[OUTLOOK]

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\boxed{\begin{array}{c}\textcircled{1}\\ \text{FREE CLASSICAL} \\ \text{OBSERVABLES} \quad \mathcal{A}\end{array}} \arrow[r, "\lambda"', "\textcircled{2}"] \arrow[d, "\hbar"] & \boxed{\begin{array}{c}\textcircled{3}\\ \text{INTERACTING CLASSICAL} \\ \text{OBSERVABLES} \quad \mathcal{A}[\![\lambda]\!]\end{array}} \arrow[d, dashed, "??"] \\
\boxed{\begin{array}{c}\textcircled{4}\\ \text{FREE QUANTUM} \\ \text{OBSERVABLES} \quad \mathcal{A}[\![\hbar]\!]\end{array}} \arrow[r, "\lambda"] & \boxed{\begin{array}{c}\textcircled{5}\\ \text{INTERACTING QUANTUM} \\ \text{OBSERVABLES} \quad \mathcal{A}[\![\hbar,\lambda]\!]\end{array}}
\end{tikzcd}
$$

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K. Rejzner: Perturbative Algebraic quantum field theory, an introduction for mathematicians. SPRINGER 2016.

ALGEBRAIC APPROACH TO CLASSICAL FIELD THEORY [PART I]

MINKOWSKI SPACETIME AND CONFIGURATIONS

[DEF]: We denote by $\mathbb{M}_d$ the d-dimensional Minkowski space, i.e. $\mathbb{R}^d$ with flat metric $\eta = \mathrm{diag}(1, -1, \dots, -1)$ in cartesian coordinates $(x^\mu)_{\mu=0,\dots,d-1}$.

[DEF]: The FORWARD and BACKWARD LIGHT CONES are defined respectively by

$$V_+ := \{ x \in \mathbb{M}_d \mid (x)^2_\eta > 0 \ , \ x^0 > 0 \}$$

$$V_- := \{ x \in \mathbb{M}_d \mid (x)^2_\eta > 0 \ , \ x^0 < 0 \}$$

where $(x)^2_\eta = \eta(x,x)$ (here we're identifying $T_x\mathbb{M}_d \cong \mathbb{M}_d \ \forall x \in \mathbb{M}_d$) and $\overline{V_+}$ and $\overline{V_-}$ will denote the closures of these open sets.

[REMARK] Eventually comment here that this definition is simplified in Minkowski spacetime, but one can define V_+ and V_- also on any globally hyperbolic spacetime.

Configurations represent the physical fields of interest for the theory under consideration. IN GENERAL fields are sections of some fiber bundle over spacetime.

For our purposes it is sufficient to consider scalar fields, namely, sections of ~~a~~ the trivial ~~fiber~~ fiber bundle $\mathbb{M}_d \times \mathbb{R} \longrightarrow \mathbb{M}_d$. These are in one-to-one correspondence with smooth real-valued functions on $\mathbb{M}_d$.

[DEF] Let $\mathcal{E}(\mathbb{M}_d) := C^{\infty}(\mathbb{M}_d, \mathbb{R})$ (be the space of smooth real-valued functions on $\mathbb{M}_d$). $\mathcal{E}(\mathbb{M}_d)$ is endowed with the FRÉCHET topology generated by the family of seminorms

$$P_{K,m}(\varphi) = \sup_{\substack{x \in K \\ |\alpha| \leq m}} |\partial^{\alpha} \varphi(x)| ,$$

where $\alpha \in \mathbb{N}^d$ is a multiindex, $m \in \mathbb{N}$ and $K \subset \mathbb{M}_d$ is a compact subset.

THE SPACE OF FUNCTIONALS (OBSERVABLES)

One of the advantages of the functional approach to algebraic field theory is that the physical observables of both the classical and quantum theories are defined in terms of the same space of functionals. The difference is then in the algebraic structures introduced on this space.

IN GENERAL, we consider functionals $F : \mathcal{E}(M_d) \longrightarrow \mathbb{C}$ SMOOTH in the sense of BASTIANI CALCULUS on LCTVS.

More specifically, in our case, we have:

[DEF]: Consider a map $F : \mathcal{E}(\mathbb{M}_d) \longrightarrow \mathbb{C}$. The (FUNCTIONAL) DERIVATIVE of F at $\varphi \in \mathcal{E}(\mathbb{M}_d)$ in the direction of $\psi \in \mathcal{E}(\mathbb{M}_d)$ is defined as:

$$< F^{(1)}[\varphi] , \psi > = \lim_{t \to 0} \frac{F(\varphi + t\psi) - F(\varphi)}{t}$$

whenever this limit exists. F is called DIFFERENTIABLE at φ if $< F^{(1)}[\varphi] , \psi >$ exists $\forall \psi \in \mathcal{E}(\mathbb{M}_d)$. $F \in C^1(\mathcal{E}(\mathbb{M}_d), \mathbb{C})$ if the map

$$\begin{aligned} F^{(1)} : \mathcal{E}(\mathbb{M}_d) \times \mathcal{E}(\mathbb{M}_d) &\longrightarrow \mathbb{C} \\ (\varphi , \psi) &\longmapsto < F^{(1)}[\varphi] , \psi > \end{aligned}$$

is continuous. Higher derivatives are defined by

$$< F^{(k)}[\varphi] , \psi_1 \otimes \cdots \otimes \psi_k > = \frac{\partial^k}{\partial t_1 \cdots \partial t_k} F(\varphi + t_1\psi_1 + \cdots + t_k \psi_k)\Big|_{t_1 = \cdots = t_k = 0}$$

where $\varphi, \psi_1, \ldots, \psi_k \in \mathcal{E}(\mathbb{M}_d)$ and

$\psi_1 \otimes \cdots \otimes \psi_k \in \mathcal{E}(\mathbb{M}_d^k)$ is the tensor product of smooth functions

namely $(\psi_1 \otimes \cdots \otimes \psi_k)(x_1, \ldots, x_k) = \psi_1(x_1) \cdots \psi_k(x_k)$

$F \in C^K(\mathcal{E}(\mathbb{M}_d), \mathbb{C})$ if $\forall j \leq K$, the maps

$F^{(j)} : \mathcal{E}(\mathbb{M}_d) \times \mathcal{E}(\mathbb{M}_d)^j \to \mathbb{C}$ are continuous maps.

Finally, $F \in C^\infty(\mathcal{E}(\mathbb{M}_d); \mathbb{C})$ if $F \in C^K(\mathcal{E}(\mathbb{M}_d), \mathbb{C})$ $\forall K \in \mathbb{N}$.

[REMARK]: It is not difficult to show that the functional derivatives are also linear in their second arguments. <u>As a consequence</u>, given $F \in C^\infty(\mathcal{E}(\mathbb{M}_d), \mathbb{C})$ and $\varphi \in \mathcal{E}(\mathbb{M}_d)$, then $\forall K \geq 1$

$F^{(K)}[\varphi] \in \mathcal{E}'(\mathbb{M}_d^K)$. On the other hand, $\langle F^{(K)}[\cdot], \psi_1 \otimes \cdots \psi_K \rangle$ is again a smooth functional $\forall \psi_1, \dots, \psi_K \in \mathcal{E}(\mathbb{M}_d)$. We say that derivatives of functionals are FUNCTIONAL-VALUED DISTRIBUTIONS.

[NOTATION] In analogy with the notation for derivatives of functions depending on a finite number of variables, we write

$$\left\langle \frac{\delta F}{\delta \phi}[\varphi], \psi \right\rangle := \langle F^{(1)}[\varphi], \psi \rangle$$ and correspondingly

$$\left\langle \frac{\delta^K F}{\delta \phi^K}[\varphi], \psi_1 \otimes \cdots \otimes \psi_K \right\rangle := \langle F^{(K)}[\varphi], \psi_1 \otimes \cdots \otimes \psi_K \rangle$$

Sometimes, in order to highlight the distributional character of functional derivatives we will use the FORMAL INTEGRAL $\underbrace{,}_{\text{KERNEL}}$ NOTATION

$$\left\langle \frac{\delta^K F}{\delta \phi^K}[\varphi], \psi_1 \otimes \cdots \otimes \psi_K \right\rangle =: \int_{\mathbb{M}_d^K} \frac{\delta^K F}{\delta\phi(x_1) \cdots \delta\phi(x_K)}[\varphi]\, \psi(x_1) \cdots \psi(x_K)\, dx_1 \cdots dx_K$$

[DEF] The SPACETIME SUPPORT of $F \in C^\infty(\mathcal{E}(\mathbb{M}_d), \mathbb{C})$ is defined as

$$\mathrm{supp}(F) = \{ x \in \mathbb{M}_d \mid \forall \text{ neighborhood } U \text{ of } x,\ \exists\, \varphi, \psi \in \mathcal{E}(\mathbb{M}_d) \text{ with } \mathrm{supp}(\psi) \subset U \text{ such that } F[\varphi+\psi] \neq F[\varphi] \}$$

[It is now possible to characterize important classes of functionals by imposing restrictions on the SUPPORT and REGULARITY of their functional derivatives.]

{EVENTUALLY INSERT AT THIS POINT A DIGRESSION TO THE WAVEFRONT SET}

The most restrictive condition gives rise to the following notion:

[DEF]: $F \in C^\infty(\mathcal{E}(\mathbb{M}_d), \mathbb{C})$ is called REGULAR if $\forall \varphi \in \mathcal{E}(\mathbb{M}_d)$ and $\forall k \in \mathbb{N}_{\geq 1}$, $WF(F^{(k)}[\varphi]) = \emptyset$, i.e. $F^{(k)}[\varphi] \in \mathcal{D}(\mathbb{M}_d^k)$

The space of regular ~~fields~~ functionals is denoted by $\mathfrak{F}_{reg}$

[EXAMPLE]: THE BASIC FIELD $\phi(f)$, $f \in \mathcal{D}(\mathbb{M}_d)$

$$\phi(f) : \mathcal{E}(\mathbb{M}_d) \longrightarrow \mathbb{C}$$

$$\varphi \longmapsto \phi(f)[\varphi] := \int_{\mathbb{M}_d} f\varphi$$

$$\begin{cases} \phi(f)^{(1)}_{[\varphi]} = f \in \mathcal{D}(\mathbb{M}_d) \\ \phi(f)^{(k)}_{[\varphi]} = 0 \quad \forall k \geq 2 \end{cases}$$

$\forall \varphi \in \mathcal{E}(\mathbb{M}_d)$,

3

The following class of functionals encompasses most (if not all) of the physical quantities that can be considered.

[DEF]: A functional $\underbrace{F \in C^\infty(\mathcal{E}(\mathbb{M}_d, \mathbb{C}))}$ is called LOCAL if $\forall \varphi_0 \in \mathcal{E}(\mathbb{M}_d)$, $\exists$ an open neighborhood V of φ_0 in $\mathcal{E}(\mathbb{M}_d)$

and $\exists K \in \mathbb{N}$ such that

$$F[\varphi] = \int_{\mathbb{M}_d} \alpha(j^K_x \varphi) \qquad \forall \varphi \in V,$$

where $j^K\varphi$ is the K-th jet prolongation of φ and α is a density-valued function on the K-th order jet bundle. The space of local functionals is denoted by $\mathcal{F}_{loc}$.

[REMARK] If $F \in \mathcal{F}_{loc}$, then $F^{(K)}[\varphi] \in \mathcal{E}'(\mathbb{M}_d^K)$ is supported on the thin diagonal $\Delta_K = \{(x_1, \dots, x_K) \in \mathbb{M}_d^K \mid x_1 = \dots = x_K\}$ and the wavefront set of $F^{(K)}[\varphi]$ is CONORMAL to $T\Delta_K$

[EXAMPLE] $\phi^2(f)$, $f \in \mathcal{D}(\mathbb{M}_d)$ (OR, MORE IN GENERAL, ANY POLYNOMIAL IN THE DERIVATIVES OF ϕ)

$$\phi^2(f) : \mathcal{E}(\mathbb{M}_d) \longrightarrow \mathbb{C}$$
$$\varphi \longmapsto \phi^2(f)[\varphi] := \int_{\mathbb{M}_d} \varphi^2(x) f(x)\, dx$$

$$\forall \varphi \in \mathcal{E}(\mathbb{M}_d) \quad \begin{cases} \phi^2(f)^{(1)}[\varphi] = 2f\varphi \in \mathcal{D}(\mathbb{M}_d) \\ \phi^2(f)^{(2)}[\varphi](x_1, x_2) = 2\delta(x - x_1, x - x_2) f(x) \\ \phi^2(f)^{(K)}[\varphi] = 0 \quad \forall K \geq 3 \end{cases}$$

in this case $\underline{V = \mathcal{E}(\mathbb{M}_d)}$

$\underline{K = 0}$

otherwise $\underline{K = \text{degree of the polynomial}}$

For the operations that one needs to perform in the construction of pAQFT models, it turns out that local functionals are not general enough.

[DEF]: $F \in C^\infty(\mathcal{E}(M_d,\mathbb{C}))$ is called MICROCAUSAL if it has compact spacetime support and if its derivatives satisfy the microlocal condition

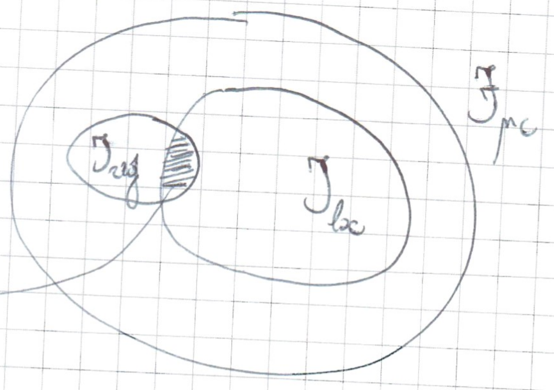
$$WF\left(F^{(n)}[\varphi]\right) \subset \Xi_n \qquad \forall n \in \mathbb{N}, \ \forall \varphi \in \mathcal{E}(M_d)$$

where

$$\Xi_n := T^*M_d^n \setminus \left\{ (x_1,\xi_1;\dots;x_n,\xi_n) \mid (\xi_1,\dots,\xi_n) \in \left(\overline{V}_+^n \cup \overline{V}_-^n\right)_{(x_1,\dots,x_n)} \right\}.$$

The space of microcausal functionals is denoted by $\mathcal{F}_{\mu c}$.

SITUATION:



local functionals of order 1

OF COURSE

$\mathcal{F}_{loc} \subsetneq \mathcal{F}_{\mu c}$

[DEF] On the space of microcausal functionals, we can introduce the following operations:

- commutative pointwise product

$$\mu : \mathcal{F}_{\mu c} \times \mathcal{F}_{\mu c} \longrightarrow \mathcal{F}_{\mu c}$$
$$(F, G) \longmapsto \mu(F \otimes G) =: FG$$

where $(FG)[\varphi] := F[\varphi]\, G[\varphi] \quad \forall \varphi \in \mathcal{E}(M_d)$

• involution $\quad {}^* : \mathfrak{F}_{\mu c} \longrightarrow \mathfrak{F}_{\mu c}$ where $F^*[\varphi] := \overline{F[\varphi]}$

$\qquad F \longmapsto F^* \qquad \forall \varphi \in \mathcal{E}(\mathbb{M}_d)$.

This way $\mathfrak{F}_{\mu c}$ becomes a $*$-algebra.

[ALGEBRAIC APPROACH TO QFT — PART II]

DYNAMICS

We have introduced the essential kinematical structure. We now introduced the dynamics, inspired by the Lagrangian formalism.

[DEF]: A GENERALIZED LAGRANGIAN on $\mathbb{M}_d$ is a map

$$L : \mathcal{D}(\mathbb{M}_d) \longrightarrow \mathcal{F}_{loc}$$ such that the following holds:

(i) ADDITIVITY: $L(f+g+h) = L(f+g) - L(g) + L(g+h)$

for $f, g, h \in \mathcal{D}(\mathbb{M}_d)$ with $supp(f) \cap supp(h) = \emptyset$;

(ii) SUPPORT: $supp(L(f)) \subseteq supp(f)$;

(iii) COVARIANCE (not necessary here)

[REMARK] The definition formalizes the idea that a generalized Lagrangian consists of a local functional obtained by integrating a Lagrangian density $\mathcal{L}(x)[\varphi]$ against a test function, which ensures convergence.

[EXAMPLE]: The FREE LAGRANGIAN L_0 on $\mathbb{M}_d$ is defined as

$$L_0(f)[\varphi] := \frac{1}{2} \int_{\mathbb{M}_d} \left(\eta^{\mu\nu} \partial_\mu \varphi \, \partial_\nu \varphi - m^2 \varphi^2 \right) f \, dx$$

[REMARK] The free Lagrangian is more than just an example. In general, the Lagrangian of ANY ~~[illegible]~~ model is given as $L = L_0 + \lambda L_{int}$, where L_{int} encodes the nature of the interactions described by the model and is treated perturbatively. $\lambda \in \mathbb{R}$ is called the BOOKKEEPING coupling constant.

[DEF]: The EULER-LAGRANGE derivative of a generalized Lagrangian L is the smooth map $S'_L : \mathcal{E}(M_d) \longrightarrow \mathcal{D}'(M_d)$ defined by

$$< S'_L[\varphi], h > = < L(f)^{(1)}[\varphi], h >$$

where $f, h \in \mathcal{D}(M_d)$ and f is chosen in such a way that $f \equiv 1$ on $supp(h)$.

[DEF]: The equation of motion induced by the generalized Lagrangian L is

$$S'_L[\varphi] = 0 \qquad (\text{in } \mathcal{D}'(M_d))$$

understood as a condition on $\varphi \in \mathcal{E}(M_d)$. The space of solutions to the EOM is denoted by $\mathcal{E}_S(M_d)$ and is a submanifold of $\mathcal{E}(M_d)$.

~~The starting point of the construction of~~ In particular, to the free Lagrangian, we can associate a fundamental DIFFERENTIAL operator

This is in practice a DIFFERENTIAL OPERATOR too

[DEF]: Consider the second functional derivative of the free Lagrangian L_0:

$$\langle L_0(f)^{(2)}[\varphi], h_1 \otimes h_2 \rangle = \int_{\mathbb{M}_d} \left(\eta^{\mu\nu} \partial_\mu h_1 \partial_\nu h_2 - m^2 h_1 h_2 \right) f \, dx$$

where $f \equiv 1$ on $\mathrm{supp}(h_1) \cup \mathrm{supp}(h_2)$. This operator extends naturally to an operator P called the WAVE OPERATOR:

$$P : \mathcal{E}(\mathbb{M}_d) \longrightarrow \cancel{\mathcal{D}'(\mathbb{M}_d)} \quad \left(\boxed{\text{in } \mathcal{E}(\mathbb{M}_d) \text{ actually}} \right)$$

$$\varphi \longmapsto P\varphi = -(\Box + m^2)\varphi = -\left(\partial^2_{x^0} - \partial^2_{x^1} - \dots - \partial^2_{x^{d-1}} + m^2 \right)\varphi$$

which acts on test functions $h \in \mathcal{D}(\mathbb{M}_d)$ as:

$$\langle P\varphi, h \rangle = \int_{\mathbb{M}_d} -\left(\partial^2_{x^0}\varphi - \partial^2_{x^1}\varphi - \dots - \partial^2_{x^{d-1}}\varphi + m^2\varphi \right) h \, dx$$

[REMARK/PROPOSITION]: P is a NORMALLY HYPERBOLIC OPERATOR hence it admits UNIQUE GREEN'S FUNCTIONS, (RETARDED AND ADVANCED)

$$\Delta^R_m, \Delta^A_m : \mathcal{D}(\mathbb{M}_d) \longrightarrow \mathcal{E}(\mathbb{M}_d)$$ defined by the requirements:

$$P \circ \Delta^{R,A}_m = id_{\mathcal{D}(\mathbb{M}_d)}$$

$$\left(\Delta^{R,A}_m \circ P \right)\Big|_{\mathcal{D}(\mathbb{M}_d)} = id_{\mathcal{D}(\mathbb{M}_d)}$$

and $$\mathrm{supp}\left(\Delta^{R,A}_m(f) \right) \subseteq J^{+,-}(\mathrm{supp}(f))$$

Alternatively, we can say that $\Delta_m^{R,A} \in \mathcal{D}'(\mathbb{M}_d)$ and satisfy

$$P\Delta_m^{R,A} = -(\Box + m^2)\,\Delta_m^{R,A} = \delta$$

$$\mathrm{supp}\,(\Delta_m^{R,A}) \subseteq \overline{V}_{+,-}$$

The difference between the retarded and advanced propagators

$$\Delta_m := \Delta_m^R - \Delta_m^A$$

is called CAUSAL PROPAGATOR. The causal propagator satisfies

$$P\Delta_m = 0 \quad \text{and} \quad \mathrm{supp}\,(\Delta_m) \subseteq (\overline{V}_+ \cup \overline{V}_-).$$

[DEF]: The PEIERL'S BRACKET of functionals with respect to the differential operator P induced by the free Lagrangian is the bilinear map

$$\{\cdot\,,\cdot\} : \mathcal{F}_{\mu c} \times \mathcal{F}_{\mu c} \longrightarrow \mathcal{F}_{\mu c} \quad \text{given by}$$

$$\{F, G\}[\varphi] := \langle F^{(1)}[\varphi],\, \Delta_m * G^{(1)}[\varphi] \rangle =$$

$$= \int_{\mathbb{M}_d^2} \frac{\delta F}{\delta\phi(x)}\, \Delta_m(x-y)\, \frac{\delta G}{\delta\phi(y)}\, dx\, dy$$

EVENTUALLY OPEN HERE A PARENTHESIS ON THE MICROLOCAL ANALYSIS INVOLVED IN THIS DEFINITION!!!

[SPACES OF DISTRIBUTIONS WITH SPECIFIED WF SET AND THEIR DUALITY]

[DEF] Let $\mathcal{D}'_{\Gamma}(\mathbb{M}_d)$ denote the space of distributions whose wavefront sets are contained in a closed cone $\Gamma \subseteq \dot{T}^* \mathbb{M}_d$.

[DEFINITION] (DABROWSKI, BROUDER) We call ~~Define~~ NORMAL TOPOLOGY τ_N the locally convex topology on $\mathcal{D}'_{\Gamma}(\mathbb{M}_d)$ given by the following system of seminorms:

(i) all seminorms on $\mathcal{D}'(\mathbb{M}_d)$ for the strong topology, that is

$$P_B(u) = \sup_{f \in B} |\langle u, f \rangle| ,$$

where $u \in \mathcal{D}'(\mathbb{M}_d)$ and where B runs over the bounded sets of $\mathcal{D}(\mathbb{M}_d)$;

(ii) the Hörmander seminorms:

$$\|u\|_{m,V,\chi} = \sup_{\xi \in V} (1 + |\xi|)^m \, |\widehat{u\chi}(\xi)| ,$$

where $m \geq 0$, $\chi \in \mathcal{D}(\mathbb{M}_d)$ and $V \subseteq \mathbb{M}_d$ is a closed cone with $(\operatorname{supp}(\chi) \times V) \cap \Gamma = \emptyset$ in $T^*\mathbb{M}_d$

[REMARK] With this topology τ_N, $\mathcal{D}'_\Gamma(\mathbb{M}_d)$ has nice functional properties. In particular, it is NUCLEAR AND COMPLETE

[LEMMA]: If Γ is a closed cone in $\dot{T}^*\mathbb{M}_d$ and $\Lambda = (\Gamma')^c :=$ $:= \{ (x,\xi) \in \dot{T}^*\mathbb{M}_d \mid (x,-\xi) \notin \Gamma \}$, then the following pairing between $\mathcal{D}'_\Gamma(\mathbb{M}_d)$ and $\mathcal{E}'_\Lambda := \{ v \in \mathcal{E}'(\mathbb{M}_d) \mid WF(v) \subseteq \Lambda \}$ is well defined:

$$\langle u, v \rangle = \frac{1}{(2\pi)^d} \int_{\mathbb{M}_d} \widehat{\chi u}(\xi)\, \hat{v}(-\xi)\, d\xi$$

where $u \in \mathcal{D}'_\Gamma(\mathbb{M}_d)$, $v \in \mathcal{E}'_\Lambda(\mathbb{M}_d)$ and $\chi \in \mathcal{D}(\mathbb{M}_d)$ is such that $\chi \equiv 1$ on ~~supp~~ a compact neighborhood of supp(v).

[PROPOSITION] $\mathcal{E}'_\Lambda(\mathbb{M}_d)$ is the DUAL of $\mathcal{D}'_\Gamma(\mathbb{M}_d)$ for its normal topology τ_N

[REMARK] From the topology τ_N on $\mathcal{D}'_\Gamma(\mathbb{M}_d)$ and this duality pairing $\mathcal{E}'_\Lambda(\mathbb{M}_d)$ inherits a topology that makes it a nuclear space.
(but not COMPLETE)

[EVENTUALLY MENTION THIS]

[DEF] The space of microcausal functionals $\mathcal{F}_{\mu c}$ can be equipped with the topology τ_{BDF} defined as the initial topology with respect to all the maps

$$\mathcal{F}_{\mu c} \longrightarrow \left(\mathcal{E}'_{\Xi_m}(\mathbb{M}_d^m),\ \tau_N^* \right) \qquad \text{where } m \in \mathbb{N}$$
$$F \longmapsto F^{(m)}_{[\varphi]} \qquad \text{and } \varphi \in \mathcal{E}(\mathbb{M}_d)$$

[REMARK] Well-posedness of the Peierls's bracket

$$WF\left(\Delta_m * G^{(1)}[\varphi]\right) \subseteq \left\{ (x+y, \xi) \mid (x,\xi) \in WF(\Delta_m) \text{ and } (y,\xi) \in WF(G^{(1)}[\varphi]) \right\}$$

this is a "theorem" by Hörmander

but from the properties of $WF(\Delta_m)$, we have that this set is ~~contained in the cone~~ ~~$(\Delta_m \times (\overline{V}_+ \cup \overline{V}_-))$~~ EMPTY, because the WF set of $G^{(1)}[\varphi]$ is never on $\overline{V}_+$ or $\overline{V}_-$

[$WF(\Delta_m)$ is given by the cone $\overline{V}_+ \cup \overline{V}_-$ over the points x with $(x)^2_\eta = 0$]

EVENTUALLY

[REMARK / PROPOSITION] With the topology τ_{BDF}, the Peierls's bracket

$$\{\cdot,\cdot\} : \mathcal{F}_{\mu c} \to \mathcal{F}_{\mu c} \to \mathcal{F}_{\mu c}$$

becomes (separately) sequentially continuous

[PROPOSITION] The Peierls bracket has the following properties:

(i) it is well-defined due to the wavefront set properties of F and G; moreover $\{F, G\}$ again satisfies the wavefront set condition;

(ii) $\{\cdot, \cdot\}$ is bilinear;

(iii) SKEW-SYMMETRY: $-\{F, G\} = +\{G, F\}$

(iv) LEIBNIZ RULE: $\{F, GH\} = \{F, G\} H + G \{F, H\}$

(v) JACOBI IDENTITY: $\{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} = 0$

[DEF] $\left(\mathcal{F}_{\mu c}, \mu, *, \{\cdot, \cdot\} \right)$ is a POISSON ALGEBRA called the Poisson algebra of FREE CLASSICAL FIELDS

[CLASSICAL INTERACTION PICTURE]

~~[PERTURBATIVE INTERACTING CLASSICAL FUNCTIONALS]~~

Suppose our Lagrangian is given by $L = L_0 + \lambda L_{int}$, $\lambda \in \mathbb{R}_{>0}$

Recall that $\mathcal{E}_{S_0}(\mathbb{M}_d) = \{ \varphi \in \mathcal{E}(\mathbb{M}_d) \mid S'_{L_0}[\varphi] = 0 \}$

and $\mathcal{E}_{S}(\mathbb{M}_d) = \{ \varphi \in \mathcal{E}(\mathbb{M}_d) \mid S'_{L_0+\lambda L_{int}}[\varphi] = 0 \}$

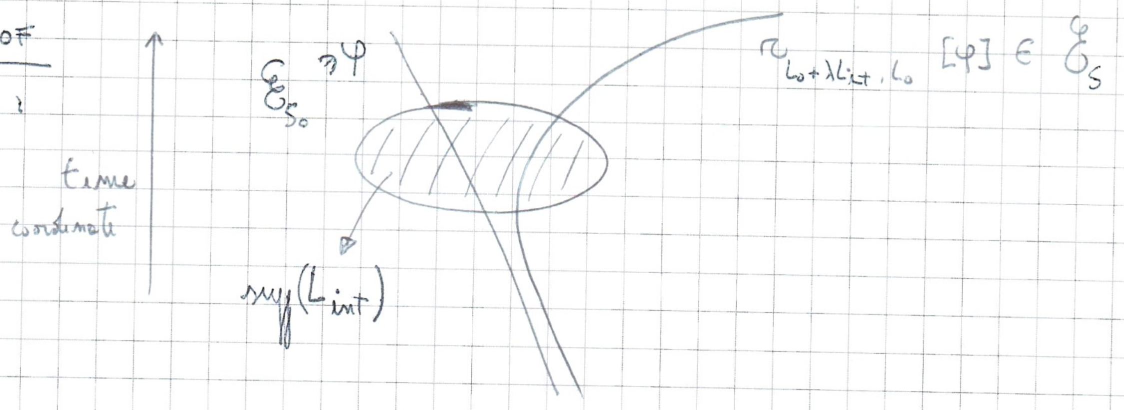
[DEF]: A RETARDED MØLLER OPERATOR is a map (SMOOTH? PROBABLY)

$$r_{L_0+\lambda L_{int}, L_0} : \mathcal{E}(\mathbb{M}_d) \longrightarrow \mathcal{E}(\mathbb{M}_d) \quad \text{such that:}$$

(i) $r_{L_0+\lambda L_{int}, L_0}[\varphi](x) = \varphi(x)$ for $x \in \mathbb{M}_d$ "sufficiently early"

(ii) $S'_{L_0+\lambda L_{int}} \circ r_{L_0+\lambda L_{int}, L_0} = S'_{L_0}$

SKETCH OF IDEA:



MOTIVATION: In analogy to some classical results, like

[JÖRGEN'S THEOREM]: Let $m>0$, $\lambda>0$ and $h, h_1 \in \mathcal{D}(\mathbb{R}^3)$ be given. Then the Cauchy problem

$$\begin{cases} (\Box + m^2) f(t,x) + 4\lambda (f(t,x))^3 = 0 \\ f(0,x) = h(x) \\ \partial_t f(0,x) = h_1(x) \end{cases}$$

has a UNIQUE SOLUTION $f \in C^\infty(\mathbb{R}^4, \mathbb{R})$

One way to define ~~an~~ INTERACTING FUNCTIONALS is the following

[DEF]: The interacting functional F_{int} associated to $F \in \mathcal{F}_{\mu c}$ is defined by taking $F_{int} := F\big|_{\mathcal{E}_S}$

We can see that this definition is sensible by considering the INTERACTING BASIC FIELD $\phi(f)_{int}$, for $\boxed{S_0}$ ~~free~~. This ~~also holds~~ implies that

$$\oint (\Box + m^2)\, \phi(f)_{int} [\varphi_{S_0}] = (\Box + m^2)\, \varphi_{S_0} = 0 \quad \text{because } \varphi_{S_0} \in \mathcal{E}_{S_0}$$

so in a sense $(\Box + m^2)\, \phi(f)_{int} = 0$, that is,

$\phi(f)_{int}$ ~~satis~~ is a ~~sol~~ functional solution of the free EOM.

The problem with working with interacting functionals is that $\mathcal{E}_S$ is a submanifold of $\mathcal{E}$, but not much is known about its structure, while $\mathcal{E}$ is a nice ~~F~~ Fréchet space.

[DEF]: Give $F \in \mathcal{F}_{\mu c}$, the RETARDED ~~FIELD~~ FUNCTIONAL F_{ret} is defined by

$$F_{ret} := F \circ r_{L_0 + \lambda L_{int}, L_0} : \mathcal{E}(M_d) \longrightarrow \mathbb{C}$$

[REMARK] Direct consequences of this definition are:

(i) $(AB)_{ret} = A_{ret} \, B_{ret}$, $\forall A, B \in \mathcal{F}_{\mu c}$

(ii) consider the retarded basic field $\phi(f)_{ret}$, $f \in \mathcal{D}(M_d)$, it satisfies:

$$-(\Box + m^2)\, \phi(f)_{ret} + \left(S'_{\lambda L_{int}}\right)_{ret} = -(\Box + m^2)\, \phi(f)$$ → CALLED OFF-SHELL FIELD EQUATION

(iii) $\phi(f)_{ret} = \phi(f)$ if $supp(f)$ is "early enough".

THIS IS WHAT IS MEANT BY RETARDED

[DEF] PERTURBATIVE EXPANSION OF RETARDED FUNCTIONALS

Give $F \in \mathcal{F}_{\mu c}$, we set:

NOTATION $\boxed{R(e_\otimes^{L_{int}}; F)}$

$$F_{ret} = \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} R_m(L_{int}^{\otimes m}; F) \in \mathcal{F}_{\mu c}[[\lambda]]$$

where $\forall m \in \mathbb{N}$

$$R_m : \mathcal{F}_{loc}^{\otimes m} \otimes \mathcal{F}_{\mu c} \longrightarrow \mathcal{F}_{\mu c}$$

are called RETARDED PRODUCTS

and are:

- LINEAR;
- SYMMETRIC IN THE FIRST m ARGUMENTS;
- satisfy: $R_m(L_{int}^{\otimes m}; AB) = R_m(L_{int}^{\otimes m}; A)\, R_m(L_{int}^{\otimes m}, B)$ $\forall A, B \in \mathcal{F}_{\mu c}$
- satisfy: $-(\Box + m^2)\, R_m(L_{int}^{\otimes m}; \phi(f)) + \underbrace{R_{m-1}(L_{int}^{\otimes (m-1)}; S'_{\lambda L_{int}})}_{\text{for } m=0 \text{ this is not defined}} = -(\Box + m^2)\, \phi(f)$ ~~$\forall m \geq 1$~~
- satisfy: $R_m(L_{int}^{\otimes m}; F) = F$ $\forall F \in \mathcal{F}_{\mu c}$ with $supp(F)$ "early enough"

[REMARKS] (1) F_{ret} is a FORMAL POWER SERIES in λ, NO CONVERGENCE IS A PRIORI ASSUMED.

(2) Intuitively one can think of the retarded products as the coefficients of the Taylor expansion of the retarded functional, that is

$$R_m(L_{int}^{\otimes m}; F) = \frac{d^m}{d\lambda^m}\Big|_{\lambda=0} F_{ret} = \frac{d^m}{d\lambda^m}\Big|_{\lambda=0} F \circ r_{L_0+\lambda L_{int}, L_0}$$

[PROPOSITION] The classical retarded product EXISTS and is UNIQUE. It can be computed in the following way: let $L_{int} \in \mathfrak{F}_{loc}$ and define the operator on $\mathfrak{F}_{\mu c}$

$$R_{L_{int}}(x) := -\int_{M_d} \frac{\delta L_{int}}{\delta \varphi(x)} \Delta^R_{m}(y-x) \frac{\delta}{\delta \varphi(y)} dy .$$

For all $F \in \mathfrak{F}_{\mu c}$, $R_{L_{int}}(x) F$ is well defined as a ~~field~~ functional-valued distribution, and

$$\int_{M_d} R_{L_{int}}(x) F \, dx \in \mathfrak{F}_{\mu c}$$

The $R_m(L_{int}^{\otimes m}; F)$ is obtained as

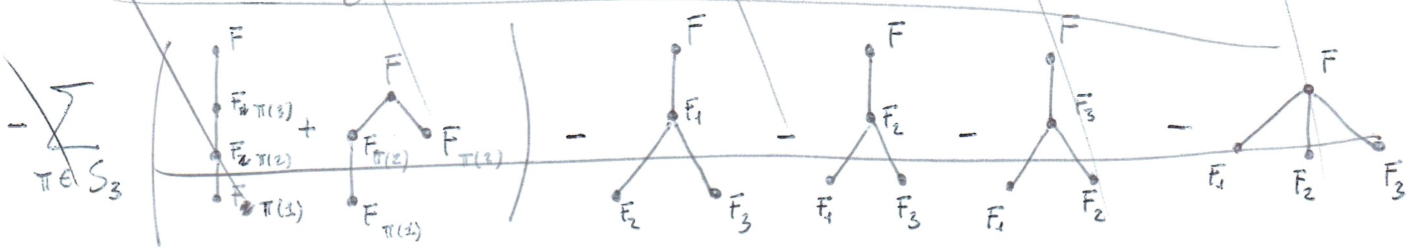
$$R_m(L_{int}^{\otimes m}; F) = m! \int_{x_1^0 \le \dots \le x_m^0} R_{L_{int}}(x_1) \cdots R_{L_{int}}(x_m) F \, dx_1 \cdots dx_m$$

CONTINUA

[EXAMPLE] It is not difficult to show that the retarded product for different entries has the form: $\forall F_1,\dots,F_m, F\in \mathfrak{F}_{loc}$

$$R_m(F_1\otimes\cdots\otimes F_m; F) = \int_{x_1^0\le\cdots\le x_m^0} \sum_{\pi\in S_m} R_{F_{\pi(1)}}(x_1)\cdots R_{F_{\pi(m)}}(x_m)\, F \; dx_1\cdots dx_m$$

Then, for $m=3$, the retarded product $R_3(F_1\otimes F_2\otimes F_3; F)$ can be written using the FEYNMAN DIAGRAM NOTATION as:



From the definition of ~~also~~ the CAUSAL PROPAGATOR, it follows that the last formula is also equivalent to the following:

$$R_m(L_{int}^{\otimes m}(g); F(f)) = \int_{\mathbb{M}_d^{m+1}} dx_1\cdots dx_m\, dy\; g(x_1)\cdots g(x_m)\, f(y)$$

$$\theta(y^0-x_m^0)\,\theta(x_m^0-x_{m-1}^0)\cdots\theta(x_2^0-x_1^0)\,\{L_{int}(x_1),\{L_{int}(x_2),\{\cdots,\{L_{int}(x_m),F(y)\}\cdots\}$$

$$=: \int_{\mathbb{M}_d^{m+1}} dx_1\cdots dx_m\, dy\; g(x_1)\cdots g(x_m)\, f(y)\; R_m(L_{int}(x_1)\otimes\cdots\otimes L_{int}(x_m); F(y))$$

[EXAMPLE (1.9.2)] Consider $F = \phi^2(f)$ and $L_{int} = \phi^4(g)$, $f, g \in \mathcal{D}(\mathbb{M}_d)$. We want to compute $R(\phi^4(g); \phi^2(f))$ up to order $n = 2$ of perturbation.

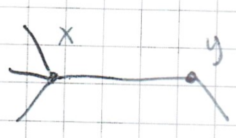
Instead of writing the functionals the usual way, we omit the test functions and regard $\phi^4(x)$, $\phi^2(y)$ and $R(\phi^4(x); \phi^2(y))$ as FUNCTIONAL - VALUED DISTRIBUTIONS.

$\underline{n=1}$: applying directly the formula for the retarded product, we get

$$R_1(\phi^4(x); \phi^2(y)) = -\int_{\mathbb{M}_d^2} \frac{\delta \phi^4(x)}{\delta \phi(w)} \Delta^R_m(z-w) \frac{\delta \phi^2(y)}{\delta \phi(z)} \, dw \, dz$$

$$= -4\phi^3(x) \Delta^R_m(y-x) \, 2\phi(y) = -8\phi^3(x) \Delta^R_m(y-x) \phi(y)$$

This term is represented by a "FEYNMAN DIAGRAM"



- where each variable is assigned a vertex;
- each basic field is a leg attached to the corresponding vertex;
- each propagator with arguments y, x represents an edge between the corresponding vertices;

$\underline{n=2}$: Using the last formula for the retarded product, we have :

$$R_2\left(\phi^4(x_1)\otimes\phi^4(x_2);\phi^2(y)\right) = \theta(y-x_2^0)\,\theta(x_2^0-x_1^0)\left\{\phi^4(x_1),\left\{\phi^4(x_2),\phi^2(y)\right\}\right\}$$
$$+\left(x_1\leftrightarrow x_2\right).$$

For the inner bracket we get, rightfully, the same result as before:

$$\theta(y^0-x_2^0)\left\{\phi^4(x_2),\phi^2(y)\right\} = -8\,\phi^3(x_2)\,\Delta_m^R(y-x_2)\,\phi(y)$$

For the ~~top~~ total nested bracket we get then 2 terms:

$$\theta(y^0-x_2^0)\,\theta(x_2^0-x_1^0)\left\{\phi^4(x_1),\phi^3(x_2)\phi(y)\right\}\Delta_m^R(y-x_2)$$

$$= -12\,\phi^3(x_1)\,\Delta_m^R(x_2-x_1)\,\phi^2(x_2)\,\phi(y)\,\Delta_m^R(y-x_2)$$
$$-4\,\phi^3(x_1)\,\Delta_m^R(y-x_1)\,\phi^3(x_2)\,\Delta_m^R(y-x_2)\,\theta(x_2^0-x_1^0)$$

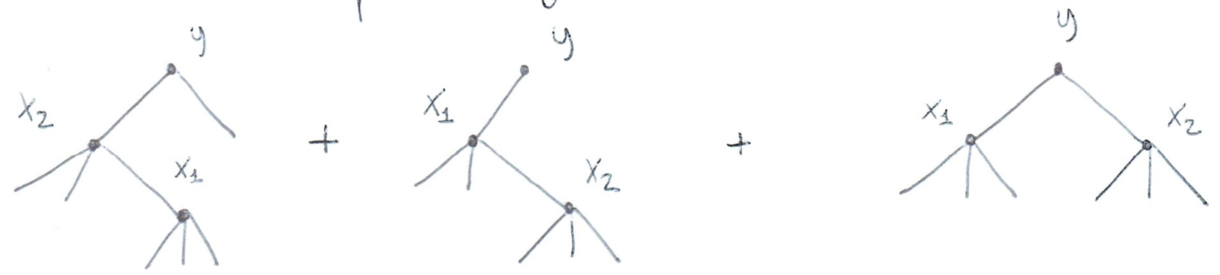
Adding the $(x_1\leftrightarrow x_2)$ terms and using that $\theta(x_2^0-x_1^0)+\theta(x_1^0-x_2^0)=1$

we obtain $R_2\left(\phi^4(x_1)\otimes\phi^4(x_2);\phi^2(y)\right)$

$$= 96\,\phi^3(x_1)\,\Delta_m^R(x_2-x_1)\,\phi^2(x_2)\,\Delta_m^R(y-x_2)\,\phi(y) + (x_1\leftrightarrow x_2)$$

$$+32\,\phi^3(x_1)\,\Delta_m^R(y-x_1)\,\phi^3(x_2)\,\Delta_m^R(y-x_2)$$

These terms can be represented by the "FEYNMAN DIAGRAMS"



[BASIC PROPERTIES OF THE RETARDED PRODUCT]

1. CAUSALITY considering the expression for the first retarded product

$$R_1(L_{int}; F) = -\int_{M_d^2} \frac{\delta L_{int}}{\delta\phi(x)} \Delta^R_m(y-x) \frac{\delta F}{\delta\phi(y)}\, dx\, dy$$

and the fact that $supp(\Delta^R_m) \subseteq \overline{V}_+$, we get that "integration over" x is restricted to $x \in (supp(F) + \overline{V}_-)$.

Therefore

$$R_1(L_{int} + \tilde{L}_{int}; F) = R_1(L_{int}; F)$$

if $(supp(F) + \overline{V}_-) \cap supp(\tilde{L}_{int}) = \phi$.

It can be shown that this relation holds at any order and implies

$$R(e_{\otimes}^{L_{int}+\tilde{L}_{int}}; F) = R(e_{\otimes}^{L_{int}}; F) \quad \text{if} \quad (supp(F) + \overline{V}_-) \cap supp(\tilde{L}_{int}) = \phi$$

2. FIELD INDEPENDENCE. taking the functional derivative of R_1, we get

$$\frac{\delta}{\delta\phi} R_1(L_{int}; F) = R_1\left(\frac{\delta L_{int}}{\delta\phi}; F\right) + R_1\left(L_{int}; \frac{\delta F}{\delta\phi}\right)$$

and proceeding analogously for any order one obtains

$$\frac{\delta}{\delta\phi} R(e_{\otimes}^{L_{int}}; F) = R\left(e_{\otimes}^{L_{int}} \otimes \frac{\delta L_{int}}{\delta\phi}; F\right) + R\left(e_{\otimes}^{L_{int}}; \frac{\delta F}{\delta\phi}\right)$$

3 GLZ RELATION

[PROPOSITION] (Glaser, Lehmann, Zimmermann) $\forall F, G, L_{int} \in \mathcal{F}_{loc} \subset \mathcal{F}_{\mu c}$

$$\left\{ R\left(e_{\otimes}^{L_{int}}; F\right), R\left(e_{\otimes}^{L_{int}}; G\right) \right\} = R\left(e_{\otimes}^{L_{int}} \otimes F; G\right) - R\left(e_{\otimes}^{L_{int}} \otimes G; F\right)$$

[REMARK]: a consequence of the GLZ relation is that if

$\forall x \in \mathrm{supp}(F)$ and $\forall y \in \mathrm{supp}(G)$ it holds $\underbrace{(x-y)^2_\eta < 0}$

then

$$\left\{ R\left(e_{\otimes}^{L_{int}}; F\right), R\left(e_{\otimes}^{L_{int}}; G\right) \right\} = 0$$

This is called being <u>SPACELIKE</u> <u>SEPARATED</u>

[DEFORMATION QUANTIZATION OF THE FREE THEORY]

By DQ, we mean a FORMAL DEFORMATION that takes the commutative Poisson $*$-algebra $(\mathfrak{F}_{\mu c}, \mu, \{\cdot,\cdot\}, {}^*)$ of the free classical functionals (or observables) and returns a non-commutative Poisson $*$-algebra $(\mathfrak{F}_{\mu c}[[\hbar]], \star, [\cdot,\cdot]_\star, {}^*)$ of corresponding quantum observables.

In other words, we seek a deformation $\star : \mathfrak{F}_{\mu c} \times \mathfrak{F}_{\mu c} \longrightarrow \mathfrak{F}_{\mu c}[[\hbar]]$ of the classical product of ~~field~~ functionals such that:

(i) $\star$ is BILINEAR;

(ii) $\star$ is an ASSOCIATIVE product;

(iii) $F \star G \xrightarrow{\hbar \to 0} FG$

(iv) $\frac{1}{i\hbar}[F, G]_\star = \frac{1}{i\hbar}(F \star G - G \star F) \xrightarrow{\hbar \to 0} \{F, G\}$

THIS GOES BACK TO DIRAC'S IDEA OF QUANTIZATION

namely "canonical" brackets are replaced by commutators

[ANSATZ]: $\phi(f) \star \phi(g) = \phi(f)\,\phi(g) + \hbar \langle f, \overset{W}{H} * g \rangle$ $\quad \forall f, g \in \mathcal{D}(\mathbb{M}_d)$

$$\langle f, \overset{W}{H} * g \rangle = \int_{\mathbb{M}_d^2} f(x)\, \overset{W}{H}(x-y)\, g(y)\, dx\, dy$$

$$\Longrightarrow \frac{1}{i\hbar}[\phi(f), \phi(g)]_\star = \frac{1}{i}\left(\langle f, \overset{W}{H} * g \rangle - \langle g, \overset{W}{H} * f \rangle \right) = \{\phi(f), \phi(g)\} = \langle f, \Delta^*_m g \rangle$$

at the level of integral Kernels, this means

$$\cancel{\mathcal{H}}^{W}(x-y) - \cancel{\mathcal{H}}^{W}(y-x) = i\,\Delta_m(x-y)$$

[PROPOSITION] There exists distributions $W \in \mathcal{D}'(\mathbb{M}_d)$, called TWO-POINT FUNCTION, and $H \in \mathcal{D}'(\mathbb{M}_d)$, called HADAMARD PARAMETRIX, such that the following decomposition holds:

$$\frac{i}{2}\Delta_m = W - H$$

where moreover;

(i) $2\,\mathrm{Im}(W) = \Delta_m$;

(ii) W is a distributional solution of the wave operator, i.e. $PW = 0$

(iii) W is <u>positive</u>, that is $\langle \bar{f}, W * f \rangle \geq 0 \quad \forall f \in \mathcal{D}(\mathbb{M}_d)$

where $\bar{f}$ is the complex conjugate of f and $W * f$ is the convolution of a distribution with a test function;

(iv) <u>WAVEFRONT SET</u> of W is

$$WF(W) = \Big\{ (x,\xi) \in T^*\mathbb{M}_d \;\Big|\; (x)^2_\eta = 0 \;,\; \underbrace{(\xi)^2_\eta}\; = 0 \text{ and}$$

$$x = \lambda\,\eta^\#(\xi) \text{ for some } \lambda \in \mathbb{R} \text{ such that } \xi^0 > 0 \Big\}$$

$$\| \quad x = \lambda\,\xi$$

<u>no ache here</u>

again here we're using that $T_x\mathbb{M}_d \cong T^*_x\mathbb{M}_d \cong \mathbb{M}_d$ so we can apply η to everything

[REMARK] The decomposition $\frac{i}{2}\Delta_m = W - H$ is not unique, depending on the choice of H. But in general the difference of two Hadamard parametrices is a smooth function $H - H' \in C^\infty(\mathbb{M}_d)$.

[DEF] For a given choice of two-point function W and of Hadamard parametrix H, we define the star product of two functionals $F, G \in \mathfrak{F}_{\mu c}$ as:

$$(F \star_H G)[\varphi] := \sum_{m=0}^{\infty} \frac{\hbar^m}{m!} \langle F^{(m)}[\varphi], (W^{\otimes m}) * G^{(m)}[\varphi] \rangle$$

$$= \sum_{m=0}^{\infty} \frac{\hbar^m}{m!} \int dx_1 \cdots dx_m \, dy_1 \cdots dy_m \frac{\delta^m F}{\delta\phi(x_1)\cdots\delta\phi(x_m)}[\varphi] \prod_{l=1}^{m} W(x_l - y_l) \frac{\delta^m G}{\delta\phi(y_1)\cdots\delta\phi(y_m)}[\varphi]$$

$$\in \mathfrak{F}_{\mu c}[[\hbar]]$$

[PROPOSITION]: The products obtained according to two different choices H, H' of Hadamard parametrices are equivalent. In other words, $\exists\ \alpha_{H-H'} : \mathfrak{F}_{\mu c}[[\hbar]] \longrightarrow \mathfrak{F}_{\mu c}[[\hbar]]$ isomorphism such that

$$F \star_{H'} G = \alpha_{H-H'}^{-1}\left(\alpha_{H-H'}(F) \star_H \alpha_{H-H'}(G)\right)$$

$$\left(\alpha_{H-H'} := e^{\frac{\hbar}{2} D_{H-H'}}, \text{ with } D_{H-H'} F[\varphi] = \langle H - H', \frac{\delta^2}{\delta\phi^2} F[\varphi] \rangle = \int_{\mathbb{M}_d^2} (H(x-y) - H'(x-y)) \frac{\delta^2 F}{\delta\phi(x)\delta\phi(y)}[\varphi]\, dx\, dy\right)$$

[DEF] or [PROPOSITION] The algebra $\left(\mathfrak{F}_{\mu c}[\![\hbar]\!], \star_H, [\cdot,\cdot]_{\star_H}, {}^* \right)$ is an associative, non commutative, Poisson $*$-algebra called the ALGEBRA OF FREE QUANTUM OBSERVABLES.

[QUANTUM INTERACTION PICTURE]

Inspired by the properties of the classical retarded product and in the spirit of constructing the quantum theory in such a way to preserve as much as possible of the classical structures, we DEFINE THE (PERTURBATIVE QUANTUM) RETARDED FUNCTIONALS AXIOMATICALLY

[DEF]: For any $L_{int} \in \mathcal{F}_{loc}$ and any $F \in \mathcal{F}_{\mu c}$, we want to construct the RETARDED FUNCTIONAL F_{ret} associated to F as:

$$F_{ret} := R\left(e_{\otimes}^{L_{int}} ; F\right) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!\,\hbar^n} R_n\left(L_{int}^{\otimes n} ; F\right) \in \mathcal{F}_{\mu c}[[\hbar, \lambda]]$$

in such a way that the following axioms are fulfilled:

BASIC AXIOMS

(a) INITIAL CONDITION: $R_0(F) = F$;

(b) LINEARITY: $R_n : \mathcal{F}_{loc}^{\otimes n} \otimes \mathcal{F}_{\mu c} \longrightarrow \mathcal{F}_{\mu c}[[\hbar, \lambda]]$ is linear;

(c) SYMMETRY: $R_n\left(F_{\pi(1)} \otimes \cdots \otimes F_{\pi(n)} ; F\right) = R_n\left(F_1 \otimes \cdots \otimes F_n ; F\right)$

$\forall F_1, \dots, F_n \in \mathcal{F}_{loc}$, $\forall F \in \mathcal{F}_{\mu c}$ and $\forall \pi \in S_n$;

(d) CAUSALITY: $R\left(e_{\otimes}^{L_{int} + \tilde{L}_{int}} ; F\right) = R\left(e_{\otimes}^{L_{int}} ; F\right)$ if

$$\left(\mathrm{supp}(F) + \overline{V}_-\right) \cap \mathrm{supp}(\tilde{L}_{int}) = \emptyset$$

(e) GLZ RELATION :

$$\frac{1}{i\hbar}\left[R(e_\otimes^{L_{int}};F),\, R(e_\otimes^{L_{int}};G)\right]_{\star_H} = R(e_\otimes^{L_{int}}\otimes F;\, G) - R(e_\otimes^{L_{int}}\otimes G;\, F)$$

RENORMALIZATION CONDITIONS

(f) FIELD INDEPENDENCE : $\forall F_1, \ldots, F_m \in \mathcal{F}_{loc}$ and $\forall F_{m+1} \in \mathcal{F}_{\mu c}$

$$\frac{\delta}{\delta\phi} R_m(F_1\otimes\cdots\otimes F_m;\, F_{m+1}) = \sum_{l=1}^{m+1} R_m\left(F_1\otimes\cdots\otimes \frac{\delta F_l}{\delta\phi}\otimes\cdots\otimes F_m;\, F_{m+1}\right)$$

(g) $*$-STRUCTURE : $R_m(F_1\otimes\cdots\otimes F_m;\, F)^* = R_m(F_1^*\otimes\cdots\otimes F_m^*;\, F^*)$

(h) OFF-SHELL FIELD EQUATION :

$$-(\square + m^2)\, R(e_\otimes^{L_{int}};\, \phi(f)) + R\left(e_\otimes^{L_{int}};\, \frac{\delta L_{int}}{\delta\phi}\right) = -(\square + m^2)\,\phi(f)$$

(i) FURTHER SYMMETRIES : (MASTER WARD IDENTITY).
(TRANSLATION INVARIANCE ALSO)

[REMARK]: Analogously to the situation in the classical theory, the GLZ relation implies that

$$\left[R(e_\otimes^{L_{int}};F),\, R(e_\otimes^{L_{int}};G)\right]_{\star_H} = 0 \quad \text{if} \quad (x-y)^2_\eta < 0$$
$$\forall\, (x,y)\in \mathrm{supp}(F)\times\mathrm{supp}(G).$$

[REMARK] Expanding the GLZ relation order by order, we get the following equivalent relations $\forall\, n \geq 1$:

$$R_n\left(G_1 \otimes \cdots \otimes G_{n-1} \otimes F ; H\right) - R_n\left(G_1 \otimes \cdots \otimes G_{n-1} \otimes H ; F\right) =$$

$$= \hbar\, J_{n-1,2}\left(G_1 \otimes \cdots \otimes G_{n-1} ; F \otimes H\right) \qquad \forall\, G_1, \dots, G_{n-1}, F, H \in \mathcal{F}_{loc}$$

where

$$J_{n-1,2}\left(G_1 \otimes \cdots \otimes G_{n-1} ; F \otimes H\right) :=$$

$$\frac{1}{i\hbar} \sum_{I \subseteq \{1, \dots, n-1\}} \left[R_{|I|}\left(G_I ; F\right), R_{|I^c|}\left(G_{I^c} ; H\right) \right]_{\star_H}$$

with $G_I := \bigotimes_{l \in I} G_l$ and $I^c := \{1, \dots, n-1\} \setminus I$

[EXAMPLE (3.1.2)] We try to determine from the preceding axioms the value of $R_1\left(\phi^4(x);\phi^2(y)\right)\underset{d=4}{=}$

[NOTATION] Instead of writing the local smeared functionals as usual as $\phi^4(f)$ and $\phi^2(g)$ for $f,g\in\mathcal{D}(\mathbb{M}_4)$, we regard ~~them~~ as $\phi^4(x)$ and $\phi^2(y)$ as FUNCTIONAL-VALUED DISTRIBUTIONS.
Hence also $R_1\left(\phi^4(x);\phi^2(y)\right)\in\mathcal{D}'(\mathbb{M}_4^2;\mathfrak{F}_{\mu c})$.

The initial condition gives $R_0\left(\phi^K(x)\right)=\phi^K(x)$, $K=2,4$.

The inductive step $R_0\longrightarrow R_1$ is obtained as follows. The GLZ relation yields the result:

$$R_1\left(\phi^4(x);\phi^2(y)\right)-R_1\left(\phi^2(y);\phi^4(x)\right)=\frac{1}{i}\left[R_0\left(\phi^4(x)\right),R_0\left(\phi^2(y)\right)\right]_{\star_H}$$

$$\hbar\,\mathfrak{I}_{0,2}\left(i\phi^4(x)\otimes\phi^2(y)\right)$$

$$=\frac{1}{i}\left[\phi^4(x),\phi^2(y)\right]_{\star_H}=$$

recalling that, by definition,
$W(x-y)-W(y-x)=$
$=i\Delta_m(x-y)$

$$\frac{1}{i}\,4\cdot2\,\hbar\,\phi^3(x)\left(W(x-y)-W(y-x)\right)\phi(y)\;+$$

$$+\frac{1}{i}\frac{\hbar^2}{2!}\,4\cdot3\,\phi^2(x)\left(W(x-y)^2-W(y-x)^2\right)\cdot2$$

$$=8\hbar\,\phi^3(x)\,\Delta_m(x-y)\,\phi(y)+\frac{12\hbar^2}{i}\,\phi^2(x)\left(W(x-y)^2-W(y-x)^2\right)$$

By the CAUSALITY AXIOM, we know that $\mathrm{supp}\left(R_1(\phi^4(x); \phi^2(y))\right) \subseteq$

$$\subseteq \{ (x,y) \in M_4^2 \mid x \in y + \overline{V}_- \}$$

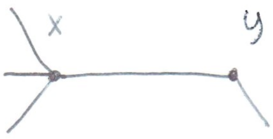
and viceversa $\mathrm{supp}\left(R_1(\phi^2(y); \phi^4(x))\right) \subseteq \{ (x,y) \mid y \in x + \overline{V}_- \}$

Then we could naively set

$$R_1(\phi^4(x); \phi^2(y)) := \hbar\, J_{0,2}(\phi^4(x) \otimes \phi^2(y))\, \vartheta(y^0 - x^0)$$

$$= -8\hbar\, \phi^3(x)\, \Delta_m^R(\cancel{x-y}\ y-x)\, \phi(y)\, \vartheta(y^0-x^0) + \frac{12\hbar^2}{i} \phi^2(x) \left(W(x-y)^2 - W(y-x)^2 \right) \vartheta(y^0-x^0)$$

These two terms are graphically represented by the "FEYNMAN DIAGRAMS"



and

where each variable x, y is assigned a vertex, each basic field represents a leg attached to the corresponding vertex, and each propagator is represented by an edge connecting two vertices.

- The first term is ok since $\Delta_m(x-y)\, \vartheta(y^0-x^0) = -\Delta_m^R(y-x)$
- The second term is defined only for $x - y \neq 0$, as a product of distributions. This is the term that needs RENORMALIZATION, that is, extension from $M_4^2 \setminus \Delta_2$ to M_4^2.

[CONSTRUCTION OF THE RETARDED PRODUCT]

To construct the general solution $(R_n)_{n \in \mathbb{N}}$ of the axioms ~~xx~~ one proceeds by induction on n. Axiom (a) gives the initial value $R_0(F) = F$. The inductive step is divided into 2 parts:

(1) first the construction of $R_m\left(A_1(x_1) \otimes \cdots \otimes A_m(x_m) \,;\, F(x_{m+1})\right)$ $\forall A_i \in \mathcal{F}_{loc}$, $i = 1, \dots, m$ and $\forall F \in \mathcal{F}_{\mu c}$ OFF THE THIN DIAGONAL $\Delta_{m+1} = \left\{ (x_1, \dots, x_{m+1}) \in \mathbb{M}_d^{m+1} \mid x_1 = \dots = x_{m+1} \right\}$

i.e. on $\mathbb{M}_d^{m+1} \setminus \Delta_{m+1}$. This part is uniquely determined by the basic axioms.

(2) then the extension to Δ_m, that is to a distribution in $\mathcal{D}'(\mathbb{M}_d^{m+1} ; \mathcal{F}_{\mu c})$. This part is non-unique, but it is strongly restricted by the renormalization conditions.

1) OFF THE THIN DIAGONAL:

Recall that the GLZ relation is equivalent at every order TB the following relation:

$$R_m(G_1\otimes\cdots\otimes G_{m-1}\otimes F;H)-R_m(G_1\otimes\cdots\otimes G_{m-1}\otimes H;F)=$$

$$=\hbar\, J_{m-1,2}(G_1\otimes\cdots\otimes G_{m-1};F\otimes H)$$

where

$$J_{m-1,2}(G_1\otimes\cdots\otimes G_{m-1};F\otimes H):=\frac{1}{i\hbar}\sum_{I\subseteq\{1,\dots,m-1\}}\left[R_{|I|}(G_I;F),\,R_{|I^c|}(G_{I^c};H)\right]_{\star_H}$$

with $G_I:=\bigotimes_{l\in I}G_l$ and $I^c=\{1,\dots,m-1\}\setminus I$.

[REMARK]: Observe that only retarded products of order at most $m-1$ appear in the expression for $J_{m-1,2}$

[PROPOSITION]: Assume the retarded products R_l, $l=0,\dots,m-1$, have been constructed such that they satisfy the basic axioms and the renormalization conditions. Then there exists an operation

$$\check{R}_m:\mathfrak{F}_{loc}^{\otimes m+1}\longrightarrow\mathcal{D}'(M_d^{m+1}\setminus\Delta_{m+1};\mathfrak{F}_{\mu c})$$

fulfilling all the axioms. Moreover

(i) $\check{R}_m$ is UNIQUELY DETERMINED on $\mathcal{D}(M_d^{m+1}\setminus\Delta_{m+1})$ by the basic axioms;

(ii) If an extension of the TOTALLY SYMMETRIC PART

$$\check{S}_m(A_1(x_1)\otimes\cdots\otimes A_m(x_m);A_{m+1}(x_{m+1}))=\frac{1}{(m+1)}\sum_{k=1}^{m+1}\check{R}_m(A_1(x_1)\cdots\widehat{A_k(x_k)}\cdots A_{m+1}(x_{m+1});A_k(x_k))$$

is continuous from $\mathcal{D}(\mathbb{M}_d^{m+1}\setminus\Delta_{m+1})$ to $\mathcal{D}(\mathbb{M}_d^{m+1})$ is constructed, the the extension R_m is completely determined.

[IDEA OF THE PROOF]: We introduce a particular gen cover of $\mathbb{M}_d^{m+1}\setminus\Delta_{m+1}$

Let
$$\Delta_{j=m+1} := \left\{ (x_1,\dots,x_{m+1})\in\mathbb{M}_d^{m+1} \;\middle|\; x_j = x_{m+1} \right\}$$

$\Delta^c_{j=m+1}$ is an open set and $\displaystyle\bigcup_{j=1}^{m} \Delta^c_{j=m+1} = \mathbb{M}_d^{m+1}\setminus\Delta_{m+1}$

Due to
$$\mathrm{supp}\Big(J_{m-1,2}\big(A_1(x_1)\otimes\cdots\otimes\widehat{A_j(x_j)}\otimes\cdots\otimes A_{m-1}(x_{m-1});\, A_j(x_j)\otimes A_{m+1}(x_{m+1})\big)$$
$$\subseteq \left\{ (x_1,\dots,x_{m+1})\in\mathbb{M}_d^{m+1} \;\middle|\; (x_j-x_{m+1})^2_\eta \geq 0 \right\}$$

and in view of the GLZ relation, we can set

$$\check{R}_m\big(A_1(x_1)\otimes\cdots\otimes A_j(x_j)\otimes\cdots\otimes A_m(x_m);\, A_{m+1}(x_{m+1})\big) :=$$
$$:= \mathrm{tr}\, J_{m-1,2}\big(A_1(x_1)\otimes\overset{\wedge}{\cdots}\otimes A_m(x_m);\, A_j(x_j)\otimes A_{m+1}(x_{m+1})\big)\,\theta(x^0_{m+1}-x^0_j)$$

on $\Delta^c_{j=m+1}$

REMARK: this distributional product is well-defined because the singularity of $\theta(x^0_{m+1}-x^0_j)$ are on $\Delta_{j=m+1}$

To express $\check{R}_m \left(A_1(x_1) \otimes \cdots \otimes A_m(x_m) ; A_{m+1}(x_{m+1}) \right)$ on $\mathbb{M}_d^{m+1} \setminus \Delta_{m+1}$

in one formula, we can introduce a PARTITION OF UNITY adapted to our special cover, that is:

$f_j \in C^\infty (\mathbb{M}_d^{m+1} \setminus \Delta_{m+1} , \mathbb{R})$ $\quad j = 1, \dots, m$, such that

$$1 = \sum_{j=1}^{m} f_j(x) \quad \forall x \in \mathbb{M}_d^{m+1} \setminus \Delta_{m+1} \quad \text{and} \quad \text{supp}(f_j) \subseteq \Delta^c_{j=m+1} .$$

With the partition of unity and the previous formula, we can write:

$$\check{R}_m \left(A_1(x_1) \otimes \cdots \otimes A_m(x_m) ; A_{m+1}(x_{m+1}) \right) :=$$

$$:= \text{tr} \sum_{j=1}^{m} f_j(x_1, \dots, x_{m+1}) \, J_{m-1, 2} \left(A_1(x_1) \otimes \overset{\wedge}{\cdots} \otimes A_m(x_m) ; A_j(x_j) \otimes A_{m+1}(x_{m+1}) \right) \theta(x^0_{m+1} - x^0_j)$$

on $\mathbb{M}_d^{m+1} \setminus \Delta_{m+1}$.

Now, the basic axioms and the renormalization conditions for $\check{R}_m$ follow from the inductive hypothesis on the retarded products of lower orders.

"q. e. d."

(2) EXTENSION TO THE THIN DIAGONAL

The extension of $\check{R}_m$ as a distribution from $\mathbb{M}_d^{m+1} \setminus \Delta_{m+1}$ to the whole $\mathbb{M}_d^{m+1}$ is the RENORMALIZATION PROBLEM

Recall that due to the last PROPOSITION, one only needs to extend the symmetric part $\check{S}_m$ of $\check{R}_m$

IMPORTANT REMARK: By construction, $\check{R}_m$ is a TRANSLATION INVARIANT DISTRIBUTION, that is: (on $\mathbb{M}_d^{m+1} \setminus \Delta_{m+1}$)

$$\check{R}_m \left(A_1(x_1) \otimes \cdots \otimes A_m(x_m) ; A_{m+1}(x_{m+1}) \right) =$$

$$= \sum_{k \geq m}^{\infty} \check{r}_{m,k} \left(x_1 - x_{m+1} , \ldots , x_m - x_{m+1} \right)$$

Translation invariant distributions on $\mathbb{M}_d^{m+1} \setminus \Delta_{m+1}$ can always be regarded as the pull-back of distributions defined on $\mathbb{M}_d^m \setminus 0$

This REMARK is to motivate the fact that the extension to the thin diagonal $\underbrace{}_{\Delta_{m+1}}$ of the unrenormalized retarded product $\check{R}_m$ can be done looking at its SCALING DEGREE.

[SCALING DEGREE OF DISTRIBUTIONS]

Heuristically speaking, it is a measure of the strength of the singularity of a distribution at the origin.

[DEF]: The (STEINMANN'S) SCALING DEGREE, with respect to the origin, of a distribution $t \in \mathcal{D}'(\mathbb{R}^d \setminus 0)$ is given by

$$sd(t) := \inf \left\{ r \in \mathbb{R} \mid \rho^r t(\rho x) = 0 \right\}$$

$$\rho^r t(\rho x) = \rho^r m_\rho^* t \qquad \text{where } m_\rho : \mathbb{R}^d \to \mathbb{R}^d,\ x \mapsto \rho x$$

By convention, we set

$$\inf \emptyset := \infty \quad \text{and} \quad \inf \mathbb{R} := -\infty .$$

[EXAMPLE]: Let δ be the Dirac delta distribution supported at the origin of $\mathbb{R}^d$ and $a \in \mathbb{N}^d$ a multiindex.

Recall that

$$\langle \partial^a \delta, f \rangle = \int_{\mathbb{R}^d} \partial^a \delta(x) f(x)\, dx = (-1)^{|a|} \partial^a f(0).$$

$\forall f \in \mathcal{D}(\mathbb{R}^d)$

Then

$$sd(\partial^a \delta) = d + |a| \quad \text{since} \quad \partial^a \delta(\rho x) = \rho^{-d-|a|} \partial^a \delta(x)$$

[REMARK]: $\forall t \in \mathcal{D}'(\mathbb{R}^d)$ it holds $sd(t) < \infty$, but for $t \in \mathcal{D}'(\mathbb{R}^d \setminus 0)$ the value $sd(t) = \infty$ is possible. An example in dimension 1 is $t(x) = \theta(x)\, e^{1/x}$ since, for a suitable $h \in \mathcal{D}(\mathbb{R} \setminus 0)$ $\lim_{\rho \to 0} \rho^r \int_0^\infty e^{1/(\rho x)} h(x)\, dx$ diverges $\forall r \in \mathbb{R}$

[PROPOSITION] (BASIC PROPERTIES OF THE SCALING DEGREE)

"Morally, differentiation increases the strength of singularity and multiplication with x^a decreases the strength of the singularity"

$$\left.\begin{array}{ll} \text{(i)}\ \forall a \in \mathbb{N}^d, & sd(\partial^a t) \leq sd(t) + |a| ; \\ \text{(ii)}\ \forall a \in \mathbb{N}^d, & sd(x^a t) \leq sd(t) - |a| ; \\ \text{(iii)} & sd(t_1 \otimes t_2) = sd(t_1) + sd(t_2) . \end{array}\right\} \quad \forall t, t_1, t_2 \in \mathcal{D}'(\mathbb{R}^d \setminus 0)$$

[DEF] Given $t \in \mathcal{D}'(\mathbb{R}^d \setminus 0)$, an EXTENSION $\bar{t} \in \mathcal{D}'(\mathbb{R}^d)$ of t is a distribution defined on $\mathbb{R}^d$ such that

$$\bar{t}(f) = t(f) \qquad \forall f \in \mathcal{D}(\mathbb{R}^d \setminus 0) \text{, that is}$$

$$\boxed{\bar{t}\,\big|_{\mathbb{R}^d \setminus 0} = t}$$

[REMARK]: From the definition of scaling degree it follows immediately that $sd(\bar{t}) \geq sd(t)$ for any extension $\bar{t}$ of t.
We are looking for extensions $\bar{t}$ which do not increase the scaling degree.

[THEOREM] Let $t \in \mathcal{D}'(\mathbb{R}^d \setminus 0)$. Then:

(a) if $sd(t) < d$, $\exists!$ extension $\bar{t} \in \mathcal{D}'(\mathbb{R}^d)$ such that

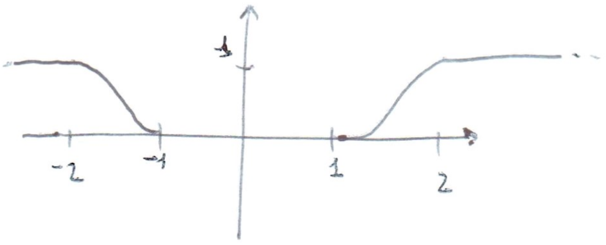
$$sd(\bar{t}) = sd(t) ;$$

(b) if $d \leqslant sd(t) < \infty$, there are several extensions $\bar{t} \in \mathcal{D}'(\mathbb{R}^d)$ satisfying the condition $sd(\bar{t}) = sd(t)$. Given a particular extension $\bar{t}$, the general solution is of the form

$$\bar{t}' = \bar{t} + \underbrace{\sum_{|a| \leqslant sd(t) - d} C_a \, \partial^a \delta}, \text{ with } C_a \in \mathbb{C}.$$

[REMARK]: In the physics jargon, a term $\uparrow$ is called a FINITE RENORMALIZATION.

[OUTLINE OF THE PROOF] (a) Let $\chi \in C^\infty(\mathbb{R}^d)$ be such that $0 \leqslant \chi(x) \leqslant 1$, $\chi(x) = 0$ for $|x| \leqslant 1$ and $\chi(x) = 1$ for $|x| \geqslant 2$



If $h \in \mathcal{D}(\mathbb{R}^d)$, then

$$\chi_\rho h : x \mapsto \chi(\rho x)\, h(x) \in \mathcal{D}(\mathbb{R}^d \setminus 0)$$

for any $\rho > 0$, hence $\langle t, \chi_\rho h \rangle$ is well-defined. Moreover, for $h_1 \in \mathcal{D}(\mathbb{R}^d \setminus 0)$ it holds that $\chi_\rho h_1 = h_1$ for ρ sufficiently large. Therefore, assuming that the limit

$$\langle \bar{t}, h \rangle := \lim_{\rho \to \infty} \langle t, \chi_\rho h \rangle \quad \forall h \in \mathcal{D}(\mathbb{R}^d) \quad \text{exists}$$

and defines a distribution $\bar{t} \in \mathcal{D}'(\mathbb{R}^d)$, this $\bar{t}$ is an extension of t.

This can be proved and moreover one can show that $sd(\bar{t}) = sd(t)$. $\bar{t}$ is called the DIRECT EXTENSION of t.
Independence on χ follows from the uniqueness.
(any other extension should differ by a δ, but there is no δ with $sd < d$)

(b) For ~~uniqueness~~ the general solution: two extensions $\bar{t}, \bar{t}'$ differ by

$$\bar{t}' - \bar{t} = \sum_a C_a \partial^a \delta$$

since

$$sd(\bar{t}' - \bar{t}) = \max\{sd(\bar{t}'), sd(\bar{t})\} = sd(t) \geq d$$

we have $|a| \leq sd(t) - d$ is admissible.

Existence: Let $\omega := sd(t) - d$ be the singular order of t. ~~Note~~ Note that ω doesn't need to be integer, let $[\omega]$ be its integer part. Introduce the space of test functions

$$\mathcal{D}_\omega(\mathbb{R}^d) := \left\{ h \in \mathcal{D}(\mathbb{R}^d) \mid \partial^a h = 0 \quad \forall |a| \leq [\omega] \right\}$$

Clearly $\mathcal{D}(\mathbb{R}^d \setminus 0) \subset \mathcal{D}_\omega(\mathbb{R}^d)$

CLAIM: t has a unique extension $\bar{t}_w$ to $\mathcal{D}_w'(\mathbb{R}^d)$

satisfying $sd(\bar{t}_w) = sd(t)$.

Each PROJECTOR $W: \mathcal{D}(\mathbb{R}^d) \longrightarrow \mathcal{D}_w(\mathbb{R}^d)$, that is, a LINEAR, CONTINUOUS, IDEMPOTENT map with range $\mathcal{D}_w$, defines an extension $t^W \in \mathcal{D}'(\mathbb{R}^d)$ by

$$\langle t^W, h \rangle := \langle t_w, Wh \rangle \qquad \forall h \in \mathcal{D}(\mathbb{R}^d).$$

Since $Wh = h$ for $h \in \mathcal{D}(\mathbb{R}^d \setminus 0)$, the relations

$$\langle t^W, h \rangle = \langle t_w, Wh \rangle = \langle t_w, h \rangle = \langle t, h \rangle$$

show that t^W is indeed an extension of t. It can be shown moreover that for any such W, $sd(t^W) = sd(t)$

Such a projector W can be defined for any set of functions $w_a \in \mathcal{D}(\mathbb{R}^d)$

$a \in \mathbb{N}^d$, $|a| \leq [w]$, satisfying

$$\partial^b w_a(0) = \delta_a^b \qquad \forall b \in \mathbb{N}^d, \ |b| \leq [w],$$

KRONECKER DELTA

By setting

$$Wh(x) := h(x) - \sum_{|a| \leq [w]} \partial^a h(0) \, w_a(x)$$

For example, one can ~~choose~~ choose

$$w_a(x) := \frac{x^a}{a!} w(x), \qquad \forall a \in \mathbb{N}^d, \ |a| \leq [w]$$

where $w \in \mathcal{D}(\mathbb{R}^d)$ satisfies $w(0) = 1$ and $\partial^b w(0) = 0$

$\forall 1 \leq |b| \leq [w]$.

"q.e.d."

[COUNTING THE RENORMALIZATION PARAMETERS]

[GOAL]: Investigate the number of indetermined parameters in the inductive step

$$R_{n-1}(L_{int}^{\otimes n-1}; F) \longrightarrow R_n(L_{int}^{\otimes n}; F)$$

as a function of n, where L_{int} is the interaction Lagrangian of the theory under consideration ($L_{int} \in \mathcal{F}_{loc}$) and $F \in \mathcal{F}_{\mu c}$.

[MOTIVATION]: This function depends on L_{int}. If it is UNBOUNDED it means that one ~~needs~~ an infinite amount of information to fix the model uniquely to all orders in the perturbation series.

In particular, if the number of indeterminate parameters having observable consequences is unbounded in n, one would need INFINITELY MANY EXPERIMENTS to determine the observable predictions of the model completely.

For example, this is the case for GRAVITATIONAL THEORIES.

[REMARK] Of course the number of indetermined parameters also depends on the renormalization conditions adopted.

We assume only: TRANSLATION INVARIANCE AND FIELD INDEPENDENCE

[PROPOSITION] (RENORMALIZABILITY BY POWER COUNTING) Let $L_{int}, F \in \mathcal{F}_{pol}$ be monomials and let

$$N(L_{int}, F, \cdot) : \mathbb{N} \longrightarrow \mathbb{N}$$
$$n \longmapsto N(L_{int}, F, n)$$

be the function that counts the number of indetermined parameters in $R_n(L_{int}^{\otimes n}; F)$ coming from the inductive step $R_{n-1} \longrightarrow R_n$. The

(a) $N(L_{int}, F, \cdot)$ is BOUNDED if and only if $\dim(L_{int}) \leq d$;

(b) The number of non-vanishing values of $N(L_{int}, F, \cdot)$ is FINITE if and only if $\dim(L_{int}) < d$.

Where the DIMENSION of L_{int} is defined as follows:

[DEF] An interaction L_{int} fulfilling (a) is called RENORMALIZABLE BY POWER COUNTING.

EXAMPLE: $L_{int} = \cancel{\phi(f)}\ \phi^4(f)$ in dimension $d = 4$.

[DEF]: If $\dim(L_{int}) < d$, only the lower orders of the perturbation series require "nontrivial" ~~ext~~ renormalization, all the higher orders can be renormalized by direct extension, that is, they are unique. Such an L_{int} is called SUPER-RENORMALIZABLE

THIS CAN BE PUT AS A REMARK AFTER [DEF]