Introduction to combinatorial species

Lectures 3 and 4

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Algebraic, analytic and geometric structures emerging from quantum field theory

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Content

Hopf algebras

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Classical Hopf algebras: Borel, Cartier, Hopf (1940-1950).

Motivation: algebraic topology, homological algebra, study of loop spaces, algebras of operations (Steenrod), homology of Eilenberg–MacLane spaces.

Joni-Rota: "*A great many problems in combinatorics are concerned in assembling, or disassembling, large objects out of pieces of prescribed shape, as in the familiar board puzzles.* "
(Joni, S. A., & Rota, G. C. (1979). *Coalgebras and bialgebras in combinatorics*. Studies in Applied Mathematics, 61(2), 93-139.)

Hopf algebras

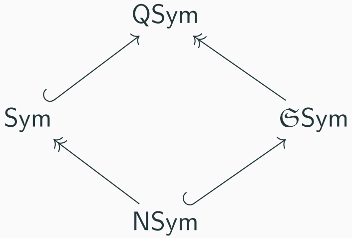
A *Hopf algebra* $(H, m, \iota, \Delta, \varepsilon, S)$ consists of

- an associative algebra (H, m, ι);
- a coassociative coalgebra (H, Δ, ε);
- compatibility between the product and the coproduct;
- the identity map id : $H \to H$ is invertible in the **convolution algebra** $(\mathrm{End}(H), *)$, where

$$f * g := m \circ (f \otimes g) \circ \Delta.$$

The inverse of id, denoted by S, is called *the antipode of H*.
Finding an optimal formula for the antipode is not easy. It provides a rich information about hidden combinatorial structures on H.

A (graded, connected) Hopf-algebraic square



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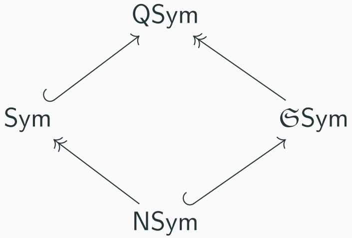
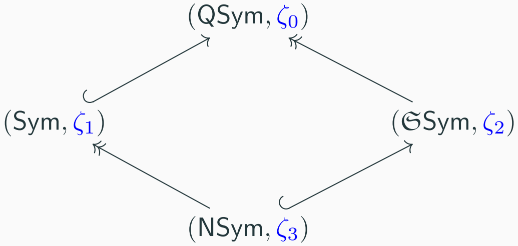


- QSym: quasisymmetric functions (compositions)
- Sym: symmetric functions (partitions)
- 𝔖Sym: free quasisymmetric functions (permutations)
- NSym: non-commutative symmetric functions (compositions)

A (combinatorial) Hopf-algebraic square



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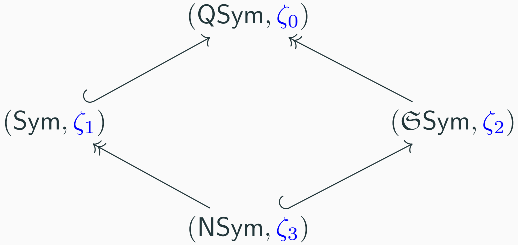

- $\mathrm{Sym} \hookrightarrow \mathrm{QSym} \hookrightarrow \mathbb{K}[x_1, x_2, \ldots]$ and
  $\mathrm{NSym} \hookrightarrow \mathfrak{S}\mathrm{Sym} \hookrightarrow \mathbb{K}\langle x_1, x_2, \ldots \rangle$

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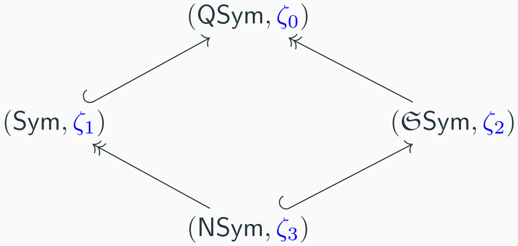

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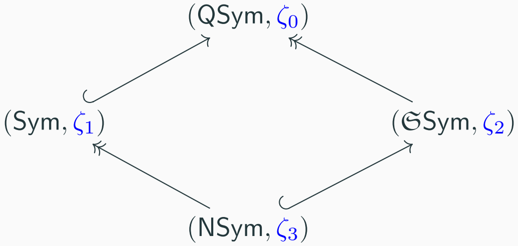

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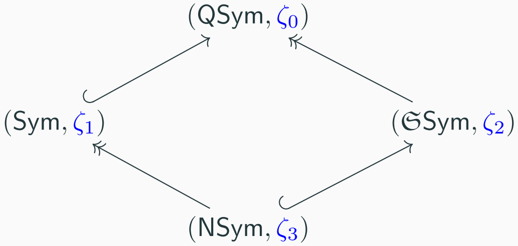

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- Theorem (Aguiar, Bergeron, Sottile): (QSym, ζ_0) is terminal.

A (combinatorial) Hopf algebras square

The map $\Psi : (H, \zeta) \to (\mathsf{QSym}, \zeta_0)$

$$\begin{array}{ccc} H & \xrightarrow{\Psi} & \mathsf{QSym} \\ & {}_{\zeta}\searrow \quad \swarrow_{\zeta_0} & \\ & \mathbb{K} & \end{array}$$

is defined, for every $h \in H_n$ and $n \geq 0$, as

$$\Psi(h) = \sum_{c \text{ composition of } n} \zeta_c(h) \mathsf{M}_c,$$

where, for $c = (c_1, c_2, \ldots, c_k)$, ζ_c is the composite

$$H \xrightarrow{\Delta^{(k-1)}} H^{\otimes k} \twoheadrightarrow H_{c_1} \otimes \cdots \otimes H_{c_k} \xrightarrow{\zeta^{\otimes k}} \mathbb{K} .$$

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The map Ψ explains the “ubiquity” of quasisymmetric functions as generating functions in combinatorics.

A combinatorial example

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The *chromatic symmetric function* of G is

$$X(G) = X(G; x_1, x_2, \ldots) := \sum_{\mathsf{col}} \prod_{v \in V(G)} x_{\mathsf{col}(v)},$$

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- If $n = |V(G)|$, then $X(G)$ is homogeneous of degree n.
- $X(G)$ is symmetric ($X(G) \in \text{Sym}$).
- Under $x_i \leftarrow 1$, for $1 \leq i \leq t$, and $x_i \leftarrow 0$, for $i < t$, written $x = 1^t$, then $X(G; 1^t)$ is the (classical) chromatic polynomial on t.

Using the universal property

Let $\mathcal{G} = \mathbb{K}\{$ isomophism classes of finite (unonriented) graphs $\}$.

If $G, H \in \mathcal{G}$, let $G \cdot H := G \sqcup H$ the disjoint union. Also, let

$$\Delta(G) := \sum_{S \subseteq V(G)} G|_S \otimes G|_{V(G)\setminus S}.$$

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Theorem: $\Psi(G)$ is the chromatic symmetric function.

$(\mathcal{G}, \cdot, \Delta)$ is called the *chromatic Hopf algebra of graphs*.

Species

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This action extends to a (covariant) functor

$$\Sigma : \mathsf{FinSet} \to \mathsf{Vect},$$

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- $I \mapsto \Sigma[I]$;
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The construction Σ is an example of a *vector species*.

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$$I \xrightarrow{\alpha} J \xrightarrow{\beta} K \implies \mathsf{p}[\beta \circ \alpha] = \mathsf{p}[\beta] \circ \mathsf{p}[\alpha],$$

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For every $n \in \mathbb{N}$, $\mathfrak{S}_{\mathfrak{n}}$ acts on $\mathsf{p}[n]$ via $\sigma \cdot x := \mathsf{p}[\sigma](x)$. Therefore,

$$\text{species } \mathsf{p} \longleftrightarrow V = (V_n)_{n \geq 0}, V_n \text{ is a } \mathfrak{S}_n\text{-module}.$$

Examples of species

- Species E of **sets**:

$$\mathsf{E}[I] := \mathbb{K}\{*_I\}.$$

- Species E_n of n-**sets**:

$$\mathsf{E}_n[I] := \begin{cases} \mathbb{K}\{*_I\}, & \text{if } |I| = n; \\ (0), & \text{if } |I| \neq n. \end{cases}$$

- Species $\mathsf{X} := \mathsf{E}_1$ of sets of one element.
- Species $\mathsf{1} := \mathsf{E}_0$.
- Species G of **graphs**:

$$\mathsf{G}[I] := \mathbb{K}\{\text{ finite graphs with vertices in } I\ \}.$$

Examples of species

- Species Π of **partitions**.
- Species L of **linear orders**.
- Species Σ of **set compositions**.
- Species B of **binary trees**.
- Species $\mathfrak{S}$ of **permutations**.
- Species Braid of **braid hyperplane arrangements**.

$$\vdots$$

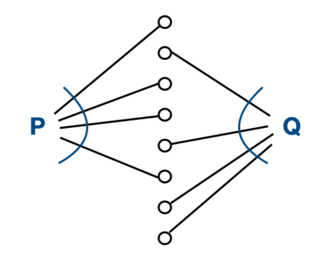
Operations on species

- **Sum of species**

$$(\mathsf{p}+\mathsf{q})[I] := \mathsf{p}[I] \oplus \mathsf{q}[I].$$

- **Product of species** (Cauchy product)

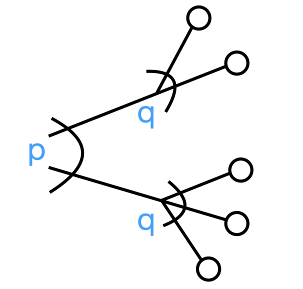
$$(\mathsf{p}\cdot\mathsf{q})[I] := \bigoplus_{I=S\sqcup T} \mathsf{p}[S] \otimes \mathsf{q}[T].$$



Operations on species

- **Composition of species**

$$(\mathsf{p} \circ \mathsf{q})[I] := \bigoplus_{\pi \in \Pi[I]} \mathsf{p}[\pi] \otimes \bigotimes_{B \in \pi} \mathsf{q}[B].$$



Generating function of a species

To every species p it is associated its **exponential generating function**:

$$\mathsf{p}(x) := \sum_{n \geq 0} \dim_{\mathbb{K}} \mathsf{p}[n] \frac{x^n}{n!}.$$

We have:

$$(\mathsf{p} + \mathsf{q})(x) = \mathsf{p}(x) + \mathsf{q}(x),$$

$$(\mathsf{p} \cdot \mathsf{q})(x) = \mathsf{p}(x) \cdot \mathsf{q}(x),$$

$$(\mathsf{p} \circ \mathsf{q})(x) = \mathsf{p}(x) \circ \mathsf{q}(x).$$

For the last identity, $\mathsf{q}[\emptyset] := (0)$.

The category Sp of vector species

A **morphism of species** $\mathsf{p} \xrightarrow{f} \mathsf{q}$ is a collection $f = (f_I)$ of linear maps such that

$$\begin{array}{ccc} \mathsf{p}[I] & \xrightarrow{f_I} & \mathsf{q}[I] \\ {\scriptstyle \mathsf{p}[\sigma]}\big\downarrow & & \big\downarrow{\scriptstyle \mathsf{q}[\sigma]} \\ \mathsf{p}[J] & \xrightarrow[f_J]{} & \mathsf{q}[J] \end{array}$$

for every $I \xrightarrow{\sigma} J$. This defines the category Sp of species.

Recall that the **Cauchy product** of two species p and q is given by

$$(\mathsf{p} \cdot \mathsf{q})[I] = \bigoplus_{I = S \sqcup T} \mathsf{p}[S] \otimes \mathsf{q}[T].$$

Recall that the **Cauchy product** of two species p and q is given by

$$(\mathsf{p} \cdot \mathsf{q})[I] = \bigoplus_{I = S \sqcup T} \mathsf{p}[S] \otimes \mathsf{q}[T].$$

Endowed with this operation, Sp is symmetric monoidal: we can speak of monoids ($\mu : \mathsf{p} \cdot \mathsf{p} \to \mathsf{p}$), comonoids ($\Delta : \mathsf{p} \to \mathsf{p} \cdot \mathsf{p}$), ..., in species.

$$\mathsf{p}[S] \otimes \mathsf{p}[T] \xrightarrow{\mu_{S,T}} \mathsf{p}[I] \qquad \mathsf{p}[I] \xrightarrow{\Delta_{S,T}} \mathsf{p}[S] \otimes \mathsf{p}[T].$$

Algebraic structures on Sp

Algebraic structures in Sp

A *monoid* in Sp is given by (a, μ, ι), where a is a species and

$$\mu : \mathsf{a} \cdot \mathsf{a} \to \mathsf{a} \qquad , \qquad \iota : 1 \to \mathsf{a}.$$

Explicitly, if $I = S \sqcup T$ then

$$\mu_{S,T} : \mathsf{a}[S] \otimes \mathsf{a}[T] \to \mathsf{a}[I].$$

The map ι is uniquely determined by its component $\iota_\emptyset : \mathbb{K} \to \mathsf{a}[\emptyset]$.

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The maps μ and ι must satisfy *associativity*, *unitality* and *naturality* axioms.

A *comonoid* in Sp is given by $(\mathsf{c}, \Delta, \varepsilon)$, where c is a species and

$$\Delta : \mathsf{c} \to \mathsf{c} \cdot \mathsf{c} \qquad , \qquad \varepsilon : \mathsf{c} \to 1.$$

Explicitly, if $I = S \sqcup T$ then

$$\Delta_{S,T} : \mathsf{c}[I] \to \mathsf{c}[S] \otimes \mathsf{c}[T].$$

The map ε is uniquely determined by its component $\varepsilon_\emptyset : \mathsf{c}[\emptyset] \to \mathbb{K}$.

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Notions of **bimonoids** and **Hopf monoids** exist, analogues to bialgebras and Hopf algebras.

Algebraic structures in Sp

Proposition: any *graded*, *locally finite* and *connected* bialgebra H is a Hopf algebra.

A species h is *connected* (resp. *positive*) if $\dim_{\mathbb{K}} \mathsf{h}[\emptyset] = 1$ (resp. $\dim_{\mathbb{K}} \mathsf{h}[\emptyset] = 0$).

Proposition: any connected bimonoid h is a Hopf monoid.

The antipode is a map $s : \mathsf{h} \to \mathsf{h}$. When a bimonoid h possess an antipode, it is unique.

Algebraic structures in Sp: examples

- $(\mathbf{E}, \mu, \Delta)$

$$\mathbf{E}[I] := \mathbb{K}\{\mathtt{H}_I\}.$$

$$\mu_{S,T}(\mathtt{H}_S \otimes \mathtt{H}_T) := \mathtt{H}_{S \sqcup T} \qquad , \qquad \Delta_{S,T}(\mathtt{H}_I) := \mathtt{H}_S \otimes \mathtt{H}_T.$$

$$s_I(\mathtt{H}_I) = (-1)^{|I|}\mathtt{H}_I.$$

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- $(\mathbf{L}, \mu, \Delta)$

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The product of $\mathbf{L}$ is **concatenation**, while the coproduct is **deshuffle**. There is also a Hopf monoid $(\mathbf{\Sigma}, \mu, \Delta)$ with analogues operations.

Algebraic structures in Sp

A *Lie monoid* in Sp is given by $(\mathfrak{g}, [\ ,\ ])$, where $\mathfrak{g}$ is a species and

$$[\ ,\ ] : \mathfrak{g} \cdot \mathfrak{g} \to \mathfrak{g}$$

satisfies

- Anticommutativity

$$[x, y]_{S,T} = -[y, x]_{T,S};$$

- Jacobi identity:

$$[[x, y]_{S,T}, z]_{S \sqcup T,R} + [[z, x]_{R,S}, y]_{R \sqcup S,T} + [[z, x]_{T,R}, x]_{T \sqcup R,S} = 0.$$

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The species $\mathrm{Prim}(\mathsf{c})$ of *primitive parts* of c is given by

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If h is connected, then $\mathrm{Prim}(\mathsf{h})$ is positive and

$$\mathrm{Prim}(\mathsf{h})[I] = \bigcap_{\substack{S \sqcup T = I \\ S, T \neq \emptyset}} \ker(\Delta_{S,T} : \mathsf{h}[I] \to \mathsf{h}[S] \otimes \mathsf{h}[T]),$$

for every $I \neq \emptyset$.

Monoids in species, revisited

Monoids in species, revisited

A *monoid* in Sp is given by (a, μ, ι), where a is a species and

$$\mu : \mathsf{a} \cdot \mathsf{a} \to \mathsf{a} \qquad , \qquad \iota : 1 \to \mathsf{a}.$$

Explicitly, if $I = S \sqcup T$ then

$$\mu_{S,T} : \mathsf{a}[S] \otimes \mathsf{a}[T] \to \mathsf{a}[I].$$

The map ι is uniquely determined by its component $\iota_\emptyset : \mathbb{K} \to \mathsf{a}[\emptyset]$.

The maps μ and ι must satisfy *associativity*, *unitality* and *naturality* axioms.

Monoids in species, revisited

Naturality: the product map behaves well with respect with the transport of structures (relabeling).

More precisely, if $I = S \sqcup T$ and $\sigma : I \to J$ is a bijection. the diagram

$$\begin{array}{ccc} \mathsf{a}[S] \otimes \mathsf{a}[T] & \xrightarrow{\mu_{S,T}} & \mathsf{a}[I] \\ {\scriptstyle \mathsf{a}[\sigma|_S] \otimes \mathsf{a}[\sigma|_T]} \downarrow & & \downarrow {\scriptstyle \mathsf{a}[\sigma]} \\ \mathsf{a}[\sigma(S)] \otimes \mathsf{a}[\sigma(T)] & \xrightarrow[\mu_{\sigma(S),\sigma(T)}]{} & \sigma[J] \end{array}$$

commutes.

Monoids in species, revisited

Unitality:

$$\begin{array}{ccc} \mathsf{a} & \xleftarrow{\ \mu\ } & \mathsf{a}\cdot\mathsf{a} \\ & {\scriptstyle =}\ \nwarrow & \uparrow {\scriptstyle \iota\cdot\mathrm{id}} \\ & & 1\cdot\mathsf{a} \end{array} \qquad \begin{array}{ccc} \mathsf{a}\cdot\mathsf{a} & \xrightarrow{\ \mu\ } & \mathsf{a} \\ {\scriptstyle \mathrm{id}\cdot\iota}\uparrow & \nearrow\ {\scriptstyle =} & \\ \mathsf{a}\cdot 1 & & \end{array}$$

The unit axiom states that for each finite set I, the diagrams

$$\begin{array}{ccc} \mathsf{a}[I] & \xleftarrow{\ \mu_{\emptyset,I}\ } & \mathsf{a}[\emptyset]\otimes\mathsf{a}[I] \\ & {\scriptstyle \cong}\ \nwarrow & \uparrow {\scriptstyle \iota_\emptyset\otimes\mathrm{id}_I} \\ & & \mathbb{K}\otimes\mathsf{a}[I] \end{array} \qquad \begin{array}{ccc} \mathsf{a}[I]\otimes\mathsf{a}[\emptyset] & \xrightarrow{\ \mu_{I,\emptyset}\ } & \mathsf{a}[I] \\ {\scriptstyle \mathrm{id}_I\otimes\iota_\emptyset}\uparrow & \nearrow\ {\scriptstyle \cong} & \\ \mathsf{a}[I]\otimes\mathbb{K} & & \end{array}$$

commute.

Monoids in species, revisited

Associativity : given a decomposition $I = R \sqcup S \sqcup T$,

$$
\begin{array}{ccc}
\mathsf{a} \cdot \mathsf{a} \cdot \mathsf{a} & \xrightarrow{\mathrm{id} \cdot \mu} & \mathsf{a} \cdot \mathsf{a} \\
{\scriptstyle \mu \cdot \mathrm{id}} \big\downarrow & & \big\downarrow {\scriptstyle \mu} \\
\mathsf{a} \cdot \mathsf{a} & \xrightarrow[\mu]{} & \mathsf{a}
\end{array}
\qquad
\begin{array}{ccc}
\mathsf{a}[R] \otimes \mathsf{a}[S] \otimes \mathsf{a}[T] & \xrightarrow{\mathrm{id} \otimes \mu_{S,T}} & \mathsf{a}[R] \otimes \mathsf{a}[S \sqcup T] \\
{\scriptstyle \mu_{R,S} \otimes \mathrm{id}} \big\downarrow & & \big\downarrow {\scriptstyle \mu_{R, S \sqcup T}} \\
\mathsf{a}[R \sqcup S] \otimes \mathsf{a}[T] & \xrightarrow[\mu_{R \sqcup S, T}]{} & \mathsf{a}[I]
\end{array}
$$

Monoids in species, revisited

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From the associativity of the collection $(\mu_{S,T})_{S,T}$, there is a unique map called the *higher product map* of a

$$\mathsf{a}[S_1]\otimes\cdots\otimes\mathsf{a}[S_k] \xrightarrow{\mu_{S_1,\ldots,S_k}} \mathsf{a}[I] \qquad \text{for every } I = S_1 \sqcup \cdots \sqcup S_k, k \geq 0,$$

obtained by iterating the product maps $\mu_{S,T}$.

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obtained by iterating the product maps $\mu_{S,T}$.

If $F = S_1 | \cdots | S_k \vDash I$, we define $\mu_F := \mu_{S_1, \ldots, S_k}$ and $\mathsf{a}(F) := \mathsf{a}[S_1] \otimes \cdots \otimes \mathsf{a}[S_k]$, so

$$\mu_F : \mathsf{a}(F) \to \mathsf{a}[I].$$

Monoids in species from higher product maps

Theorem(Aguiar-Mahajan): Let a be a connected species equipped with a collection of maps

$$\mu_F : \mathsf{a}(F) \to \mathsf{a}[I], \qquad \text{for every } F \vDash I, I \text{ finite set .}$$

Then a is a connected monoid with higher products maps μ_F if and only if the naturality axiom holds and the diagram

$$\begin{array}{ccc} \mathsf{a}(G) & \xrightarrow{\mu_G} & \mathsf{a}[I] \\ {\scriptstyle \mu_{G/F}}\downarrow & \nearrow {\scriptstyle \mu_F} & \\ \mathsf{a}(F) & & \end{array}$$

commutes, for each compositions F and G of I with $F \leq G$.

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Here, $\leq$ refers to the *refinement partial order* on set compositions. Also, G/F is a set composition of I constructed from G and F.

The combinatorics of set compositions encode algebraic properties of connected monoids in species.

The Tits monoid of set compositions

Let I be a finite set.

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Let $\Sigma[I]$ be the set of all compositions of I. If $F \in \mathsf{S}[I]$, we write $F \vDash I$. There is a unique set composition on the empty set, so $|\Sigma[\emptyset]| = 1$.

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For example,

$$2|569|3|1478 \in \Sigma[10].$$

Operations on set compositions

- **Concatenation**

  Let $I = \{a, b, c, d, e, f, g, h\}$ and let $I = S \sqcup T$, with

  $$S = \{a, b, c, d, e\} \quad \text{and} \qquad T = \{f, g, h\}.$$

  Consider

  $$F = de|abc \qquad \text{and } G = fg|h.$$

  The **concatenation** of F and G is

  $$F \odot G := de|abc|fg|h \vDash I.$$

If p is species and $F, G \vDash I$, there is a canonical isomorphism

$$\mathsf{p}(F) \otimes \mathsf{p}(G) \cong \mathsf{p}(F \odot G).$$

Operations on set compositions

- **Tits product** (Jacques Tits - 1974; Coxeter groups, Buildings)

  Let $I = \{a, b, c, d, e, f, g, h\}$ and consider

  $$F = cdfg|ah|be \vDash I \qquad \text{and} \qquad G = adefh|bcg \vDash I.$$

  The **Tits product** of F and G is

  $$F \cdot G := df|cg|ah|\ |e|b \equiv df|cg|ah|e|b \vDash I.$$

  $(\Sigma[I], \cdot)$ is a monoid (with unit (I)), called the **Tits monoid on** I.
  The Tits product is *strongly* non-commutative:

  $$G \cdot F = df|ah|e|cg|\ |b \equiv df|ah|e|cg|b.$$

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  The Tits product is intimately related to the *refinement order* on set compositions.

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Let I be a finite set and $F, G \vDash I$.

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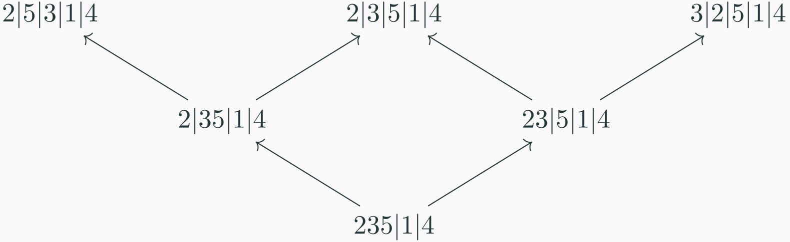
$F \leq G$ if each block of F is a reunion of **adyacent** blocks of G.



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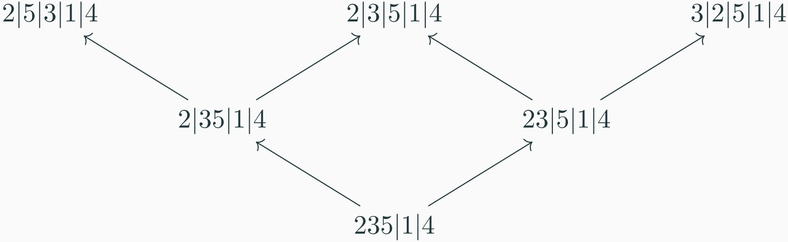

Minimal element: $\widehat{0}_I := (I)$
Maximal elements: permutations in $\mathfrak{S}_I$.

Refinement order on set partitions

Let I be a finite set and let $\Pi[I]$ be the set of all set partitions of I.

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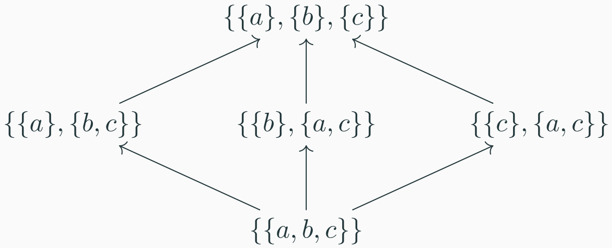

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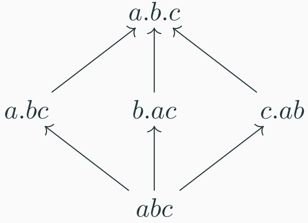

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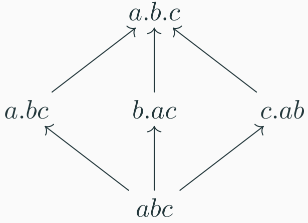

Minimal element: $\widehat{0}_I := \{I\}$
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Let $F, G \vDash I$. We have:

1. $F \leq F \cdot G$.
2. $F \leq G \iff F \cdot G = G$.
3. $F^2 = F$.
4. $F \cdot G \cdot F = F \cdot G$ (The monoid $(\Sigma[I], \cdot)$ is a *left regular band*)
5. $\text{supp}(F \cdot G) = \text{supp}_I(F) \vee \text{supp}(G)$.
6. $G \cdot F = G \iff \text{supp}(F) \leq \text{supp}(G)$.

Back to species

Splitting operation

Let $F, G \vDash I$.

$F \leq G \iff \exists!$ "splitting" $G/F := G_1|\cdots|G_k$ of G with $\begin{cases} G_j \vDash I_j \\ F = I_1|\cdots|I_j \end{cases}$

Let (a, μ, ι) be a monoid. Define $\mu_{G/F} : \mathsf{a}(G) \to \mathsf{a}(F)$ by means of the diagram

$$\begin{array}{ccc}
\mathsf{a}(G) & \overset{\mu_{G/F}}{\dashrightarrow} & \mathsf{a}(F) \\
\cong \downarrow & & \parallel \\
\mathsf{a}(G_1) \otimes \cdots \otimes \mathsf{a}(G_k) & \xrightarrow[\mu_{G_1} \otimes \cdots \otimes \mu_{G_k}]{} & \mathsf{a}[I_1] \otimes \cdots \otimes \mathsf{a}[I_k]
\end{array}$$

Monoids in species from higher product maps

Theorem(Aguiar-Mahajan): Let a be a connected species equipped with a collection of maps

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The combinatorics of set compositions encode algebraic properties of connected monoids in species.

Comonoids in species, revisited

A *comonoid* in Sp is given by $(\mathsf{c}, \Delta, \varepsilon)$, where c is a species and

$$\Delta : \mathsf{c} \to \mathsf{c} \cdot \mathsf{c} \qquad , \qquad \varepsilon : \mathsf{c} \to 1.$$

Explicitly, if $I = S \sqcup T$ then

$$\Delta_{S,T} : \mathsf{c}[I] \to \mathsf{c}[S] \otimes \mathsf{c}[T].$$

The map ε is uniquely determined by its component $\varepsilon_\emptyset : \mathsf{c}[\emptyset] \to \mathbb{K}$.

The maps Δ and ε must satisfy *coassociativity*, *counitality* and *naturality* axioms.

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Exercise: write explicitely the naturality axiom, counitality axiom and coassociative axiom for the coproduct in a comonoid.

Comonoids in species, revisited

Given a decomposition $I = S_1 \sqcup \cdots \sqcup S_k$, there is a unique map

$$\mathsf{c}[I] \xrightarrow{\Delta_{S_1,\ldots,S_k}} \mathsf{a}[S_1] \otimes \cdots \otimes \mathsf{a}[S_k].$$

For $k = 1$, this map is defined to be the identity of $\mathsf{c}[I]$, and for $k = 0$ to be the counit map $\varepsilon_\emptyset$.

The map $\Delta_{S_1,\ldots,S_k}$ is called the *higher coproduct map* of a.

As before, if $F = S_1|\cdots|S_k \vDash I$, we define write $\Delta_F := \Delta_{S_1,\ldots,S_k}$.

Hence,

$$\Delta_F : \mathsf{a}[I] \to \mathsf{a}(F).$$

Bimonoids in species

Definition/Theorem(Aguiar-Mahajan): Let h be a connected species equipped with two collections of maps

$$\mu_F : \mathsf{h}(F) \to \mathsf{h}[I] \quad \text{and} \quad \Delta_F : \mathsf{h}[I] \to \mathsf{h}(F),$$

one map for each composition F of a nonempty finite set I. Then h is a bimonoid with higher product maps μ_F and higher coproduct maps Δ_F if and only if the following conditions hold:

- naturality,
- higher associativity,
- higher coassociativy,
- higher compatitiblity: the diagram commutes for any pair of compositions $F, G \vDash I$.

$$\begin{array}{ccccc} \mathsf{h}(FG) & & \xrightarrow{\quad\beta\quad} & & \mathsf{h}(GF) \\ \uparrow{\scriptstyle \Delta_{FG/F}} & & & & \downarrow{\scriptstyle \mu_{GF/F}} \\ \mathsf{h}(F) & \xrightarrow[\mu_F]{} & \mathsf{h}[I] & \xrightarrow[\Delta_G]{} & \mathsf{h}(G) \end{array}$$

Questions?

Bimonoids in species

A bimonoid h is at the same time a monoid (h, μ, ι) and a comonoid $(\mathsf{h}, \Delta, \varepsilon)$, which are *related* in the following way: the maps

$$\mu : \mathsf{h} \cdot \mathsf{h} \to \mathsf{a} \qquad , \qquad \iota : 1 \to \mathsf{h}$$

are *morphism of comonoids*. This is equivalent to ask that the maps

$$\Delta : \mathsf{h} \to \mathsf{h} \cdot \mathsf{h} \qquad , \qquad \varepsilon : \mathsf{h} \to 1$$

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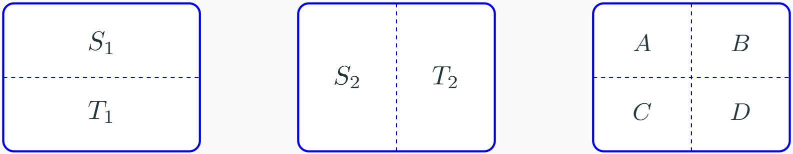
In order to describe the compatibility rule, fix decompositions $S \sqcup T = I = S' \sqcup T'$, and consider the resulting pairwise intersections:

$$A = S_1 \cap S_2,\ B = S_1 \cap T_2,\ C = T_1 \cap S_2,\ D = T_1 \cap T_2.$$



$$
\begin{array}{ccccc}
\mathsf{h}[A] \otimes \mathsf{h}[B] \otimes \mathsf{h}[C] \otimes \mathsf{h}[D] & & \xrightarrow{\mathrm{id} \otimes \beta \otimes \mathrm{id}} & & \mathsf{h}[A] \otimes \mathsf{h}[C] \otimes \mathsf{h}[B] \otimes \mathsf{h}[D] \\
\Delta_{A,B} \otimes \Delta_{C,D} \Big\uparrow & & & & \Big\downarrow \mu_{A,C} \otimes \mu_{B,D} \\
\mathsf{h}[S_1] \otimes \mathsf{h}[S_2] & \xrightarrow[\mu_{S_1,S_2}]{} & \mathsf{h}[I] & \xrightarrow[\Delta_{T_1,T_2}]{} & \mathsf{h}[T_1] \otimes \mathsf{h}[T_2]
\end{array}
$$

$$
\begin{array}{ccc}
\mathsf{h}[\emptyset] \otimes \mathsf{h}[\emptyset] & \xrightarrow{\varepsilon_\emptyset \otimes \varepsilon_\emptyset} & \mathbb{K} \otimes \mathbb{K} \\
\mu_{\emptyset,\emptyset} \Big\downarrow & & \Big\downarrow \cong \\
\mathsf{h}[\emptyset] & \xrightarrow[\varepsilon_\emptyset]{} & \mathbb{K}
\end{array}
\qquad
\begin{array}{ccc}
\mathbb{K} & \xrightarrow{\iota_\emptyset} & \mathsf{h}[\emptyset] \\
\cong \Big\downarrow & & \Big\downarrow \Delta_{\emptyset,\emptyset} \\
\mathbb{K} \otimes \mathbb{K} & \xrightarrow[\varepsilon_\emptyset \otimes \varepsilon_\emptyset]{} & \mathsf{h}[\emptyset] \otimes \mathsf{h}[\emptyset]
\end{array}
$$

$$
\begin{array}{ccccc}
 & & \mathsf{h}[\emptyset] & & \\
 & \overset{\iota_\emptyset}{\nearrow} & & \overset{\varepsilon_\emptyset}{\searrow} & \\
\mathbb{K} & & \xrightarrow[\cong]{} & & \mathbb{K}
\end{array}
$$