Free Rota-Baxter family algebras and free (tri)dendriform family algebras

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joint work with Xing Gao and Dominique Manchon

Algebraic, analytic and geometric structures emerging from quantum field theory, Chengdu

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Outline

1. Free Rota-Baxter family algebras (RBFA)
   - Typed angularly decorated planar rooted trees
   - A multiplication on $\mathbf{k}\mathcal{T}(X, \Omega)$
   - Construction of free RBFA
2. Embedding free DFAs (resp. TFAs) into free RBFAs
   - Embedding free DFAs into free RBFAs
   - Embedding free TFAs into free RBFAs
   - Universal enveloping algebras of (tri)dendriform family algebras

Motivation

- In 1998, Loday and Ronco proved the free dendriform algebra on one generator can be described as an algebra over the set of planar binary trees.
- In 2004, Loday and Ronco showed the free tridendriform algebra on one generator can be described as an algebra over the set of planar trees.

J. -L. Loday and M. O. Ronco, Hopf algebra of the planar binary trees, *Adv. Math.* **39** (1998), 293-309.

J. -L. Loday and M. O. Ronco, Trialgebras and families of polytopes, in "Homotopy theoty: relations with algebraic geometry, group cohomology, and algebraic K-theory", *Contemp. Math.* **346** (2004), 369-398.

Motivation

- In 2007, K. Ebrahimi-Fard, J. Gracia-Bondia and F. Patras studied about algebraic aspects of renormalization in Quantum field theory. The first example about Rota-Baxter family algebras of -1 appeared in this paper and Guo suggested them to use this name.
- In 2008, K. Ebrahimi-Fard and L. Guo, they used rooted trees and forests to give explicit constructions of free noncommutative Rota-Baxter algebras on modules and sets.

K. Ebrahimi-Fard, J. Gracia-Bondia and F. Patras, A Lie theoretic approach to renormalization, *Comm. Math. Phys.* **276** (2007), 519-549.

K. Ebrahimi-Fard and L. Guo, Free Rota-Baxter algebras and rooted trees, *J. Algebra Appl.* **7** (2008), 167-194.

Motivation

- In 2009, L. Guo named Rota-Baxter family algebras of weight λ.
- In 2012, E. Panzer studied algebraic aspects of renormalization in Quantum Field Theory. They proved the Taylor expansion operators fulfil for Rota-Baxter family algebras of weight -1 .

L. Guo, Operated monoids, Motzkin paths and rooted trees, *J. Algebraic Combin.* **29** (2009), 35-62.

D. Kreimer and E. Panzer, Hopf-algebraic renormalization of Kreimer's toy model, Master thesis, Handbook.

Motivation

- In 2018, Bruned, Haier and Zambotti gave a systematic description of a canonical renormalisation procedure of stochastic PDEs, in which their construction is based on bialgebras of typed decorated forests in cointeraction.
- In 2018, L. Foissy studied multiple prelie algebras and related operads. He proved the free T-multiple prelie algebra generated by a set D.

Y. Bruned, M. Hairer and L. Zambotti, Algebraic renormalisation of regularity structures, *Invent. math.* **215** (2019), 1039-1156.

L. Foissy, Algebraic structures on typed decorated planar rooted trees, *Symmetry Integr. Geom. Methods Appl.* **17** (2021), 28pp.

Recent related work

- Y. Bruned, M. Hairer and L. Zambotti, Algebraic renormalisation of regularity structures, *Invent. math.* **215** (2019), 1039-1156.
- M. Aguiar, Dendriform algebras relative to a semigroup, *Symmetry Integr. Geom. Methods Appl.* **16** (2020) 066.
- L. Foissy, Algebraic structures on typed decorated planar rooted trees, *Symmetry Integr. Geom. Methods Appl.* **17** (2021), 28pp.
- L. Foissy, Typed binary trees and generalized dendriform algebras and typed binary trees, *J. Algebra*, **586** (2021), 1-61.

Recent related work

- L. Foissy and X. Peng, Typed angularly decorated planar rooted trees and generalized Rota-Baxter algebras, *J. Algebr. Comb.* **57** (2023), 271-303.
- X. Gao, L. Guo and Y. Zhang, Commutative matching Rota-Baxter operators, shuffle products with decorations and matching Zinbiel algebras, *J. Algebra*, **586** (2021), 402-432.
- Y. Zhang, X. Gao and L. Guo, Matching Rota-Baxter algebras, matching dendriform algebras and matching pre-Lie algebras, *J. Algebra* **552** (2020), 134-170.
- Y. Zhang and S. J. Guo, Matching Rota-Baxter systems and Gröbner-Shirshov bases, accepted by *Algebra Colloq.*

Recent related work

- Y. Y. Zhang and X. Gao, Free Rota-Baxter family algebras and (tri)dendriform family algebras, *Pacific J. Math.* **301** (2019), 741-766.
- Y. Y. Zhang, X. Gao and D. Manchon, Free (tri)dendriform family algebras, *J. Algebra* **547** (2020), 456-493.
- Y. Y. Zhang, X. Gao and D. Manchon, Free Rota-Baxter family algebras and free (tri)dendriform family algebras, *Algebr. Represent. Theory*, **301** (2023), 741-766.
- Y. Y. Zhang, J. Zhao and G. Q. Liu, Rota-Baxter Ω-associative conformal algebras and their cohomology, *J. Math. Phys.* **64**, 101705 (2023).

Recent related work

- Y. Y. Zhang and D. Manchon, Free pre-Lie family algebras, *Ann. Inst. Heri Poincare Comb. Phys. Interact.* (2023), doi:10.4171/AIHPD/162.
- Y. Y. Zhang, H. H. Zhang and X. Gao, Free Ω-Rota-Baxter systems and Gröbner-Shirshov bases, *J. Algebra Appl.*, doi: 10.1142/So219498825501622.
- L. Foissy, D. Manchon and Y. Y. Zhang, Families of algebraic structures, submitted.
- S. Chao, K. Wang and Y. Y. Zhang*, The cohomology theory of Rota-Baxter Ω-associative algebras and derived bracket, submitted.

Recent related work

- A. Das, Twisted Rota-Baxter families and NS-family algebras, *J. Algebra* 612, (2022), 577-615.
- A. Das and S. Sen, Diassociative family algebras and averaging family operators, *J. Geom. Phys.* **193** (2023), 104964, 15 pp.
- T. S. Ma and J. Li, Matching Rota-Baxter BiHom-algebras and related algebraic structures, *Rocky Mountain J. Math.* **52** (2022), 17411765.
- Y. Bruned and K. Ebrahimi-Fard, Bogoliubov type recursions for renormalisation in regularity structures, *Ann. Inst. Heri Poincare Comb. Phys. Interact.*, doi:10.4171/AIHPD/186 (2024).

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Questions

1. How to construct the free Rota-Baxter family algebras by using typed angularly decorated planar rooted trees?
2. How to construct the free (tri)dendriform family algebra by using typed decorated planar (binary) rooted trees?
3. What's the relationship between the free Rota-Baxter family algebra and the free (tri)dendriform family algebra?

1. Free Rota-Baxter family algebras (RBFA)
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   - Construction of free RBFA

2. Embedding free DFAs (resp.TFAs) into free RBFAs
   - Embedding free DFAs into free RBFAs
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   - Universal enveloping algebras of (tri)dendriform family algebras

Definition and Example

Definition (Ebrahimi-Fard et al. 2007, Guo 2009)

Let Ω be a semigroup and $\lambda \in \mathbf{k}$ be given. A **Rota-Baxter family** of weight λ on an algebra R is a collection of linear operators $(P_\omega)_{\omega\in\Omega}$ on R such that

$$P_\alpha(a)P_\beta(b) = P_{\alpha\beta}\left(P_\alpha(a)b + aP_\beta(b) + \lambda ab\right),$$

where $a, b \in R$ and $\alpha, \beta \in \Omega$. Then the pair $(R, (P_\omega)_{\omega\in\Omega})$ is called a **Rota-Baxter family algebra** of weight λ.

Example 1

The algebra of Laurent series $R = \mathbf{k}\left[z^{-1}, z\right]\!]$ is a Rota-Baxter family algebra of weight -1 , with $\Omega = (\mathbb{Z}, +)$, where the operator P_ω is the projection onto the subspace $R_{<\omega}$ generated by $\left\{z^k, k < \omega\right\}$ parallel to the supplementary subspace $R_{\geq\omega}$ generated by $\left\{z^k, k \geq \omega\right\}$.

Example 2

Let $\Omega = (\mathbb{R}, +)$ be a semigroup, and R is the $\mathbb{R}$-algebra consisting of all continous functions from $\mathbb{R}$ to $\mathbb{R}$, the multiplication is defined by

$$(fg)(x) := f(x)g(x), \text{ for } f, g \in R.$$

And for any $\alpha \in \Omega$, define a family linear operator $P_\alpha : R \to R$ as follows

$$P_\alpha(f)(x) = e^{-\alpha A(x)} \int_0^x e^{\alpha A(t)} f(t)\, dt, \text{ for } \alpha \in \Omega,$$

where A is a fixed nonzero element of R. Then $(P_\alpha)_{\alpha \in \Omega}$ is a Rota-Baxter family of weight 0.

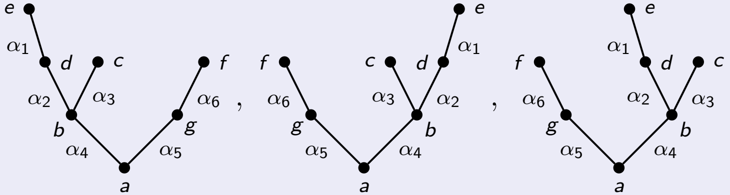
Typed angularly decorated planar rooted trees

Definition (Bruned-Hairer-Zambotti, Invent. math.)

Let X and Ω be two sets. An X-**decorated** Ω-**typed (abbreviated typed decorated) rooted tree** is a triple $T = (T, \mathrm{dec}, \mathrm{type})$, where

1. T is a rooted tree.
2. $\mathrm{dec} : V(T) \to X$ is a map.
3. $\mathrm{type} : E(T) \to \Omega$ is a map.

Example 3



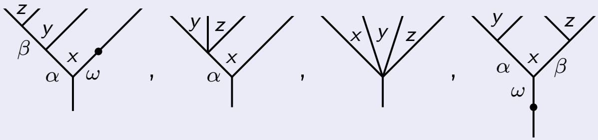
Typed angularly decorated planar rooted trees

Definition

Let X and Ω be two sets. An X**-angularly decorated** Ω**-typed (abbreviated typed angularly decorated) planar rooted tree** is a triple $T = (T, \mathrm{dec}, \mathrm{type})$, where

1. T is a planar rooted tree.
2. $\mathrm{dec} : A(T) \to X$ is a map.
3. $\mathrm{type} : IE(T) \to \Omega$ is a map.

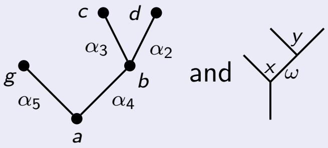
Example 4



Typed angularly decorated planar rooted trees

Remark

The graphical representation of (planar) rooted trees



in Example 4 and Example 3 is slightly different.

- Here the root and the leaves are now edges rather than vertices.
- The set $E(T)$ must be replaced by the set $IE(T)$ of internal edges.

Typed angularly decorated planar rooted trees

If a semigroup Ω has no identity element, we consider the monoid $\Omega^1 := \Omega \sqcup \{1\}$ obtained from Ω by adjoining an identity:

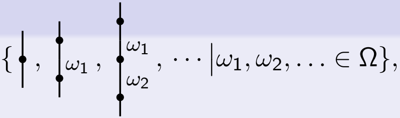
$$1\omega := \omega 1 := \omega, \ \text{ for } \ \omega \in \Omega \ \text{ and } \ 11 := 1.$$

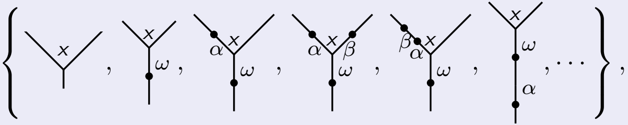
For $n \geq 0$, let $\mathcal{T}_n(X, \Omega)$ denote the set of X-angularly decorated Ω^1-typed planar rooted trees with $n+1$ leaves such that leaves are decorated by the identity 1 in Ω^1 and internal edges are decorated by elements of Ω. Note that the root is not decorated. Denote by

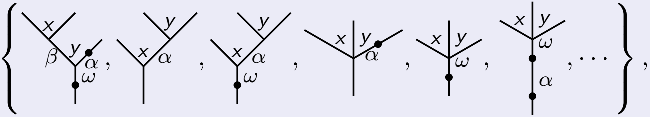
$$\mathcal{T}(X, \Omega) := \bigsqcup_{n \geq 0} \mathcal{T}_n(X, \Omega) \ \text{ and } \ \mathbf{k}\mathcal{T}(X, \Omega) := \bigoplus_{n \geq 0} \mathbf{k}\mathcal{T}_n(X, \Omega).$$

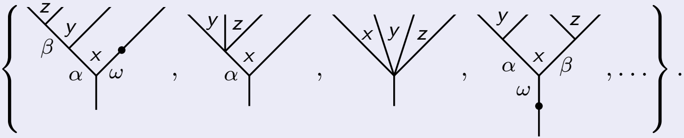
Typed angularly decorated planar rooted tree

Example 5

$\mathcal{T}_0(X,\,\Omega) =$ 

$\mathcal{T}_1(X,\,\Omega) =$ 

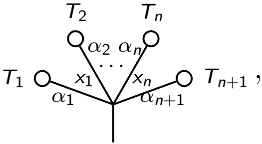
$\mathcal{T}_2(X,\,\Omega) =$ 

$\mathcal{T}_3(X,\,\Omega) =$ 

where $\alpha, \beta, \omega \in \Omega$ and $x, y, z \in X$

Typed angularly decorated planar rooted tree

Graphically, an element $T \in \mathcal{T}(X, \Omega)$ is of the form:

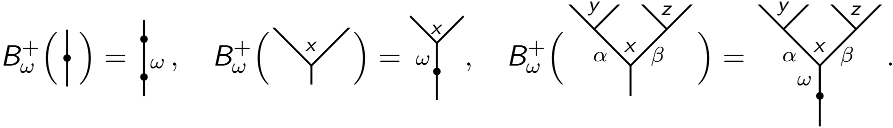
$T =$ 

, with $n \geq 0$,

i.e., $\alpha_j \in \Omega$ if $T_j \neq |$; $\alpha_j = 1$ if $T_j = |$. For each $\omega \in \Omega$, define a linear operator

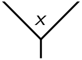
$$B_\omega^+ : \mathbf{k}\mathcal{T}(X, \Omega) \to \mathbf{k}\mathcal{T}(X, \Omega),$$

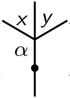
by adding a new root and a new internal edge decorated by ω connecting the new root and the root of T. For example,



Typed angularly decorated planar rooted tree

The **depth** $\mathrm{dep}(T)$ of a rooted tree T is the maximal length of linear chains from the root to the leaves of the tree. For example,

$\mathrm{dep}\Big($  $\Big) = \mathrm{dep}\Big($ 

$\Big) = 1$ and $\mathrm{dep}\Big($ 

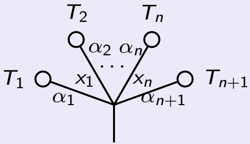
$\Big) = 2.$

We add the "zero-vertex tree" $|$ to the picture, and set $\mathrm{dep}(|) = 0$. Note that the operators B^+_ω are not defined on $|$.

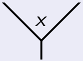
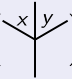
Typed angularly decorated planar rooted tree

Remark

For any $T \in \mathcal{T}(X, \Omega) \sqcup \{|\}$, if $\mathrm{dep}(T) = 0$, then $T = |$. Define the number of branches $\mathrm{bra}(T)$ of T to be 0 in this case. Otherwise, $\mathrm{dep}(T) \geq 1$ and T is of the form

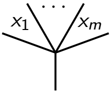
$T =$  with $n \geq 0$.

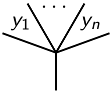
Here any branch $T_j \in \mathcal{T}(X, \Omega) \sqcup \{|\}, j = 1, \ldots, n+1$ is of depth at most one less than the depth of T, and equal to zero if and only if $T_j = |$. We define $\mathrm{bra}(T) := n + 1$. For example,

$\mathrm{bra}($  $) = 1$, $\mathrm{bra}($  $) = 2$ and $\mathrm{bra}($  $) = 3$.

A multiplication on $\mathbf{k}\mathcal{T}(X, \Omega)$

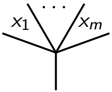
We define $T \diamond T'$ by induction on $\mathrm{dep}(T) + \mathrm{dep}(T') \geq 2$. For the initial step $\mathrm{dep}(T) + \mathrm{dep}(T') = 2$, we have $\mathrm{dep}(T) = \mathrm{dep}(T') = 1$ and T, T' are of the form

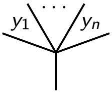
$T =$ 

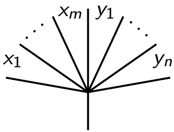
and $T' =$ 

, with $m, n \geq 0$.

Define

$T \diamond T' :=$ 

$\diamond$ 

$:=$ 

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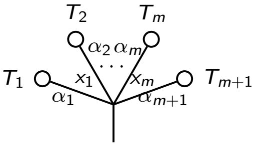
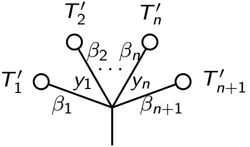
A multiplication on $\mathbf{k}\mathcal{T}(X, \Omega)$

For the induction step $\operatorname{dep}(T) + \operatorname{dep}(T') \geq 3$, the trees T and T' are of the form

$T =$ 

and $T' =$ 

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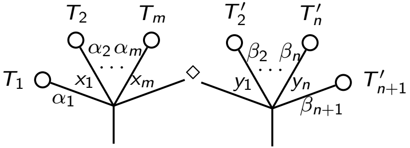
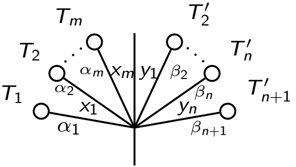
There are four cases to consider.

Case 1: $T_{m+1} = | = T'_1$. Define

$T \diamond T' :=$ 

$:=$ 

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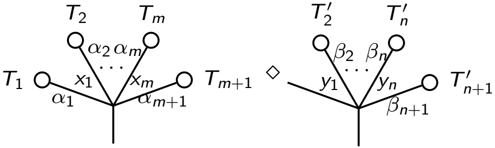
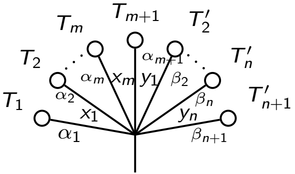
A multiplication on $\mathbf{k}\mathcal{T}(X, \Omega)$

Case 2: $T_{m+1} \neq | = T'_1$. Define

$T \diamond T' :=$



$:=$



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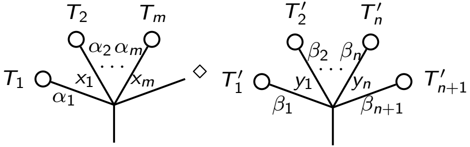
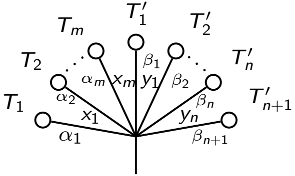
A multiplication on $\mathbf{k}\mathcal{T}(X, \Omega)$

Case 3: $T_{m+1} = | \neq T'_1$. Define

$T \diamond T' :=$



$:=$



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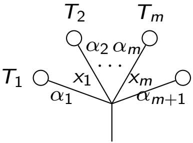
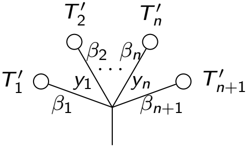
A multiplication on $\mathbf{k}\mathcal{T}(X,\Omega)$

Case 4: $T_{m+1} \neq | \neq T'_1$. Define

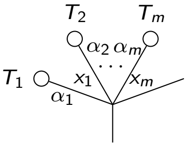
$T \diamond T' :=$



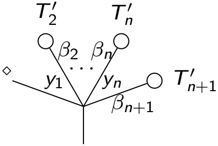
$\diamond$



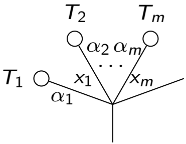
$:= \Bigg($



$\diamond \left(B^+_{\alpha_{m+1}}(T_{m+1}) \diamond B^+_{\beta_1}(T'_1)\right)\Bigg) \diamond$

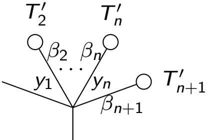


$:= \Bigg($



$\diamond B^+_{\alpha_{m+1}\beta_1}\left(B^+_{\alpha_{m+1}}(T_{m+1}) \diamond T'_1 + T_{m+1} \diamond B^+_{\beta_1}(T'_1) + \lambda T_{m+1} \diamond T'_1\right)\Bigg)$

$\diamond$



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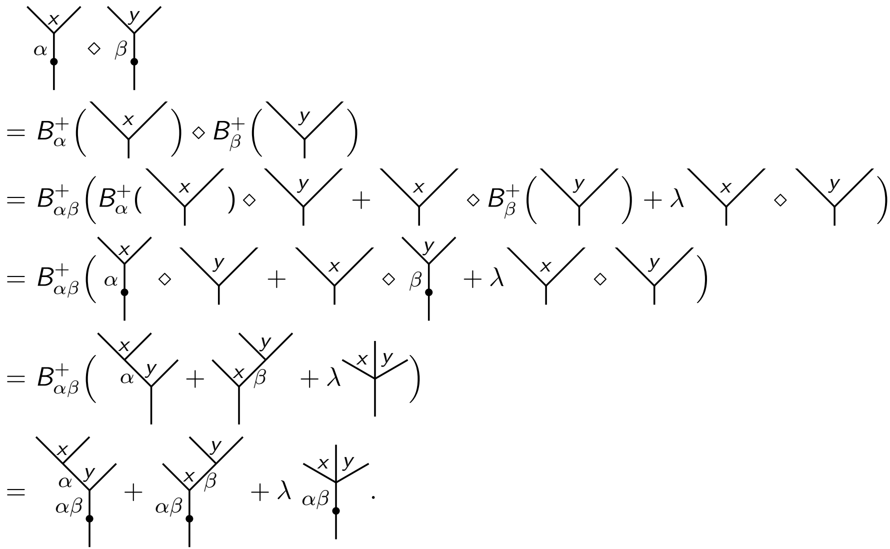
Example 6



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Problems

1. About the Rota-Baxter family algebras on typed-angularly decorated planar rooted trees constructed above: are they free?
2. What's the relationship between typed-angularly decorated planar rooted trees and bracketed words?

Free FRBA on bracketed words

Denote by S the following vector subspace of $\boldsymbol{k}\mathfrak{M}(\Omega, X)$:

$$S := \left\{ \lfloor x \rfloor_\alpha \lfloor y \rfloor_\beta - \lfloor \lfloor x \rfloor_\alpha y \rfloor_{\alpha\beta} - \lfloor x \lfloor y \rfloor_\beta \rfloor_{\alpha\beta} - \lambda \lfloor xy \rfloor_{\alpha\beta} \right\},$$

where $\alpha, \beta \in \Omega, x, y \in \mathfrak{M}(\Omega, X)$.

Theorem (Z.-Gao, Pacific J. Math.)

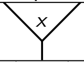
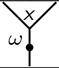
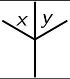
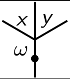
Let X be a set and let Ω be a semigroup.

1. *S is a Gröbner-Shirshov basis in $\mathbf{k}\mathfrak{M}(\Omega, X)$ with respect to a monomial order.*
2. *The set $\mathfrak{X}_\infty$ is a $\mathbf{k}$-basis of the free Rota-Baxter family algebra $(\mathbf{k}X_\infty, \diamond_w, (\lfloor\rfloor\omega)_{\omega\in\Omega}) = \mathbf{k}\mathfrak{M}(\Omega, X)/\operatorname{Id}(S)$ of weight λ, where $\operatorname{Id}(S)$ is the operated ideal generated by S in $\mathrm{kM}(\Omega, X)$.*

Y. Y. Zhang and X. Gao, Free Rota-Baxter family algebras and (tri)dendriform family algebras, *Pacific J. Math.* **301** (2019), 741-766.

The relationship

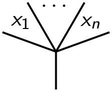
Built up a one-one correspondence between $\mathbf{k}\mathcal{T}(X,\Omega)$ and $\mathbf{k}\mathfrak{X}_\infty$.

Typed angularly decorated planar rooted trees $\mathcal{T}(X,\Omega)$	$\mathfrak{X}_\infty$
	1
	$\lfloor 1\rfloor_\omega$
	x
	$\lfloor x\rfloor_\omega$
	xy
	$\lfloor xy\rfloor_\omega$

Construction of the isomorphism map

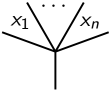
Define a linear map

$$\phi : \mathbf{k}\mathcal{T}(X, \Omega) \to \mathbf{k}\mathfrak{X}_\infty, \quad T \mapsto \phi(T)$$

by induction on $\operatorname{dep}(T) \geq 1$. If $\operatorname{dep}(T) = 1$, then $T =$ 

with $n \geq 0$ and define

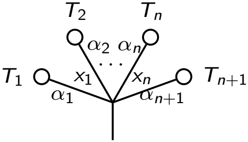
$$\phi(T) := \phi(\ \text{[tree]}\ ) := x_1 \cdots x_n,$$



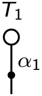
with $x_1 \cdots x_n := 1$ in the case $n = 0$.

Construction of the isomorphism map

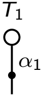
For the induction step $\operatorname{dep}(T) \geq 2$, T is the form



with some $T_i \neq |$.

Then we define $\phi(T)$ by the induction on $\operatorname{bra}(T) \geq 1$. For the initial step $\operatorname{bra}(T) = 1$, we have $T =$ 

and define

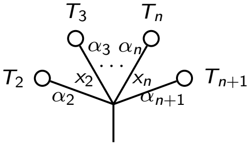
$$\phi\Big(\ $$


$$\Big) := \phi(B^+_{\alpha_1}(T_1)) := \lfloor \phi(T_1) \rfloor_{\alpha_1}.$$

Construction of the isomorphism map

For the induction step $\mathrm{bra}(T) \geq 2$, there are two cases to consider.
Case 1: $T_1 = |$. Define

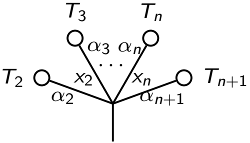
$\phi(T) := x_1 \phi\Big($



$\Big).$

Case 2: $T_1 \neq |$. Define

$\phi(T) := \lfloor \phi(T_1) \rfloor_{\alpha_1} x_1 \phi\Big($



$\Big).$

Construction of the isomorphism map

Conversely, define a linear map

$$\psi : \mathbf{k}X_\infty \to \mathbf{k}\mathcal{T}(X,\Omega), \quad w \mapsto \psi(w)$$

by induction on $\mathrm{dep}(w) \geq 1$.

- If $\mathrm{dep}(w) = 1$, then $w = x_1 \cdots x_n \in M(X)$ with $n \geq 0$.
- If $\mathrm{dep}(w) > 1$, we apply induction on $\mathrm{bre}(w) \geq 1$.

Write $w = w_1 \cdots w_n$ with $\mathrm{bre}(w) = n \geq 1$.
If $\mathrm{bre}(w) = 1$, then $w = \lfloor \overline{w} \rfloor_\alpha$ for $\overline{w} \in \mathfrak{X}_\infty$ and $\alpha \in \Omega$ by $\mathrm{dep}(w) \geq 1$, and define

$$\psi(w) = \psi\left(\lfloor \overline{w} \rfloor_\alpha\right) := B_\alpha^+(\psi(\overline{w})).$$

If $\mathrm{bre}(w) \geq 2$, then define

$$\psi(w) := \psi\left(w_1\right) \diamond \psi\left(w_2 \cdots w_n\right).$$

Construction of the isomorphism map

Proposition (Z.-Gao-Manchon, Algebr. Represent. Theory)

We have $\psi \circ \phi = \mathrm{id}$ *and* $\phi \circ \psi = \mathrm{id}$.

Lemma (Z.-Gao-Manchon, Algebr. Represent. Theory)

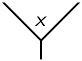
For T *and* T' *in* $\mathcal{T}(X, \Omega)$, *we have*

$$\phi(T \diamond T') = \phi(T) \diamond_w \phi(T').$$

Y. Y. Zhang, X. Gao and D. Manchon, Free Rota-Baxter family algebras and free (tri)dendriform family algebras, *Algebr. Represent. Theory*, **301** (2023), 741-766.

Main results

Let j_X be the embedding given by

$$j_X : X \to \mathbf{k}\mathcal{T}(X, \Omega), x \mapsto$$ 

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Theorem (Z.-Gao-Manchon, Algebr. Represent. Theory)

Let X be a set and let Ω be a semigroup. The triple $\big(\mathbf{k}\mathcal{T}(X, \Omega), \diamond, (B^{+}_{\omega})_{\omega \in \Omega}\big)$, together with the j_X, is the free Rota-Baxter family algebra of weight λ on X.

Corollary (Z.-Gao-Manchon, Algebr. Represent. Theory)

Let X be a set and let Ω be a trivial semigroup with one element. Then the triple $(\mathbf{k}\mathcal{T}(X), \diamond, B^{+})$, together with the j_X, is the free Rota-Baxter algebra of weight λ on X.

Problems

1. How to construct free (tri)dendriform family algebras via typed decorated planar rooted trees?
2. What's the relationship between free Rota-Baxter family algebras and free (tri)dendriform family algebras?

Embedding free DFAs into free RBFAs

Definition

Let Ω be a semigroup. A **dendriform family algebra** is a **k**-module D with a family of binary operations $(\prec_\omega, \succ_\omega)_{\omega\in\Omega}$ such that for $x, y, z \in D$ and $\alpha, \beta \in \Omega$,

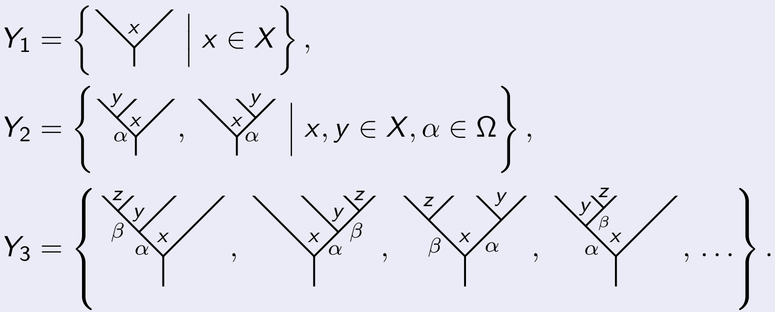
$$\begin{aligned}
(x \prec_\alpha y) \prec_\beta z &= x \prec_{\alpha\beta} (y \prec_\beta z + y \succ_\alpha z),\\
(x \succ_\alpha y) \prec_\beta z &= x \succ_\alpha (y \prec_\beta z),\\
(x \prec_\beta y + x \succ_\alpha y) \succ_{\alpha\beta} z &= x \succ_\alpha (y \succ_\beta z).
\end{aligned}$$

Embedding free DFAs into free RBFAs

For $n \geq 1$, let

$$Y_n := Y_{n,X,\Omega} := \mathcal{T}_n(X,\Omega) \cap \{\text{planar binary trees}\}.$$

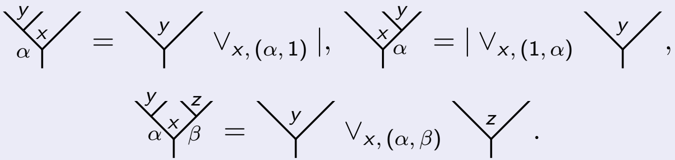
Example 7



Embedding free DFAs into free RBFAs

The grafting $\vee_{x,(\alpha,\beta)}$ over x and (α,β) is defined to be $T = T^l \vee_{x,(\alpha,\beta)} T^r$ for some $x \in X$ and $\alpha, \beta \in \Omega^1$.

Example 8



Denote by

$$\mathrm{DD}(X,\Omega) := \bigoplus_{n\geq 1} \mathbf{k}\, Y_n.$$

Embedding free DFAs into free RBFAs

Definition

Let X be a set and let Ω be a semigroup. Define binary operations

$$\prec_\omega, \succ_\omega\colon \Big(\mathrm{DD}(X,\Omega)\otimes \mathrm{DD}(X,\Omega)\Big) \oplus \Big(\mathbf{k}|\otimes \mathrm{DD}(X,\Omega)\Big) \oplus \Big(\mathrm{DD}(X,\Omega)\otimes \mathbf{k}|\Big) \to \mathrm{DD}(X,\Omega), \text{ for } \omega\in\Omega$$

recursively on $\mathrm{dep}(T)+\mathrm{dep}(U)$ by

1. $|\succ_\omega T := T \prec_\omega | := T$ and $|\prec_\omega T := T\succ_\omega | := 0$ for $\omega\in\Omega$ and $T\in Y_n$ with $n\geq 1$.
2. For $T = T^l \vee_{x,(\alpha_1,\alpha_2)} T^r$ and $U = U^l\vee_{y,(\beta_1,\beta_2)} U^r$, define

$$T\prec_\omega U := T^l \vee_{x,(\alpha_1,\alpha_2\omega)} (T^r\prec_\omega U + T^r \succ_{\alpha_2} U),$$
$$T\succ_\omega U := (T\prec_{\beta_1} U^l + T\succ_\omega U^l)\vee_{y,(\omega\beta_1,\beta_2)} U^r.$$

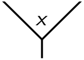
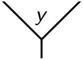
Embedding free DFAs into free RBFAs

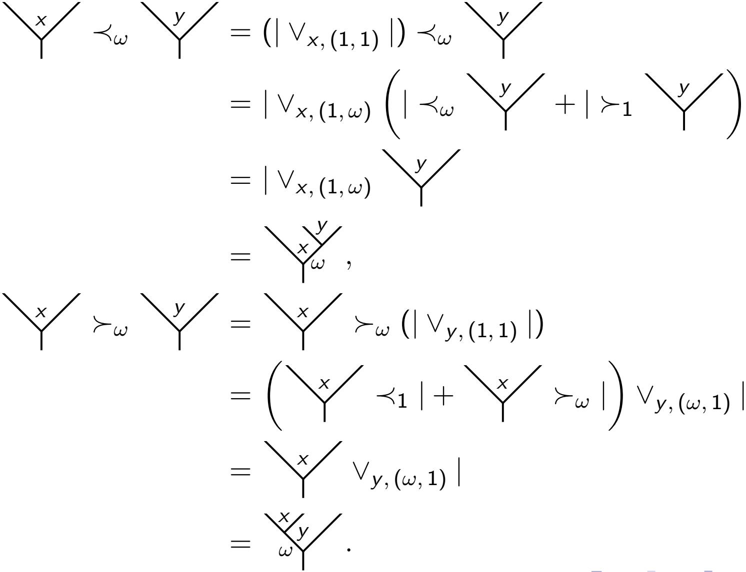
Remark

Note that $| \prec_\omega |$ and $| \succ_\omega |$ are not defined for $\omega \in \Omega^1$. Here we apply the convention that

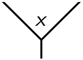
$$| \succ_1 T := T \prec_1 | := T \text{ and } | \prec_1 T := T \succ_1 | := 0.$$

Example 9

Let $T =$  and $U =$  with $x, y \in X$. For $\omega \in \Omega$,



Embedding free DFAs into free RBFAs

Let $j : X \to \mathrm{DD}(X,\Omega)$ be the natural embedding map defined by $j(x) =$  for $x \in X$.

Theorem (Z.-Gao-Manchon, Algebr. Represent. Theory)

Let X be a set and let Ω be a semigroup. Then $\big(\mathrm{DD}(X,\Omega), (\prec_\omega, \succ_\omega)_{\omega\in\Omega}\big)$, together with the map j, is the free dendriform family algebra on X.

Lemma (Z.-Gao, Pacific J. Math.)

Let X be a set and let Ω be a semigroup. The Rota-Baxter family algebra $\big(\mathbf{k}\mathcal{T}(X,\Omega), \diamond, (B_\omega^+)_{\omega\in\Omega}\big)$ of weight 0 induces a dendriform family algebra $\big(\mathbf{k}\mathcal{T}(X,\Omega), (\prec'_\omega, \succ'_\omega)_{\omega\in\Omega}\big)$, where

$$T \prec'_\omega U := T \diamond B_\omega^+(U) \text{ and } T \succ'_\omega U := B_\omega^+(T) \diamond U,$$

with $T, U \in \mathcal{T}(X,\Omega)$.

Embedding free DFAs into free RBFAs

Now, we give the relationship between the free dendriform family algebra and the free Rota-Baxter family algebra.

Theorem (Z.-Gao-Manchon, Algebr. Represent. Theory)

Let X be a set and let Ω be a semigroup. The free dendriform family algebra $\big(\mathrm{DD}(X,\,\Omega),(\prec_\omega,\succ_\omega)_{\omega\in\Omega}\big)$ on X is a dendriform family subalgebra of the free Rota-Baxter family algebra $\big(\mathbf{k}\mathcal{T}(X,\,\Omega),\diamond,(B^+_\omega)_{\omega\in\Omega}\big)$ of weight 0.

Embedding free TDFAs into free RBFAs

Definition

Let Ω be a semigroup. A **tridendriform family algebra** is a **k**-module T equipped with a family of binary operations $(\prec_\omega, \succ_\omega)_{\omega\in\Omega}$ and a binary operation $\cdot$ such that for $x, y, z \in T$ and $\alpha, \beta \in \Omega$,

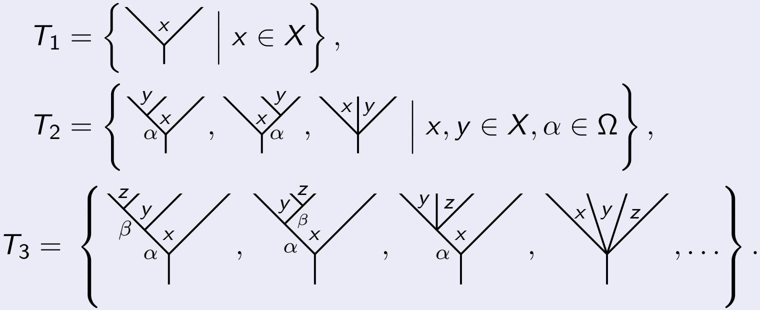
$$
\begin{aligned}
(x \prec_\alpha y) \prec_\beta z &= x \prec_{\alpha\beta} (y \prec_\beta z + y \succ_\alpha z + y \cdot z),\\
(x \succ_\alpha y) \prec_\beta z &= x \succ_\alpha (y \prec_\beta z),\\
(x \prec_\beta y + x \succ_\alpha y + x \cdot y) \succ_{\alpha\beta} z &= x \succ_\alpha (y \succ_\beta z),\\
(x \succ_\alpha y) \cdot z &= x \succ_\alpha (y \cdot z),\\
(x \prec_\alpha y) \cdot z &= x \cdot (y \succ_\alpha z),\\
(x \cdot y) \prec_\alpha z &= x \cdot (y \prec_\alpha z),\\
(x \cdot y) \cdot z &= x \cdot (y \cdot z).
\end{aligned}
$$

Embedding free TDFAs into free RBFAs

For $n \geq 1$, let

$$T_n := T_{n,X,\Omega} := \mathcal{T}_n(X, \Omega) \cap \{\text{Schröder trees}\}.$$

Example 10



Embedding free TDFAs into free RBFAs

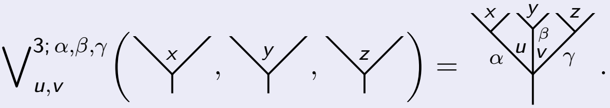
Denote by

$$\mathrm{DT}(X,\Omega) := \bigoplus_{n\geq 1} \mathbf{k}\, T_n$$

The grafting $\bigvee$ of $T^{(i)}, 1 \leq i \leq k$ over $(x_1,\ldots,x_k)$ and $(\alpha_0,\ldots,\alpha_k)$ is

$$T = \bigvee_{x_1,\ldots,x_k}^{k+1;\, \alpha_0,\ldots,\alpha_k} (T^{(0)},\ldots,T^{(k)})$$

Example 11



Embedding free TDFAs into free RBFAs

Definition

Let X be a set and let Ω be a semigroup. Define binary operations

$$\prec_\omega, \succ_\omega, \cdot \ : \Big(\mathrm{DT}(X,\Omega)\otimes \mathrm{DT}(X,\Omega)\Big) \oplus \Big(\mathbf{k}|\otimes \mathrm{DT}(X,\Omega)\Big) \oplus \Big(\mathrm{DT}(X,\Omega)\otimes \mathbf{k}|\Big) \to \mathrm{DT}(X,\Omega), \ \text{for } \omega\in\Omega$$

recursively on $\mathrm{dep}(T)+\mathrm{dep}(U)$ by

1. $| \succ_\omega T := T \prec_\omega | := T$, $| \prec_\omega T := T \succ_\omega | := 0$ and $| \cdot T := T \cdot | := 0$ for $\omega \in \Omega$ and $T \in \mathcal{T}_n$ with $n \geq 1$.
2. Let

$$T = \bigvee\nolimits_{x_1,\ldots,x_m}^{m+1;\,\alpha_0,\ldots,\alpha_m}(T^{(0)},\ldots,T^{(m)}) \in \mathcal{T}_m,$$
$$U = \bigvee\nolimits_{y_1,\ldots,y_n}^{n+1;\,\beta_0,\ldots,\beta_n}(U^{(0)},\ldots,U^{(n)}) \in \mathcal{T}_n,$$

Definition

$$T \prec_\omega U := \bigvee^{m+1;\,\alpha_0,\dots,\alpha_{m-1},\alpha_m\omega}_{x_1,\dots,x_{m-1},x_m} (T^{(0)}, \dots, T^{(m-1)}, T^{(m)} \succ_{\alpha_m} U + T^{(m)} \prec_\omega U + T^{(m)} \cdot U),$$

$$T \succ_\omega U := \bigvee^{n+1;\,\omega\beta_0,\beta_1,\dots,\beta_n}_{y_1,y_2,\dots,y_n} (T \succ_\omega U^{(0)} + T \prec_{\beta_0} U^{(0)} + T \cdot U^{(0)}, U^{(1)}, \dots, U^{(n)}),$$

$$T \cdot U := \bigvee^{m+n+1;\,\alpha_0,\dots,\alpha_{m-1},\alpha_m\beta_0,\beta_1,\dots,\beta_n}_{x_1,\dots,x_{m-1},x_m,y_1,\dots,y_n} (T^{(0)}, \dots, T^{(m-1)}, T^{(m)} \succ_{\alpha_m} U^{(0)} + T^{(m)} \prec_{\beta_0} U^{(0)} + T^{(m)} \cdot U^{(0)}, U^{(1)}, \dots, U^{(n)}).$$

Remark

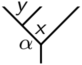
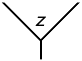
Note that $| \prec_\omega |, | \succ_\omega |$ and $| \cdot |$ are not defined. We employ the convention that

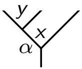
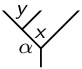
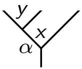
$$| \prec_1 | + | \succ_1 | + | \cdot | := |,$$

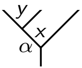
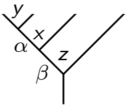
and

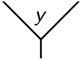
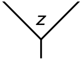
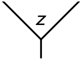
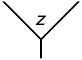
$$| \succ_1 T := T \prec_1 | := T \text{ and } | \prec_1 T := T \succ_1 | := 0.$$

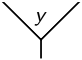
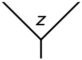
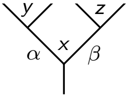
Example 12

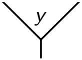
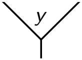
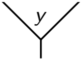
Let $T =$  $, U =$  with $x, y, z \in X$ and $\alpha \in \Omega$. For $\beta \in \Omega$,

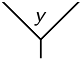
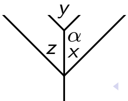
$T \succ_\beta U = = \bigvee_z^{2;\,\beta,1}\Big($  $\succ_\beta | +$  $\prec_1 | +$  $\cdot |, |\Big)$

$= \bigvee_z^{2;\,\beta,1}\Big($  $, |\Big) =$  .

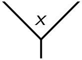
$T \prec_\beta U = \bigvee_x^{2;\,\alpha,\beta}\Big($  $, | \prec_\beta$  $+ | \succ_1$  $+ | \cdot$  $\Big)$

$= \bigvee_x^{2;\,\alpha,\beta}\Big($  $,$  $\Big) =$  .

$U \cdot T = = \bigvee_{z,x}^{3;\,1,\alpha,1}\Big(|, | \succ_1$  $+ | \prec_\alpha$  $+ | \cdot$  $, |\Big)$

$= \bigvee_{z,x}^{3;\,1,\alpha,1}\Big(|,$  $, |\Big) =$  .

Embedding free TDFAs into free RBFAs

Let $j : X \to \mathrm{DT}(X, \Omega)$ be the natural embedding map defined by $j(x) =$ 

for $x \in X$.

Theorem (Z.-Gao-Manchon, Algebr. Represent. Theory)

Let X be a set and let Ω be a semigroup. Then $\left(\mathrm{DT}(X, \Omega), (\prec_\omega, \succ_\omega)_{\omega\in\Omega}, \cdot\right)$, together with the map j, is the free tridendriform family algebra on X.

Lemma (Z.-Gao, Pacific J. Math.)

Let X be a set and let Ω be a semigroup. The Rota-Baxter family algebra $\left(\mathbf{k}\mathcal{T}(X, \Omega), \diamond, (B_\omega^+)_{\omega\in\Omega}\right)$ of weight 1 induces a tridendriform family algebra $\left(\mathbf{k}\mathcal{T}(X, \Omega), (\prec'_\omega, \succ'_\omega)_{\omega\in\Omega}, .'\right)$, where

$$T \prec'_\omega U := T \diamond B_\omega^+(U),\ T \succ'_\omega U := B_\omega^+(T) \diamond U \text{ and}$$
$$T .' U := T \diamond U, \text{ for } T, U \in \mathcal{T}(X, \Omega).$$

Embedding free TDFAs into free RBFAs

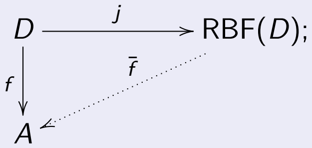
Theorem (Z.-Gao-Manchon, Algebr. Represent. Theory)

Let X be a set and let Ω be a semigroup. The free tridendriform family algebra $\big(\mathrm{DT}(X, \Omega), (\prec_\omega, \succ_\omega)_{\omega\in\Omega}, \cdot\big)$ on X is a tridendriform family subalgebra of the free Rota-Baxter family algebra $\big(\mathbf{k}\mathcal{T}(X, \Omega), \diamond, (B_\omega^+)_{\omega\in\Omega}\big)$ of weight 1.

Universal enveloping algebras

Definition

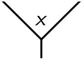
Let D be a DFA (resp. TDFA). A universal enveloping Rota-Baxter family algebra of weight λ of D satisfies the following commutative diagram:



The pair $(\mathrm{RBF}(D), j)$ is the universal enveloping RBFA of weight λ of D if

- j is a DFA (resp. TDFA) morphism (embedding map).
- A is any RBFA of weight λ, f is any DFA (resp. TDFA) morphism.
- $\exists\,!$ RBFA morphism $\bar{f}$.

Universal enveloping algebras

Let $j : X \to \mathrm{DD}(X, \Omega)$ be the natural embedding map defined by
$j(x) =$ 

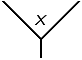
for $x \in X$.

Theorem (Z.-Gao-Manchon, Algebr. Represent. Theory)

The pair $(\mathbf{k}\mathcal{T}(X, \Omega), j)$ is the universal enveloping weight 0 Rota-Baxter family algebra of the free dendriform family algebra $\mathrm{DD}(X, \Omega)$, satisfying the following commutative diagram:

$$\begin{array}{ccc} \mathrm{DD}(X,\, \Omega) & \xrightarrow{\quad j \quad} & \mathbf{k}\mathcal{T}(X,\, \Omega) \\ {\scriptstyle g}\downarrow & \swarrow {\scriptstyle \bar{g}} & \\ A & & \end{array}$$

Universal enveloping algebras of (tri)dendriform family algebras

Let $j : X \to \mathrm{DT}(X, \Omega)$ be the natural embedding map defined by $j(x) =$ 

for $x \in X$.

Theorem (Z.-Gao-Manchon, Algebr. Represent. Theory)

The pair $\left(\mathbf{k}\mathcal{T}(X, \Omega), \lambda^{-1} j\right)$ is the universal enveloping weight λ Rota-Baxter family algebra of the free tridendriform family algebra $\mathrm{DT}(X, \Omega)$, satisfying the following commutative diagram:

$$
\begin{array}{ccc}
\mathrm{DT}(X, \Omega) & \xrightarrow{\lambda^{-1} j} & \mathbf{k}\mathcal{T}(X, \Omega) \\
{\scriptstyle f}\downarrow & \swarrow{\scriptstyle \bar{f}} & \\
A & &
\end{array}
$$

Thank you for your attention!