Yang-Baxter relations and Stokes phenomenon

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Algebraic, analytic, geometric structures emerging from quantum field theory
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$$ {}_1F_1(a,b;z) \sim \frac{\Gamma(b)}{\Gamma(a)} e^{-z} z^{b-a} + \frac{\Gamma(b)}{\Gamma(b-a)} (-z)^a, \quad as\ z \to 0. $$

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• A fundamental subject in differential equations, special functions, integrable systems. It has deep relation with Gromov-Witten theory, stability conditions, symplectic and complex geometry, cluster algebras, TQTF and so on. However, very hard to study.

Stokes matrices of ODEs with second order poles

- Consider the linear system on z-plane

$$\frac{dF}{dz} = \left(\frac{u}{z^2} + \frac{A}{z}\right) F,$$

where $F(z) \in \mathfrak{gl}_n$, $u = \mathrm{diag}(u_1, ..., u_n)$, and $A \in \mathfrak{gl}_n$.

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where $F(z) \in \mathfrak{gl}_n$, $u = \mathrm{diag}(u_1, ..., u_n)$, and $A \in \mathfrak{gl}_n$.

- Any fundamental solution $F(z) \in \mathrm{GL}_n$ has asymptotics

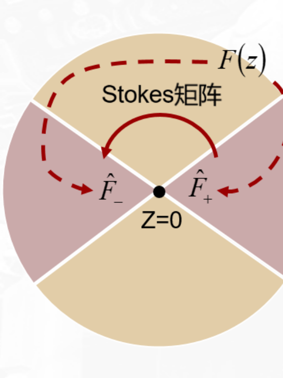
$$e^{\frac{u}{z}} z^{-[A]} \cdot F(z) \sim T_\pm \quad as\ z \to\ 0\ in\ left/right\ planes\ \mathbb{H}_\pm,$$

for some invertible constant matrices $T_\pm$.

- The different asymptotics of $F(z)$ are measured by the ratio

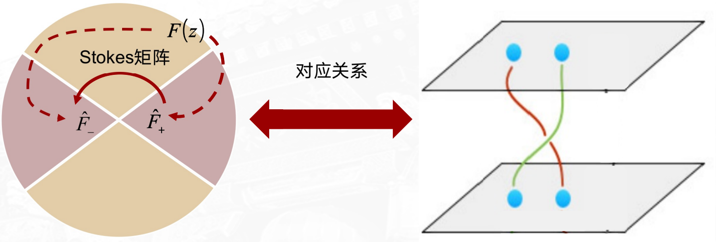
$$S_+(A,u) = T_+ \cdot T_-^{-1},$$

called Stokes matrix, similarly define $S_-(A,u)$.



Idea: study the transcendental Stokes phenomenon via quantum algebras

为什么可以用表示论的工具研究复分析和微分方程中的Stokes现象？



渐近行为跳跃性

量子群定义中的Yang-Baxter关系

Part I

Quantum group and the Stokes phenomenon at second order pole

Quantum groups and canonical basis

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$$[e_i, f_j] = \delta_{ij} \frac{q^{h_i - h_{i+1}} - q^{-h_i + h_{i+1}}}{q - q^{-1}};$$

- for $|i - j| = 1$,

$$e_i^2 e_j - (q + q^{-1}) e_i e_j e_i + e_j e_i^2 = 0.$$

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• (Crystal basis) A finite set $\mathbb{B}_\lambda$ equipped with operators $\tilde{e}_i$ and $\tilde{f}_i$ model on a canonical weight basis of $L(\lambda)$: if $v \in L_\mu$, then $\tilde{e}_i(v) \in L_{\mu+\alpha_i}$.

- $U(\mathfrak{gl}_n)$: generator $\{e_{ij}\}$, relation $[e_{ij}, e_{kl}] = \delta_{jk}e_{il} - \delta_{li}e_{kj}$.

Stokes matrices of ODEs in noncommutative rings

- $U(\mathfrak{gl}_n)$: generator $\{e_{ij}\}$, relation $[e_{ij}, e_{kl}] = \delta_{jk}e_{il} - \delta_{li}e_{kj}$.
- n by n matrix $T = (T_{ij})$ with entries valued in $U(\mathfrak{gl}_n)$

$$T_{ij} = e_{ij}, \qquad \text{for } 1 \leq i, j \leq n.$$

- For any $u \in \mathfrak{h}_{\mathrm{reg}}(\mathbb{R})$ n by n diagonal matrices with distinct real eigenvalues, any representation $L(\lambda)$ of $U(\mathfrak{gl}_n)$, consider

$$\frac{dF}{dz} = h\left(\frac{u}{z^2} + \frac{T}{z}\right) \cdot F,$$

for a function $F(z) \in \mathrm{Mat}_n \otimes \mathrm{End}(L(\lambda))$.

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- The quantum Stokes matrices $S_{h\pm}(u) = (S_{h\pm}(u)_{ij})$, with entries $S_{h\pm}(u)_{ij}$ in $\mathrm{End}(L(\lambda))$.

Representations of quantum group from Stokes matrices

Theorem (Xu)

For any fixed $h \in \mathbb{C}^*$ *and* $u \in \mathfrak{h}_{\mathrm{reg}}$*, the map (with* $q = e^{h/2}$*)*

$$\mathcal{S}_q(u) : U_q(\mathfrak{gl}_n) \to \mathrm{End}(L(\lambda)) \ ; \ e_i \mapsto S_{h+}(u)_{i,i+1}, \ f_i \mapsto S_{h-}(u)_{i+1,i}$$

defines a representation of the Drinfeld-Jimbo quantum group $U_q(\mathfrak{gl}_n)$ *on the vector space* $L(\lambda)$*.*

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- Equivalently, take standard R-matrix $R \in \mathrm{End}(\mathbb{C}^n) \otimes \mathrm{End}(\mathbb{C}^n)$,

$$R = \sum_{i \neq j, i,j=1}^{n} E_{ii} \otimes E_{jj} + e^{\pi \mathrm{i} h} \sum_{i=1}^{n} E_{ii} \otimes E_{ii} + (e^{\pi \mathrm{i} h} - e^{-\pi \mathrm{i} h}) \sum_{1 \leq j < i \leq n} E_{ij} \otimes E_{ji}.$$

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Then

$$R^{12} {S_\pm}^{(1)} {S_\pm}^{(2)} = {S_\pm}^{(2)} {S_\pm}^{(1)}.$$

A dictionary

Table: A dictionary

	Stokes phenomenon at 2nd order pole	Quantum group $U_q(\mathfrak{gl}_n)$ with $q = e^{\pi i h}$
1	Nonresonant case $h \notin \mathbb{Q}$	Realization of $U_q(\mathfrak{gl}_n)$ at a generic q
2	Resonant case $h \in \mathbb{Q}$	Representation at roots of unity
3	WKB approximation as $h \to \infty$	$\mathfrak{gl}_n$-Crystals
4	Wall-crossing in WKB approximation	Cactus group actions on crystals
5	Whitham dynamics	HKRW covers on eigenbasis
6	Analytic branching rules	Braching rules/ Gelfand-Tsetlin theory
7	Asymptotic Riemann-Hilbert problem	An explicit Drinfeld isomorphism
8	Involution of equations	Quantum symmetric pairs
9	Formal power series solutions	Yangians/ Trigonometric R-matrix
10	Semiclassical limit	Dual Poisson Lie groups

WKB approximation and crystal limits

- A $\mathfrak{gl}_n$-crystal is a finite set which models a weight basis for a representation of $\mathfrak{gl}_n$, and crystal operators $\tilde{e}_i$ and $\tilde{f}_i$ indicate the leading order behaviour of the simple root vectors on the basis under the crystal limit $q \to 0$ in quantum group $U_q(\mathfrak{gl}_n)$ ($q = e^{h/2}$).

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• to study the limits of q-Stokes matrices $S_{h\pm}(u) = (S_{h\pm}(u)_{ij})$ as $h \to -\infty$, where $S_{h\pm}(u)_{ij} \in \mathrm{End}(L(\lambda))$.

WKB analysis and crystals

- The algebraic characterization of the $h \to \infty$ asymptotics of $S_{h\pm}(u) \in \mathrm{End}(L(\lambda)) \otimes \mathrm{End}(\mathbb{C}^n)$ of $\frac{1}{h}\frac{dF}{dz} = \left(\frac{u}{z^2} + \frac{T}{z}\right) \cdot F$.

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- The action of the off-diagonal entry $S_{h+}(u)_{k,k+1}$ on certain canonical basis $\{v_i(u)\}_{i\in I}$ of $L(\lambda)$ has,

$$S_{h+}(u)_{k,k+1} \cdot v_i(u) = \sum_{j\in I} e^{h\phi_{ij}^{(k)}(u) + \sqrt{-1} g_{ij}^{(k)}(u,h)} \big(v_j(u) + O(h^{-1})\big),$$

where $\phi_{ij}^{(k)}(u)$, $g_{ij}^{(k)}(u,h)$ are real valued functions for all $1 \leq i, j \leq k \leq n-1$.

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• The WKB approximation of $S_{h+}(u)_{k,k+1}$ naturally defines an operator $\tilde{e}_k$ by picking the unique leading term

$$\tilde{e}_k(v_i(u)) := v_j(u), \quad \text{if} \quad \phi_{ij}^{(k)}(u) = \max\{\phi_{il}^{(k)}(u) \mid l \in I\}.$$

A transcendental realization of crystals

Conjecture (Xu, Proved under the WKB asmptotic assumption)

For any $u \in \mathfrak{h}_{\text{reg}}(\mathbb{R})$*, there exists a canonical basis* $\{v_I(u)\}$ *of* $L(\lambda)$*, operators* $\tilde{e}_k(u)$ *and* $\tilde{f}_k(u)$ *for* $k = 1, ..., n-1$ *such that there exists constants* c*,* c'

$$\lim_{q=e^{\pi i h}\to 0} q^{c} S_{h+}(u)_{k,k+1} \cdot v_I(u) = \tilde{e}_k(v_I(u)),$$

$$\lim_{q=e^{\pi i h}\to 0} q^{c'} S_{h-}(u)_{k+1,k} \cdot v_I(u) = \tilde{f}_k(v_I(u)).$$

Furthermore, the datum $(\{v_I(u)\}, \tilde{e}_k(u), \tilde{f}_k(u))$ *is a* $\mathfrak{gl}_n$*−crystal.*

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Theorem (Xu)

The conjecture is true as $u_n \gg u_{n-1} \gg \cdots \gg u_1$*. And the WKB datum coincides with the known* $\mathfrak{gl}_n$*-crystal structure on semistandard Young tableaux.*

Part II

Arbitrary order pole and quantization of Riemann-Hilbert mpas

Quantum Stokes matrices at pole of order $k+1$

- The universal enveloping algebra $U\big(\mathfrak{gl}_n(\mathbb{C}[t]/t^k)\big)$ generated by $\{e_{ij}t^{m-1}\}$ for $i,j=1,...,n$ and $m=1,...,k$ subject to the relation

$$[e_{ij}t^a, e_{kl}t^b] = \begin{cases} \delta_{jk}e_{il}t^{a+b} - \delta_{li}e_{kj}t^{a+b}, & \text{if } a+b \le k \\ 0, & \text{if } a+b > k. \end{cases}$$

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- Consider the equation

$$\frac{dF}{dz} = h\left(\frac{u}{z^{k+1}} + \frac{T_{[k]}}{z^k} + \cdots + \frac{T_{[2]}}{z^2} + \frac{T_{[1]}}{z}\right)\cdot F,$$

where $u \in \mathfrak{h}_{\text{reg}}$, h is a complex parameter, each $T_{[m]}$ is an $n\times n$ matrix with entries valued in $U\big(\mathfrak{gl}_n(\mathbb{C}[t]/t^k)\big)$

$$(T_{[m]})_{ij} = e_{ij}t^{m-1}, \qquad \text{for } 1\le i,j\le n, \ \ 1\le m\le k.$$

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- $2k$ quantum Stokes matrices

$$S_i(u) \in \widehat{U}\big(\mathfrak{gl}_n(\mathbb{C}[t]/t^k)\big)\otimes \mathrm{End}(\mathbb{C}^n)\ \ for\ i=1,...,2k$$

Here S_{2i+1} is upper triangular and S_{2i} is lower triangular.

- Take the standard R-matrix $R \in \mathrm{End}(\mathbb{C}^n) \otimes \mathrm{End}(\mathbb{C}^n)$,

$$R = \sum_{i \neq j, i,j=1}^{n} E_{ii} \otimes E_{jj} + e^{\pi \mathrm{i} h} \sum_{i=1}^{n} E_{ii} \otimes E_{ii} + (e^{\pi \mathrm{i} h} - e^{-\pi \mathrm{i} h}) \sum_{1 \leq j < i \leq n} E_{ij} \otimes E_{ji}.$$

- Introduce

$$\mathbb{S}_{[i]}^{(1)} := S_1^{(1)} S_2^{(1)} \cdots S_i^{(1)} \in \widehat{U}\big(\mathfrak{gl}_n(\mathbb{C}[t]/t^k)\big) \otimes \mathrm{End}(\mathbb{C}^n) \otimes \mathrm{End}(\mathbb{C}^n),$$
$$\mathbb{S}_{[i]}^{(2)} := S_{i+1}^{(2)} S_{i+2}^{(2)} \cdots S_{2k}^{(2)} \in \widehat{U}\big(\mathfrak{gl}_n(\mathbb{C}[t]/t^k)\big) \otimes \mathrm{End}(\mathbb{C}^n) \otimes \mathrm{End}(\mathbb{C}^n).$$

Here the indices are taken modulo $2k$.

Theorem (Xu)

For any $u \in \mathfrak{h}_{\mathrm{reg}}$, the quantum Stokes matrices satisfy the algebraic relations ($RL...L = L...LR$)

$$\mathbb{R}^{12} \mathbb{S}_{[i]}^{(1)} \mathbb{S}_{[i]}^{(2)} = \mathbb{S}_{[i]}^{(2)} \mathbb{S}_{[i]}^{(1)} \mathbb{R}^{12}, \quad i = 1, ..., 2k-1.$$

Part III

Quantization of Riemann-Hilbert maps

Irregular Riemann-Hilbert maps at pole of order $k+1$

- Consider the differential equations for a function $f(z) \in \mathrm{GL}_n$

$$\frac{df}{dz} = \left(\frac{u}{z^{k+1}} + \frac{A_k}{z^k} + \cdots + \frac{A_2}{z^2} + \frac{A_1}{z}\right) \cdot f,$$

where $u \in \mathfrak{h}_{\mathrm{reg}}$, and $A_i \in \mathfrak{gl}_n$.

- For fixed u, the moduli space is the dual $A(t) \in \mathfrak{gl}_n(\mathbb{C}[t]/t^k)^*$

$$A(t) = A_1 + A_2 t + \cdots + A_k t^{k-1}.$$

- The space of Stokes matrices is $\mathcal{M}^{(k)} := \{(U_- \times U_+)^k\}$.

Theorem (Boalch)

For fixed $u \in \mathfrak{h}_{\mathrm{reg}}$, the irregular Riemann-Hilbert map

$$\mathcal{S}(u) : \mathfrak{gl}_n(\mathbb{C}[t]/t^k)^* \to \mathcal{M}^{(k)} \ ; \ A(t) \mapsto (S_1, ..., S_{2k})$$

is a locally analytic Poisson isomorphism.

- Each $S_i(A(t); u)$ is in $\widehat{Sym}(\mathfrak{gl}_n(\mathbb{C}[t]/t^a)) \otimes \mathrm{End}(\mathbb{C}^n)$.

Quantum RH maps at arbitrary order poles

For the case of pole of order $k+1$, we have the commutative diagram

$$\begin{array}{ccc} U_\hbar^{(k)} & \xrightarrow{\text{q-Stokes matrices } \{S_i\}} & U(\mathfrak{gl}_n(\mathbb{C}[t]/t^k))[\![\hbar]\!] \\ {\scriptstyle h\to 0}\downarrow & & {\scriptstyle h\to 0}\downarrow \\ Fun(\mathcal{M}^{(k)}) & \xrightarrow{\nu(u)^*} & Sym(\mathfrak{gl}_n(\mathbb{C}[t]/t^k)) \end{array}$$

Here recall

$$\nu(u) : \mathfrak{gl}_n(\mathbb{C}[t]/t^k)^* \to \mathcal{M}^{(k)} \ ; \ A(z) \mapsto (S_1, ..., S_{2k}).$$

Quantum RH maps at a second order pole

$$\begin{array}{ccc} \text{Associative algebra } U_\hbar & \xrightarrow{\text{q-Riemann-Hilbert map}} & \text{Underformed algebra } U \\ h \to 0 \Big\downarrow & & h \to 0 \Big\downarrow \\ Fun(\mathcal{M}_{Betti}) & \xrightarrow{\text{Pull back of RH map}} & Fun(\mathcal{M}_{deRham}) \end{array}$$

Here in some context, $Fun(\mathcal{M}_{deRham})$ is the Poisson algebra of functions on the moduli space of connections, $Fun(\mathcal{M}_{Betti})$ is the Poisson algebra of functions on space of monodromy data.

In a very special case, $\mathcal{M}_{Betti}$ is the dual Poisson Lie group, $U_\hbar$ the quantum group and $\mathcal{M}_{Betti} = \mathfrak{g}^*$ the dual Lie algebra, and $U = U(\mathfrak{g})$. Theorem 1.2 states that the RH map is a Poisson map. Thus a quantum analog of Theorem 1.2 would be an associated algebra isomorphism between $U(\mathfrak{g})$ and $U_\hbar(\mathfrak{g})$, constructed in a transendental way (from a study of some quantum differential equation).

Thank you very much!