rangle_g$$

where $T_g$ is the two-covariant tensor $T_{ij} := \partial_i X^\mu \, \partial_j X^\mu - \frac{1}{2}\mathrm{tr}_g(\partial_i X^\mu \partial_j X^\mu)\, g$ called the **energy-momentum tensor**.

3. Check that $\mathrm{tr}_g(T_g) = 0$ as a consequence of the conformal invariance of $\mathcal{A}(X,\cdot)$.

*The Teichmüller space:* Up to the fact that the action is isometric only for $\mathcal{D}_0(M)$, a discrepancy we argued is only a minor difficulty when applying the above theorem because the action of the Weyl group is rather straightforward, we can apply the slice theorem. The quotient space, called the Teichmüller space

$$\mathcal{T}(M) := \mathcal{M}(M)/\mathcal{G}(M)$$

is a smooth finite dimensional manifold (its dimension over $I\!R$ is $6p-6$) and we have the following diffeomorphisms of finite dimensional manifolds [AJPS], [Tr]:

$$\mathcal{T}(M) \simeq \mathcal{J}(M)/\mathcal{D}_0(M) \simeq Conf(M)/\mathcal{D}_0(M).$$

## References

[Ad1] J.F. Adams, *Lectures on Lie groups*, Benjamin-Cummings 1967

[Ad2] R.A. Adams, *Sobolev spaces*, Academic Press 1975

[At] M: F: Atiyah, *K-theory*, Benjamin, New York 1967

[AM] R. Abraham, E. Marsden, *Foundations of mechanics*, Benjamin-Cummings 1978

[AJPS] S.Albeverio, J. Jost, S. Paycha, S. Scarlatti, *A mathematical introduction to string theory- variational problems, geometric and probabilistic methods*, London Math. Soc. lecture Notes Series **225**, Cambridge University Press 1997

[B] L. Baggett, *Functional Analysis, A primer*, Dekker 1992

[BGV] N. Berline, E.Getzler, M. Vergne, *Heat-Kernels and Dirac Operators*, Grundlehren Band Springer Verlag 1991, 2nd. ed. 1997

[BDIP] J.Bertin, J.-P. Demailly, L. Illusie, C.Peters, *Introduction à la théorie de Hodge*, Panoramas et Synthèses, N. ß, Soc.Math.France 1996

[Br] G.E. Bredon, *Introduction to compact transformation groups*, Academic Press 1972

[Bre] H. Brézis, *Analyse fonctionnelle, Théorie et Applications*, Masson 1983

[CT] H-M. Chan, S-T. Tsun, *Elementary Gauge Theory Concepts*, World Scientific 1993

[D] P. Deligne, P. Etingof, D. Freed, L. Jeffrey, D. Kazhdan, J. Morgan, D. Morrison, E. Witten, *Quantum Fields and Strings: A course for Mathematicians*, Vol I,II, Am. Math. Soc. 1999

[DHP] E. D'Hoker, D. Phong, *The geometry of string perturbation theory*, Reviews of Modern Physics, Vol **60**, n.**4** 1988

[Dr] B. Driver, *Analytic tools with applications*, Springer math.ucsd.edu/ driver/231-02-03/LectureNotes/PDE.../analpde1.pdf

[E] D.G. Ebin, *The manifold of Riemannian metrics*, Proc. Symp. Pure Math. Amer. Math. Soc. **XV** (1970) p.11-40

[FU] D. Freed, K. Uhlenbeck, *Instantons and four-manifolds*, Springer Verlag 1984

[F] T. Friedrich, *Dirac-Operatoren in der Riemanschen Geometrie*, Vieweg, Braunschweig-Wiesbaden, 1997

[GHL] S. Gallot, D.Hulin, J. Lafontaine, *Riemannian geometry*, Springer Verlag 1980, 2nd ed. 2002

[Ge] I. M. Gel'fand , *On elliptic equations*, Russ. Math.Surv. 15 (3) 1960 p. 113123 (reprinted in volume 1 of his collected works, 6575)

[Gi] P.B. Gilkey, *Invariance Theory, the Heat Equation and the Atiyah-Singer Index Theorem*, 2nd ed. Studies in Advanced Mathematics, CRC Press 1995

[Go] S. Goldberg, *Curvature and Homology*, Academic Press 1962

[Gr] M. Greenberg, *Lectures on algebraic topology*, Mathematics Lecture Notes, W.A. Benjamin 1967

[H] O. Hijazi, *Spectral properties of the Dirac operator and geometrical structures*, in [OPR]

[Hi] M.W. Hirsch, *Differential topology*, Graduate Texts in Mathematics, Springer Verlag 1976

[HL] F. Hirsch, G. Lacombe, *Elements of functional analysis*, Graduate Texts in Mathematics **192**, Springer Verlag 1999

[Hit] N. Hitchin, *Harmonic spinors*, Adv. in Math. **14** (1994) 1-55

[Hu] S.T. Hu, *Differentiable Manifolds*, Holt, Rinehart and Winston 1969

[HUs] D. Husemoller, *Fibre bundles*, McGraw-Hill Series 1966; 2nd edition, GTM vol. **20**, Springer Verlag 1975

[J] J. Jost, Riemannian geometry and geometric analysis, Springer Verlag, Berlin 1995; 2nd ed. 1998; 3rd ed. 2002

[Kac] V. Kac,*Can you hear the shape of a drum?*, Amer. Math. Montly **73** (1966) 1-23

[1] M. Karoubi, *K-theory,: an introduction*, Springer Verlag, Berlin 1978

[K] W. Klingenberg, *Riemannian geometry*, de Gruyter Studies in Mathematics 1982

[KM] A. Kriegl, P.W. Michor, *The convenient setting for global analysis* A.M.S. Math. Surveys and Monographs **53**, American Mathematical Society, Providence (1997)

[KN] S. Kobayashi, N. Nomizu, *Foundations of differential geometry I, II*, Interscience, John Wiley 1963, 2nd ed. 1969

[KR] R. Kadison, J. Ringrose, *Fundamentals of the theory of operator algebras*, Volume 1., Academic Press 1983

[KRo] W. Kondracki, J. Rogulski, *On the stratification of the orbit space*, Dissertationes Mathematicae **CCL**, Warsaw 1986

[L] S. Lang, *Differential and Riemannian Manifolds*, Springer 1995 (3rd edition of "Differential Manifolds", originally published by Addison-Wesley 1962)

[LM] B. Lawson and M. Michelson, *Spin geometry*, Princeton University Press, Princeton 1989

[Ma] M. Marcolli, *Seiberg-Witten gauge theory*, Hindustan Book Agency 1999

[M] Y. Matsushima, *Differentiable manifolds*, Pure and Applied Mathematics 1972

[MM] K. Marathe, G. Martucci, *The geometry of gauge fields*, Journ. Geom. Phys. **6** p.1-106 1989

[Mo] J. Morgan, *The Seiberg-Witten Equations and Applications to the Topology of Smooth Manifolds*, Mathematical Notes 44, Princeton University Press 1996

[Mu] G. Murphy, *$C^*$-algebras and operator theory*, Academic Press, London, 1990

[MS] J. Milnor, J. Stasheff, *Characteristic classes*, Princeton University Press 1974

[MT] I. Masden, J. Tornehave, *From calculus to cohomology*, Cambridge University Press 1997

[Na] M. Nakahara, *Geometry, topology and Physics*, Graduate Student Series in Physics, IOP Publishing 1990

[N] C. Nash, *Differential topology and quantum field theory*, Academic Press 1991

[O] H. Omori, *Infinite dimensional Lie groups*, AMS Transl. of Math. Monographs **158** Am. Math. Soc. , Providence, R.I. (1997)

[OPR] H. Ocampo, S. Paycha, A. Reyes, editors of *Geometric Methods for Quantum Field Theory*, Proceedings of a Summer School, World Scientific, 2001

[P] R.S. Palais, *Foundations of global non linear analysis*, Benjamin Company Inc. , New York 1968

[RS] M. Reed, B. Simon, *Methods of modern mathematical physics I, II*, Academic Press, London and New York, 1972

[R] S. Rosenberg, *The Laplacian on a Riemanian manifold*, London Mathematical Society, Student Texts 1997

[RLL] M. Røodam, F. Larsen, N.J. Lautsen, *An introduction to $K$-theory for $C^*$-algebras*, London Math. Soc. Student texts **49** 2000

[Ru] W. Rudin, *Functional Analysis*, Mac Graw Hill 1973

[S] N. Steenrod, *The topology of fibre bundles*, Princeton University Press , 1951

[Sh] M. Shubin, *Pseudodifferential operators and spectral theory*, Springer Verlag 1987, 2nd ed.
2001

[T] M. Taylor, *Pseudodifferential operators*, Lect. Notes in Math. **416**, Springer Verlag 1974

[Tr] A. Tromba, *Teichmüller theory in Riemannian geometry*, Lectures in Mathematics, ETH Zürich, Birkhäuser, Basel, 1992

[Ts] S.T.Tsou, *Concepts in gauge theory leading to electric-magnetic duality*, in [OPR]

[We] N.E. Wegge-Olsen, *K-theory and $C^*$-algebras*, A firendly approach, Oxford University Press 1993

[W] J. Weidmann, *Linear operators in Hilbert spaces*, Graduate texts in Mathematics, Springer Verlag 1980

[Wu] T. Wurzbacher, *Introduction to differentiable manifolds and symplectic geometry*, in [OPR].