Introduction to combinatorial species

Course 2

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Algebraic, analytic and geometric structures emerging from quantum field theory

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Sequences with a combinatorial/probabilistic flavor

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Moment problem: $(a_n)_n$ is the sequence of moments of some measure if and only if the Hankel matrices associated to the sequence are positive definite.

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be the *formal Laplace transform*. It is the exponential generating function for moments. We can write this series as

$$\mathcal{F}(z) = e^{K(z)},$$

where

$$K(z) = \sum_{n \geq 1} \kappa_n \frac{z^n}{n!}$$

is the exponential generating function for cumulants.

By the *exponential formula*,

$$\text{since } \mathcal{F}(z) = e^{K(z)}, \text{ then we have } m_\pi = \sum_{\pi \leq \tau} \kappa_\tau,$$

where $m_\pi = m_{|B_1|} m_{|B_2|} \cdots m_{|B_k|}$ and $\kappa_\pi = \kappa_{|B_1|} \kappa_{|B_2|} \cdots \kappa_{|B_k|}$ if $\pi = \{B_1, B_2, \ldots, B_k\}$.

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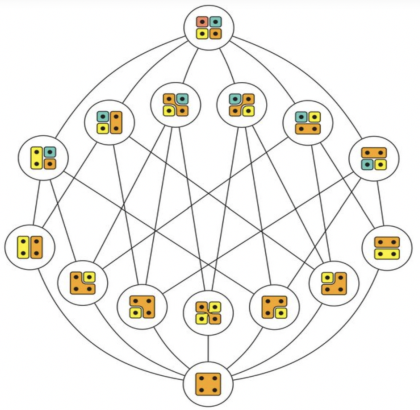
Here, $\leq$ corresponds to the poset of partitions $\Pi(n)$ of the set $[n] := \{1, 2, \ldots, n\}$ with the refinement order.

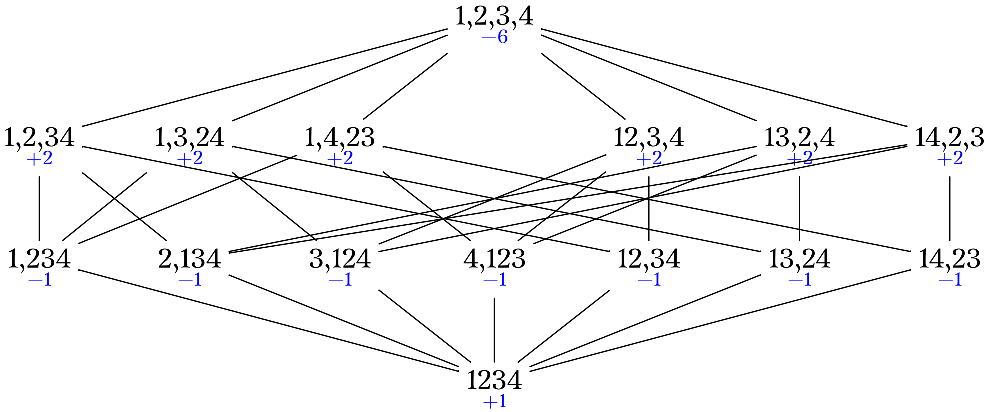
Hence,

$$m_n = \sum_{\pi \in \Pi(n)} \kappa_\pi$$

$$\updownarrow$$

$$\kappa_n = \sum_{\pi \in \Pi(n)} \mu(\widehat{0}, \pi) m_\pi.$$





$$\kappa_4 := \kappa_{|1234|} = m_4 - 4m_1m_3 - 3m_2m_2 + 12m_1m_1m_2 - 6m_1m_1m_1m_1.$$

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- the (classical) cumulant sequence $(k_n(f))_{n\geq 0}$:

$$k_n(f) := \sum_{\pi \in \Pi(n)} \mu(\widehat{0}, \pi) f(\pi);$$

- the free cumulant sequence $(c_n(f))_{n\geq 0}$:

$$c_n(f) := \sum_{\pi \in \mathsf{NC}(n)} \mu(\widehat{0}, \pi) f(\pi);$$

- the boolean sequence $(b_n(f))_{n\geq 0}$:

$$b(f)_n := \sum_{\pi \in \mathsf{NC}_{\mathsf{int}}(n)} \mu(\widehat{0}, \pi) f(\pi).$$

For example,

$$f\left(\left\{\{3,8,9\},\{1,2\},\{6\},\{4,6,7\}\right\}\right) = a_{|\{3,8,9\}|} \cdot a_{|\{1,2\}|} \cdot a_{|\{6\}|} \cdot a_{|\{4,6,7\}|} = a_1 a_2 a_3^2.$$

Non-crossing partitions

A set partition $\pi = \{B_1, \ldots, B_k\}$ of $[n] := \{1, 2, \ldots, n\}$ is **non-crossing** if we do not have

$$p_1 < q_1 < q_2 < p_2 \qquad \text{and} \qquad p_1, p_2 \in B_i, q_1, q_2 \in B_j, i \neq j.$$

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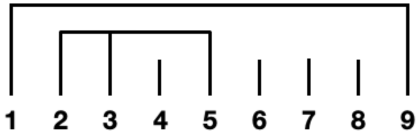
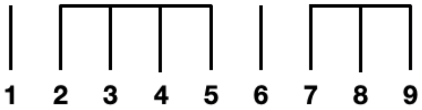
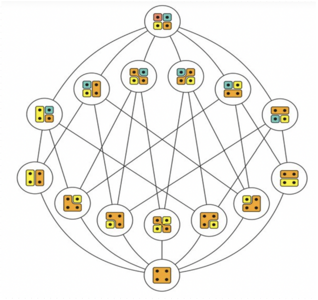
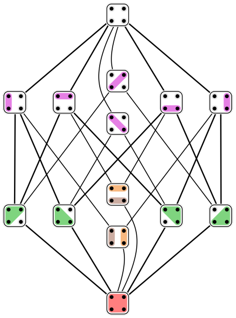
$NC(n)$ = set of non-crossing partitions of $[n]$.
$NC_{int}(n)$ = set of interval non-crossing partitions of $[n]$.

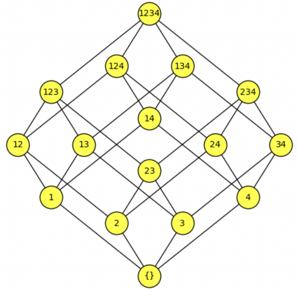




Cumulant	Poset	Size	Möbius function $\mu(\widehat{0}, \widehat{1})$
Classical	$\Pi(n)$	Bell_n	$(-1)^{n-1}(n-1)!$
Free	$\mathrm{NC}(n)$	Cat_n	$(-1)^{n-1}\mathrm{Cat}_{n-1}$
Boolean	$\mathrm{NC}_{\mathrm{int}}(n)$	2^{n-1}	$(-1)^{n-1}$







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These formulas seem to encode "connected" structures of certain kind.