Asymptotic expansion conjecture for Seifert fibered integral homology sphere via resurgence theory

Algebraic, analytic and geometric structures emerging from quantum field theory, Chengdu

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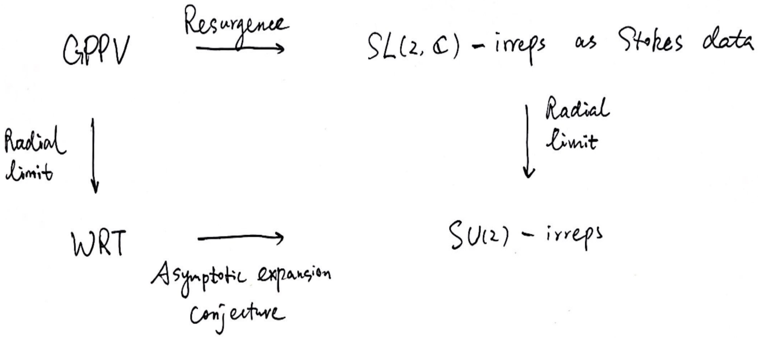
1. Outline
2. Seifert fibered integral homology sphere (SFIHS)
3. Chern-Simons theory and resurgence

Outline


GPPV
Resurgence
SL(2, ℂ) - irreps as Stokes data
Radial limit
Radial limit
WRT
Asymptotic expansion conjecture
SU(2) - irreps

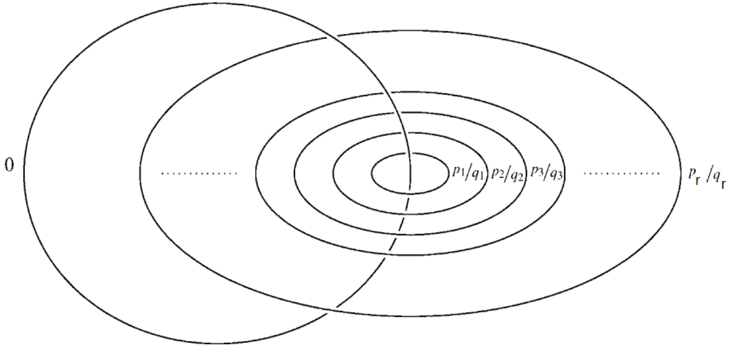
1. Outline
2. Seifert fibered integral homology sphere (SFIHS)
3. Chern-Simons theory and resurgence

Definition: Seifert fibered integral homology sphere (SFIHS)

- $r \geq 3$ (the number of the singular fibers),
- $p_1, \cdots, p_r$ pairwise coprime, $P := \prod p_j$, $\widehat{p}_j := \frac{P}{p_j}$. Let $\widehat{p}_1 = p_2 \cdots p_r$ be odd.
- Choose $\{q_j\}_j^r$ s.t. $(p_j, q_j) = 1$ and $\sum_{j=1}^{r} \widehat{p}_j q_j = 1$.

Definition: SFIHS

Rational surgery in S^3 along the following link:



SFIHS

After rational surgery, the resulting 3-manifold (closed, oriented) is called Seifert fibered integral homology sphere (SFIHS). And denote it by $\sum(p_1, \cdots, p_r)$.

例 (Example)

- $\sum(2,3,5)$ Poincaré homology sphere
- $\sum(2,3,7)$
- $\sum(2,3,5,7)$

Character variaty - The space of irreducible G-representation of the fundamental group

Let X be a SFIHS $\sum(p_1, \cdots, p_r)$.

$$\pi_1(X) = \left\langle x_1, \cdots, x_r, h \;\middle|\; h \text{ center}, \; x_j^{p_j} h^{-q_j}, \; x_1 \cdots x_r \right\rangle$$

Let G be $SU(2)$ or $SL(2, \mathbb{C})$. Consider the character variaties $\mathcal{M}(X, G)$, i.e. the space of of irreducible G-representation of the fundamental group

$$\mathcal{M}(X, G) := Hom^{irre}\left(\pi_1(X), G\right) \Big/ G - conjugation$$

An irreducible G-represention is a homomorphism

$$\begin{aligned}\rho_{\underline{\ell}} : \pi_1(X) &\to G \\ h &\mapsto (-1)^{p_1} I \\ x_j &\mapsto S_j \begin{pmatrix} e^{\frac{i\pi\ell_j}{p_j}} & \\ & e^{-\frac{i\pi\ell_j}{p_j}} \end{pmatrix} S_j^{-1}\end{aligned}$$

for some $\{S_j\}_{j=1}^{r} \subset G$ with an overall factor in G.

Proposition (Andersen, Mistegård)

$$\pi_0(\mathcal{M}(X, SL(2,\mathbb{C})) \cong \mathfrak{L}_{\underline{p}}$$

where

$$\mathfrak{L}_{\underline{p}} = \left\{ \underline{\ell} = (\ell_1, \cdots, \ell_r) \in \mathbb{Z}^r \;\middle|\; \begin{array}{l} 0 \leq \ell_j \leq p_j, \\ \ell_j \text{ even for } j = 2, \cdots, r, \\ \text{at least } 3 \text{ of } \ell_j \text{ satisfy } \dfrac{\ell_j}{p_j} \notin \mathbb{Z} \end{array} \right\}$$

Example: $\sum(2,3,5)$, $\sum(2,3,7)$, $\sum(2,3,5,7)$

$$\mathfrak{L}_{2,3,5} = \{(1,2,2),(1,2,4)\}$$

$$\mathfrak{L}_{2,3,7} = \{(1,2,2),(1,2,4),(1,2,6)\}$$

$\mathfrak{L}_{2,3,5,7} =$

(0,2,2,2)	(0,2,2,4)	(0,2,2,6)	(0,2,4,2)	(0,2,4,4)	(0,2,4,6)
(2,2,2,2)	(2,2,2,4)	(2,2,2,6)	(2,2,4,2)	(2,2,4,4)	(2,2,4,6)
(1,0,2,2)	(1,0,2,4)	(1,0,2,6)	(1,0,4,2)	(1,0,4,4)	(1,0,4,6)
(1,2,0,2)	(1,2,0,4)	(1,2,0,6)			
(1,2,2,0)	(1,2,4,0)				
(1,2,2,2)	(1,2,2,4)	(1,2,2,6)	(1,2,4,2)	(1,2,4,4)	(1,2,4,6)

$\sharp\mathfrak{L}_{2,3,5,7} = 29.$

Proposition (Andersen, Mistegård)

$$\pi_0(\mathcal{M}(X, SL(2,\mathbb{C}))) = \pi_0(\mathcal{M}(X, SU(2))) \bigsqcup \pi_0(\mathcal{M}(X, SL(2,\mathbb{R})))$$

Let's enlarge $\mathfrak{L}$

$$\mathfrak{H} := \{(h_1, \cdots, h_r) \in \mathbb{Z}^r \mid 0 \le h_j \le p_j, \text{at least } 3 \text{ of } \ell_j \text{ satisfy } \frac{h_j}{p_j} \notin \mathbb{Z}\}.$$

Let σ_i be a switch at i-th position

$$\begin{aligned}\sigma_i : \mathfrak{H} &\to \mathfrak{H} \\ (h_1, \cdots, h_r) &\mapsto (h_1, \cdots, p_i - h_i, \cdots, h_r)\end{aligned}$$

and σ be the set of all compositions of odd number of σ_i's.

定理 (Theorem - AHLMSS)

$$\pi_0(\mathcal{M}(X, SU(2))) \cong \mathfrak{L}_{\underline{p}}^{SU(2)} = \{\underline{\ell} \in \mathfrak{L} \mid \sum_{j=1}^{r} \frac{h_j}{p_j} > 1 \textit{ for all } \underline{h} \in \sigma(\underline{\ell})\}$$

$$\pi_0(\mathcal{M}(X, SL(2,\mathbb{R}))) \cong \mathfrak{L}_{\underline{p}}^{SL(2,\mathbb{R})} = \{\underline{\ell} \in \mathfrak{L} \mid \sum_{j=1}^{r} \frac{h_j}{p_j} < 1 \textit{ if exists } \underline{h} \in \sigma(\underline{\ell})\}$$

Example: $\sum(2,3,5)$

$$\mathfrak{L}_{2,3,5} = \{(1,2,2),(1,2,4)\} = \mathfrak{L}_{2,3,5}^{SU(2)}. \quad \mathfrak{L}_{2,3,5}^{SL(2,\mathbb{R})} = \emptyset$$

Here, $(1,2,2)$ represents an component of $SU(2)$ character variety since

$$\sigma_1(1,2,2) = (1,2,2) \Longrightarrow \frac{1}{2}+\frac{2}{3}+\frac{2}{5} = \frac{47}{30} > 1,$$
$$\sigma_2(1,2,2) = (1,1,2) \Longrightarrow \frac{1}{2}+\frac{1}{3}+\frac{2}{5} = \frac{37}{30} > 1,$$
$$\sigma_3(1,2,2) = (1,2,3) \Longrightarrow \frac{1}{2}+\frac{2}{3}+\frac{3}{5} = \frac{53}{30} > 1,$$
$$\sigma_1 \circ \sigma_2 \circ \sigma_3(1,2,2) = (1,1,3) \Longrightarrow \frac{1}{2}+\frac{1}{3}+\frac{3}{5} = \frac{43}{30} > 1.$$

Exmaple: $\sum(2,3,7)$

$$\mathfrak{L}_{2,3,7}^{SU(2)} = \{(1,2,2),(1,2,4)\}\,. \qquad \mathfrak{L}_{2,3,7}^{SL(2,\mathbb{R})} = \{(1,2,6)\}$$

Here, $(1,2,6)$ represents an component of $SL(2,\mathbb{R})$ character variety since

$$\sigma_1 \circ \sigma_2 \circ \sigma_3(1,2,6) = (1,1,1) \Longrightarrow \frac{1}{2} + \frac{1}{3} + \frac{1}{7} = \frac{41}{42} < 1.$$

Example - $\sum(2,3,5,7)$

(0,2,2,2)	(0,2,2,4)	(0,2,2,6)	(0,2,4,2)	(0,2,4,4)	(0,2,4,6)
(2,2,2,2)	(2,2,2,4)	(2,2,2,6)	(2,2,4,2)	(2,2,4,4)	(2,2,4,6)
(1,0,2,2)	(1,0,2,4)	(1,0,2,6)	(1,0,4,2)	(1,0,4,4)	(1,0,4,6)
(1,2,0,2)	(1,2,0,4)	(1,2,0,6)			
(1,2,2,0)	(1,2,4,0)				
(1,2,2,2)	(1,2,2,4)	(1,2,2,6)	(1,2,4,2)	(1,2,4,4)	(1,2,4,6)

$$\sharp\mathfrak{L}_{2,3,5,7} = 29, \quad \sharp\mathfrak{L}^{SU(2)}_{2,3,5,7} = 22, \quad \sharp\mathfrak{L}^{SL(2,\mathbb{R})}_{2,3,5,7} = 7.$$

Chern-Simons theory

Let G be $SU(2)$ and $\pi : P \to X$ be a trivial G-bundle. The Chern-Simons action is

$$S(A) = \frac{1}{8\pi^2}\int_M \mathrm{Tr}(A \wedge dA + \frac{2}{3}A \wedge A \wedge A) \mod \mathbb{Z}.$$

Here A is a gauge connection.

The partition function is

$$Z_k(X) = \int_{A_{SU(2)}} DAe^{2\pi ikS(A)},$$

which is also known as Witten-Reshetikhin-Turaev (WRT) invariant with level $k \in \mathbb{N}^*$.

- $$\frac{\delta S}{\delta A} = 0 \Longleftrightarrow \text{Curvature} = 0$$

The solutions of EOM are flat G-connections.

- $$\{\text{Flat } G\text{-connections}\} \cong \{G\text{-represention s of } \pi_1(X)\}$$

If X is a SFIHS,
$\{A \mid \frac{\delta S}{\delta A} = 0\} \cong \{\text{trivial representation}\} \sqcup \mathcal{M}(X, G)$.

- $$CS(X) := \{S(A) \mid A \text{ is solution of } \frac{\delta S}{\delta A} = 0\}$$

If $X = \sum(p_1, \cdots, p_r)$ with p_j's prime, then $\sharp CS(X) = 1 + \sharp \mathcal{L}_{\underline{p}}^{SU(2)}$.

Conjecture: Asymptotic expansion conjecture of WRT invariant

Let M be a closed oriented 3-manifold. For each $S \in CS(M)$ there exists a rational $d_S \in \mathbb{Q}$ and a series $Z_S \in \mathbb{C}[[x]]^\times$ such that the WRT invariant of M have the following Poincare asymptotic expansion for k tending to infinity

$$Z_k(M) \sim \sum_{S \in CS(M)} e^{2\pi i k S} k^{d_S} Z_S(k^{-1/2}). \tag{1}$$

定理 (Theorem - AHLMSS)

This conjecture is correct when M is a SFIHS.

Gukov-Pei-Putrov-Vafa (GPPV) invariant on SFIHS

- The GPPV invariant is supposed to be the partition function of $SL(2,\mathbb{C})$ Chern-Simons theory and the expression in the case of $\sum(p_1,\cdots,p_r)$ is

$$\Psi(q)=\sum_{j>0}\chi_j q^{\frac{j^2}{4P}}$$

where χ_j is generated by

$$G(z)=\frac{1}{(z^P-z^{-P})^{r-2}}\prod_{j=1}^{r}\left(z^{\widehat{p}_j}-z^{-\widehat{p}_j}\right)=(-1)^r\sum_{j\geq j_0}\chi_j z^j.$$

GPPV invariant on SFIHS

Then

$$\Psi(q) = \sum_{\substack{\nu,s\geq 0 \\ \nu+s\leq r-3}} C(\nu,s)\Theta(\tau;\nu,m^s f,2P), \quad q = e^{2\pi i\tau}$$

where $C(\nu,s)$ are some certain coefficients, and

$$\Theta(\tau;\nu,f,2P) := \sum_{n>0} n^\nu f(n) e^{\frac{2\pi i n^2 \tau}{4P}}$$

with $2P$-periodic function f.

Ralation between GPPV and WRT - Radial limit conjecture

定理 (Theorem - AM)

For a SFIHS X,

$$\lim_{q \to e^{\frac{2\pi i}{k}}} \Psi(q) = Z_k(X)$$

as k even.

定理 (Theorem - AHLMSS)

For a SFIHS X,

$$\lim_{q \to e^{\frac{2\pi i}{k}}} \Psi(q) = Z_k(X)$$

as k integer.

Recall the resurgence property of partial theta series by HLSS

$$f \text{ is even} \Longrightarrow$$

$$\Theta(\tau;1,f) = \mathcal{L}_{\nearrow}\mathcal{B}\widetilde{\Theta}(\tau;1,f) - i\left(\frac{\tau}{i}\right)^{-\frac{3}{2}}\Theta(-\tau^{-1};1,\widehat{f}),$$

$$f \text{ is odd} \Longrightarrow$$

$$\Theta(\tau;0,f) = \mathcal{L}_{\nearrow}\mathcal{B}\widetilde{\Theta}(\tau;0,f) - \left(\frac{\tau}{i}\right)^{-\frac{1}{2}}\Theta(-\tau^{-1};0,\widehat{f}).$$

For ν large, we may use the relation between the normal derivative w.r.t. τ and the so-called "alien operators".

$$\Delta_{\xi_n}\frac{d}{d\tau} = \left(\frac{d}{d\tau} + \xi_n\tau^{-2}\right)\Delta_{\xi_n}.$$

Resurgence on GPPV

A GPPV invariant $\Psi(q)$ for a SFIHS $\sum(p_1, \cdots, p_r)$ satisfies

$$\begin{aligned}
&\Psi(q) - \mathcal{L}_{\nearrow}\mathcal{B}[\text{Ohtsuki series}] \\
&= \sum_{\substack{\nu,s\geq 0\\ \nu+s\leq r-3}} Q_{\nu,s}\left(\tau^{-\frac{1}{2}}\right)\Theta(-\tau^{-1};\nu,\widehat{m^s f}) \\
&= \sum_{\underline{\ell}\in\mathfrak{L}_{\underline{p}}} \sum_{\substack{\nu,s\geq 0\\ \nu+s\leq r-3}} Q_{\nu,s}\left(\tau^{-\frac{1}{2}}\right)\Theta(-\tau^{-1};\nu,\widehat{m^s f}\Big|_{\mathfrak{S}_{\underline{p}}^{\sigma(\underline{\ell})}})
\end{aligned}$$

where Q is a polynomial, Θ is a partial theta series, $m^s f$ are some fixed $2P$-periodic functions, $\widehat{\bullet}$ is discrete Fourier transform, and $\mathfrak{S}_{\underline{p}}^{\sigma(\underline{\ell})}$ is a Hikami set.

Hikami sets for any $\underline{h} \in \mathfrak{H}$

$$\mathfrak{S}_{\underline{p}}^{\underline{h}} := \left\{ \sum_{j=1}^{r} \varepsilon_j h_j \widehat{p}_j + P \ [2P] \right\}, \quad \{\varepsilon_j = \pm 1\}.$$

- $\mathfrak{S}_{\underline{p}}^{\underline{h}} = \mathfrak{S}_{\underline{p}}^{\underline{h}'}$ if $\underline{h}, \underline{h}' \in \sigma(\underline{\ell})$.
- $\widehat{m^s f}$ is supported on $\bigsqcup_{\underline{h} \in \mathfrak{H}} \mathfrak{S}_{\underline{p}}^{\underline{h}}$.
- $\widehat{m^s f}(n) = \sum_{\underline{h}} \widehat{m^s f}|_{\mathfrak{S}_{\underline{p}}^{\underline{h}}}(n)$.
- $n^2 \equiv m^2 \ [4P]$ if $n, m \in \mathfrak{S}_{\underline{p}}^{\underline{h}}$.
- $CS(\underline{\ell}) = -\frac{n^2}{4P} \ [\mathbb{Z}]$ for $n \in \mathfrak{S}_{\underline{p}}^{\sigma(\underline{\ell})}$.

Example: $\sum(2,3,7)$

The GPPV invariant in the case of $\sum(2,3,7)$ is

$$\Psi_{2,3,7}(q) = \Theta(\tau;0,f) := \sum_{n>0} f(n) e^{\frac{2\pi i n^2 \tau}{168}}$$

where f is a 84-periodic odd function

n	1	13	29	41	43	55	71	83
f(n)	1	-1	-1	1	-1	1	1	-1

The asymptotic behavior of Ψ as τ goes to 0 is the Ohstuki series

$$\widetilde{\Theta}(\tau,0,f) = \sum_{n\geq 0} \frac{1}{n!} L(-2n,f) \left(\frac{\pi i}{2P}\right)^n \tau^n.$$

Example: $\sum(2,3,7)$

$$\Psi(q) - \mathcal{L}_{\nearrow}\mathcal{B}\widetilde{\Theta}(\tau,0,f) = \left(\frac{\tau}{i}\right)^{-\frac{1}{2}} \Theta(-\tau^{-1};0,\widehat{f})$$

$$= \left(\frac{\tau}{i}\right)^{-\frac{1}{2}} \left(d_1\Theta(-\tau^{-1};0,f_1) + d_2\Theta(-\tau^{-1};0,f_2) + d_3\Theta(-\tau^{-1};0,f_3)\right)$$

where

n	1	13	29	41	43	55	71	83	$\mathfrak{S}_{2,3,7}^{\sigma(1,2,6)}$
f_1	1	-1	-1	1	-1	1	1	-1	

n	5	19	23	37	47	61	65	79	$\mathfrak{S}_{2,3,7}^{\sigma(1,2,2)}$
f_2	1	1	1	1	-1	-1	-1	-1	

n	11	17	25	31	53	59	67	73	$\mathfrak{S}_{2,3,7}^{\sigma(1,2,4)}$
f_3	1	1	1	1	-1	-1	-1	-1	

Example: $\sum(2,3,7)$

$$
\begin{aligned}
&\Psi(q) - \mathcal{L}_{\nearrow}\mathcal{B}\widetilde{\Theta}(\tau, 0, f) \\
&= d_1 \left(\frac{\tau}{i}\right)^{-\frac{1}{2}} \Theta(-\tau^{-1}; 0, \widehat{f}\Big|_{\mathfrak{S}_{2,3,7}^{\sigma(1,2,6)}}) \\
&+ d_2 \left(\frac{\tau}{i}\right)^{-\frac{1}{2}} \Theta(-\tau^{-1}; 0, \widehat{f}\Big|_{\mathfrak{S}_{2,3,7}^{\sigma(1,2,2)}}) \\
&+ d_3 \left(\frac{\tau}{i}\right)^{-\frac{1}{2}} \Theta(-\tau^{-1}; 0, \widehat{f}\Big|_{\mathfrak{S}_{2,3,7}^{\sigma(1,2,4)}}) \\
&= \sum_{\underline{\ell} \in \mathfrak{L}_{2,3,7}} d_{\bullet} \left(\frac{\tau}{i}\right)^{-\frac{1}{2}} \Theta(-\tau^{-1}; 0, \widehat{f}\Big|_{\mathfrak{S}_{2,3,7}^{\sigma(\underline{\ell})}})
\end{aligned}
$$

Theorem - AHLMSS

Since GPPV goes to WRT as q goes to roots of unity $e^{\frac{2\pi i}{k}}$, it is natural to explain the "Asymptotic Expansion Conjecture" in the following sense:

$$\lim_{q\to e^{\frac{2\pi i}{k}}} \left(\Psi(q) - \mathcal{L}_{\nearrow}\mathcal{B}[\text{Ohtsuki series}]\right)$$

$$= \sum_{\underline{\ell}\in\mathfrak{L}_{\underline{p}}^{SU(2)}} \sum_{\substack{\nu,s\geq 0\\ \nu+s\leq r-3}} Q_{\nu,s}\left(k^{\frac{1}{2}}\right)\Theta^{n.t.}(-k;\nu,\widehat{m^s f}\Big|_{\mathfrak{S}_{\underline{p}}^{\sigma(\underline{\ell})}})$$

$$= \sum_{\underline{\ell}\in\mathfrak{L}_{\underline{p}}^{SU(2)}} e^{2\pi i k CS(\underline{\ell})} \sum_{\substack{\nu,s\geq 0\\ \nu+s\leq r-3}} Q_{\nu,s}\left(k^{\frac{1}{2}}\right)\Theta^{n.t.}(0;\nu,\widehat{m^s f}\Big|_{\mathfrak{S}_{\underline{p}}^{\sigma(\underline{\ell})}})$$

Theorem - AHLMSS

Equivalently,

$$\Theta^{n.t.}(0;\nu,\widehat{m^s f}|_{\mathfrak{S}_{\underline{p}}^{\sigma(\underline{\ell})}})=0$$

if $\nu+s\leq r-3$ and $\underline{\ell}\in\mathfrak{L}_{\underline{p}}^{SL(2,\mathbb{R})}$.

Example: $\sum(2,3,7)$

$$\lim_{q\to e^{\frac{2\pi i}{k}}} \left(\Psi(q) - \mathcal{LB}_{\nearrow}[\text{Ohtsuki series}]\right)$$

$$= d_2 \left(\frac{1}{ik}\right)^{-\frac{1}{2}} \Theta^{n.t.}(-k; 0, \widehat{f}\Big|_{\mathfrak{S}^{\sigma(1,2,2)}_{2,3,7}})$$

$$+ d_3 \left(\frac{1}{ik}\right)^{-\frac{1}{2}} \Theta^{n.t.}(-k; 0, \widehat{f}\Big|_{\mathfrak{S}^{\sigma(1,2,4)}_{2,3,7}})$$

$$= d_2 (ik)^{\frac{1}{2}} e^{-\frac{25}{168} 2\pi i k} L(0, \widehat{f}\Big|_{\mathfrak{S}^{\sigma(1,2,2)}_{2,3,7}}) + d_3 (ik)^{\frac{1}{2}} e^{-\frac{121}{168} 2\pi i k} L(0, \widehat{f}\Big|_{\mathfrak{S}^{\sigma(1,2,4)}_{2,3,7}})$$

Since

$$\Theta^{n.t.}(-k; 0, \widehat{f}\Big|_{\mathfrak{S}^{\sigma(1,2,6)}_{2,3,7}}) = \Theta^{n.t.}(0; 0, \widehat{f}\Big|_{\mathfrak{S}^{\sigma(1,2,6)}_{2,3,7}}) = 0.$$

Example: $\sum(2,3,7)$

$$Z_k(X) = \lim_{q \to e^{\frac{2\pi i}{k}}} \Psi(q)$$

$$= \mathcal{LB}_{\nearrow}[\text{Ohtsuki series}])(\frac{1}{k})$$

$$+ d_2 (ik)^{\frac{1}{2}} e^{-\frac{25}{168} 2\pi i k} L(0, \widehat{f}\Big|_{\mathfrak{S}^{\sigma(1,2,2)}_{2,3,7}}) + d_3 (ik)^{\frac{1}{2}} e^{-\frac{121}{168} 2\pi i k} L(0, \widehat{f}\Big|_{\mathfrak{S}^{\sigma(1,2,4)}_{2,3,7}})$$

$$\Longrightarrow$$

$$Z_k(X) \sim [\text{Ohtsuki series}](\frac{1}{k}) + \sum_{\underline{\ell} \in \mathfrak{L}^{SU(2)}_{2,3,7}} d_{\bullet} L(0, \widehat{f}\Big|_{\mathfrak{S}^{\sigma(\underline{\ell})}_{2,3,7}}) e^{2\pi i k CS(\underline{\ell})} (ik)^{\frac{1}{2}}$$

as $k \to +\infty$.

Thanks for your attention!