On the physics of degenerate dynamical systems

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We start with a first order dynamical system:

$$I[z;1,2] = \int_{t_1}^{t_2} [A_i(z)\dot{z}^i + A_0(z)]dt, \quad \text{with } i = 1, 2, \cdots 2n,$$

From which: $F_{ij}\dot{z}^j + E_i = 0$

$$F_{ij} \equiv \partial_i A_j - \partial_j A_i, \quad \text{and} \quad E_i \equiv \partial_i A_0.$$

Usual dynamical systems:

$$F_{ij}\dot{z}^j + E_i = 0 \longrightarrow \text{rank } \rho(F_{ij}) \text{ is constant}$$

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maximal rank

$$\rho(F_{ij}) = 2n$$

nonmaximal rank

$$\rho(F_{ij}) = 2m < 2n$$

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There exists degenerate surfaces

$$\Sigma = \{z \in \Gamma \,|\, \Delta = 0\} \qquad \Delta(z) = \det[F_{ij}(z)]$$

J. Saavedra, R. Troncoso and J. Zanelli,
J. Math. Phys. 42 (2001) 4383

We can block diagonalize the matrix

$$F_{ij} = \begin{bmatrix} 0 & f_1 & & & \\ -f_1 & 0 & & & \\ & & 0 & f_2 & \\ & & -f_2 & 0 & \\ & & & & \ddots \end{bmatrix}$$

$$\Delta(z) = [f_1(z) f_2(z) \cdots f_n(z)]^2 > 0$$

Then, as it is a closed form $dF = 0$,

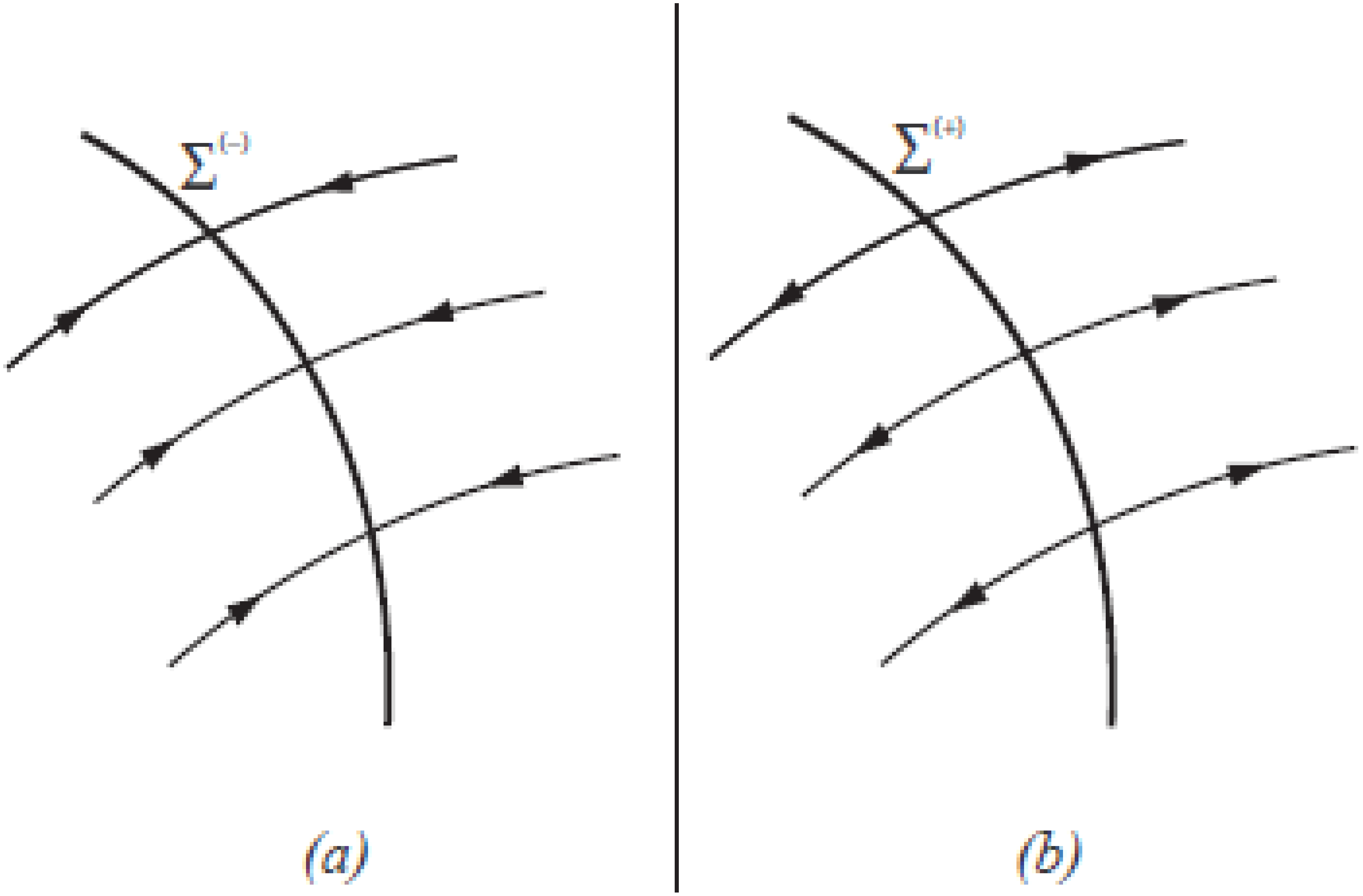
$$F = \sum_{r=1}^{n} f_r(z^{2r-1}, z^{2r})\, dz^{2r-1} \wedge dz^{2r}$$

Therefore, we can decouple our equations

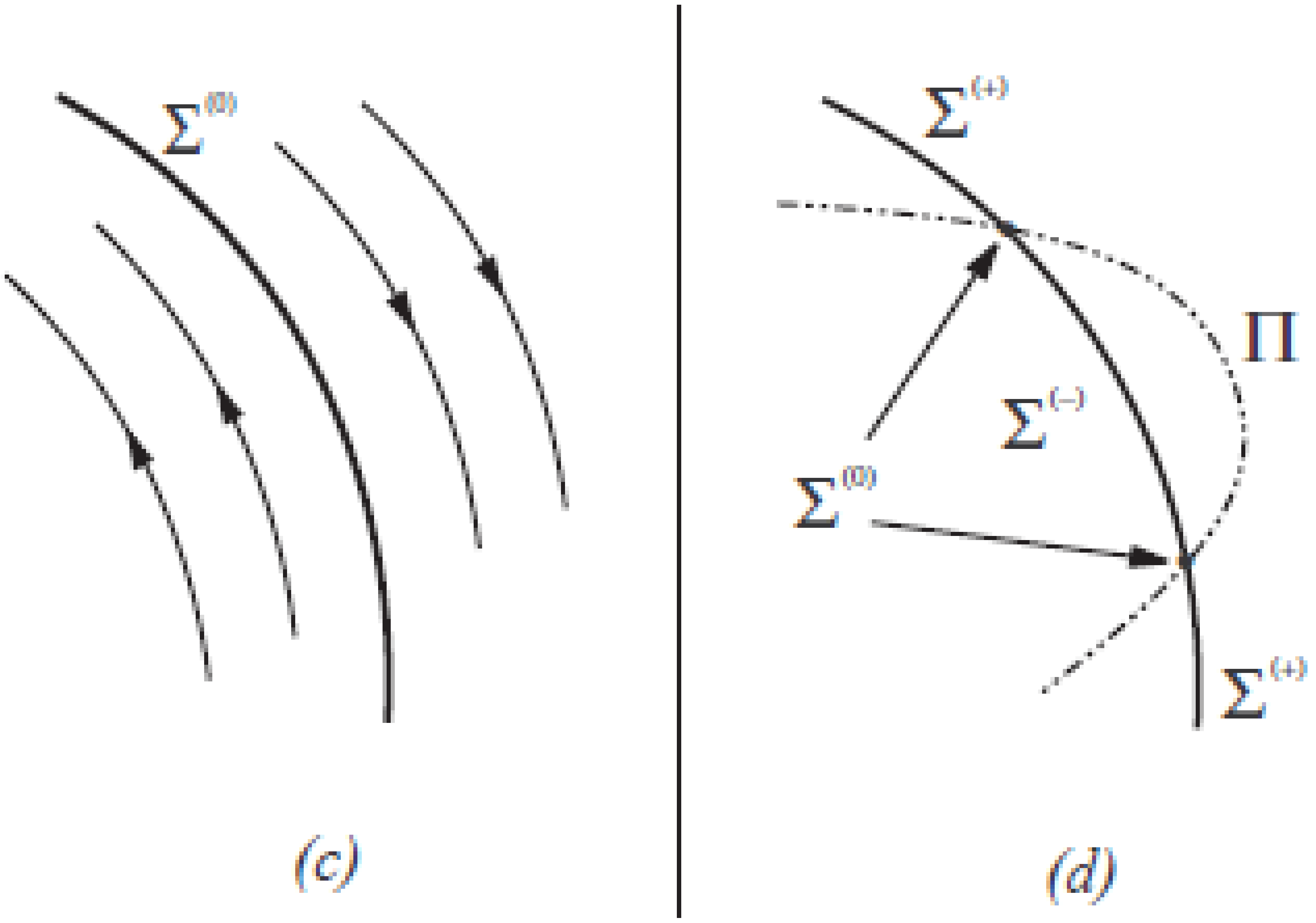
$$\begin{aligned} f(z^1, z^2)\, \dot{z}^1 &= -\partial_2 A_0(z^1, z^2; z^a) \\ -f(z^1, z^2)\, \dot{z}^2 &= -\partial_1 A_0(z^1, z^2; z^a)\,. \end{aligned}$$

Moreover, the character of the degeneracy surface is given by the flux

$$\Phi = j^i n_i = -F^{1/2} F^{ij} E_j \partial_i F^{1/2}$$



$\Sigma^{(-)}$
$\Sigma^{(+)}$
(a)
(b)


$\Sigma^{(0)}$
(c)
$\Sigma^{(+)}$
Π
$\Sigma^{(-)}$
$\Sigma^{(0)}$
$\Sigma^{(+)}$
(d)

Possible in a plethora of interesting physical systems

$\longrightarrow$ Extentions of GR (Lovelock, Horndeski...)

$\longrightarrow$ Nonlinear Electrodynamics

$\longrightarrow$ K-essence scalar fields

$\longrightarrow$ Classical time crystals

Lagrangians with non-canonical kinetic terms

Hamiltonians multi-valued in the momenta

Singularities in the dynamical equations

$$\ddot{q}^j \frac{\partial p_i}{\partial \dot{q}^j} + \dot{q}^j \frac{\partial p_i}{\partial q^j} - \frac{\partial L}{\partial q^i} = 0,$$

Lagrangians with non-canonical kinetic terms

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Singularities in the dynamical equations

$$\ddot{q}^j \frac{\partial p_i}{\partial \dot{q}^j} + \dot{q}^j \frac{\partial p_i}{\partial q^j} - \frac{\partial L}{\partial q^i} = 0,$$

The simplest system: $L = \frac{\beta}{4}\dot{\phi}^4 - \frac{\kappa}{2}\dot{\phi}^2 - V(\phi),$

$$p_\phi = \beta\dot{\phi}^3 - \kappa\dot{\phi},$$

$$(3\beta\dot{\phi}^2 - \kappa)\ddot{\phi} = -V'(\phi),$$

M. Henneaux, C. Teitelboim, and J. Zanelli Phys. Rev. **36**, 4417 (1987)

A. Shapere and F. Wilczek, Phys. Rev. Lett. **109**, 160402 (2012)

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$$p_\phi = \beta\dot{\phi}^3 - \kappa\dot{\phi},$$

$$\underbrace{(3\beta\dot{\phi}^2 - \kappa)}_{=0}\ddot{\phi} = -V'(\phi),$$

$$\rightarrow \quad \dot{\phi}^2 = \kappa/3\beta,$$

M. Henneaux, C. Teitelboim, and J. Zanelli Phys. Rev. **36**, 4417 (1987)

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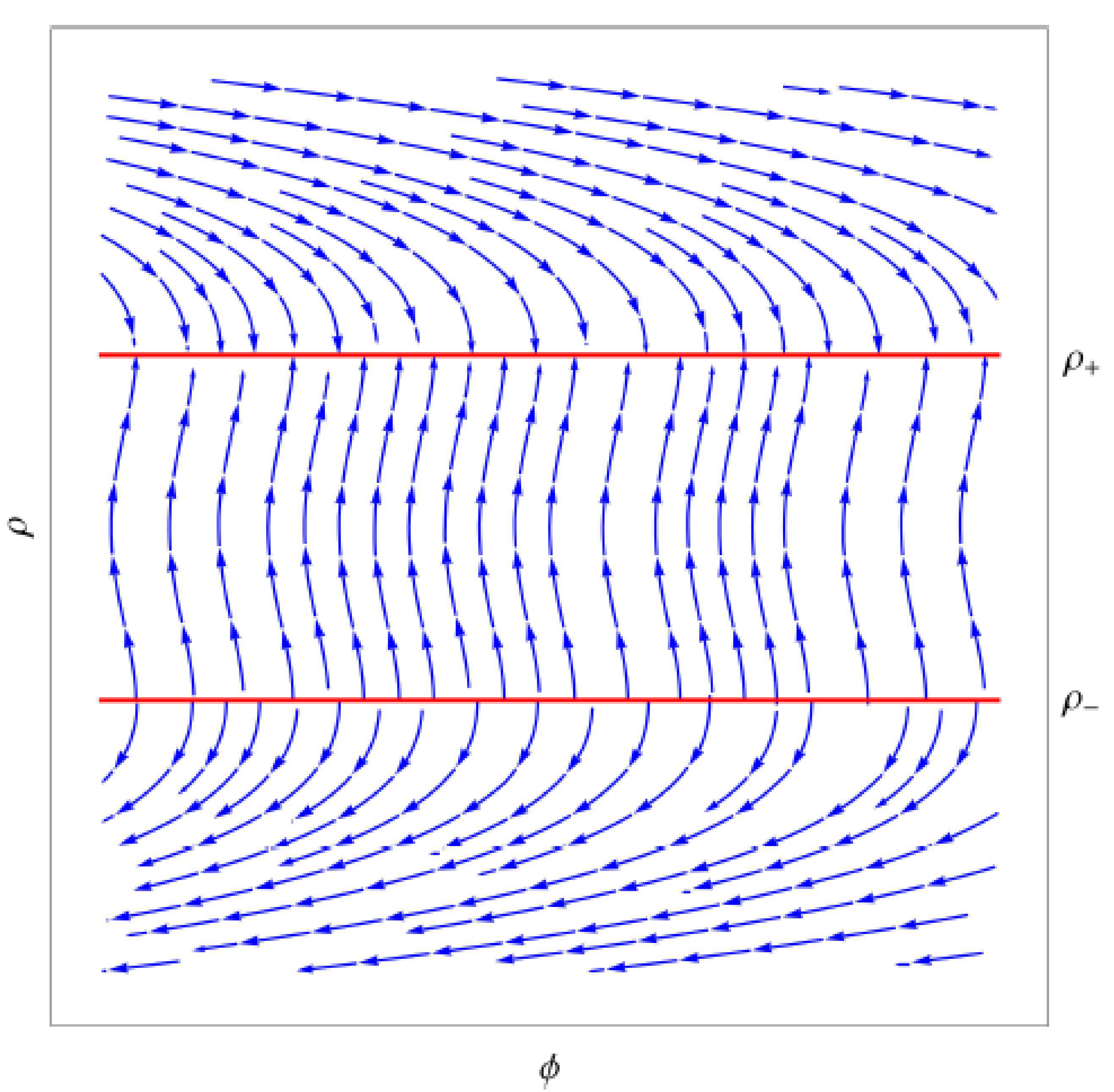
In the first order formalism

$$L = (\beta\rho^3 - \kappa\rho)\dot{\phi} - H = (\beta\rho^3 - \kappa\rho)\dot{\phi} - \frac{3\beta}{4}\rho^4 + \kappa\rho^2 - V(\phi),$$

$$\Delta\dot{\rho} = -V'(\phi), \quad \Delta(\dot{\phi} - \rho) = 0,$$

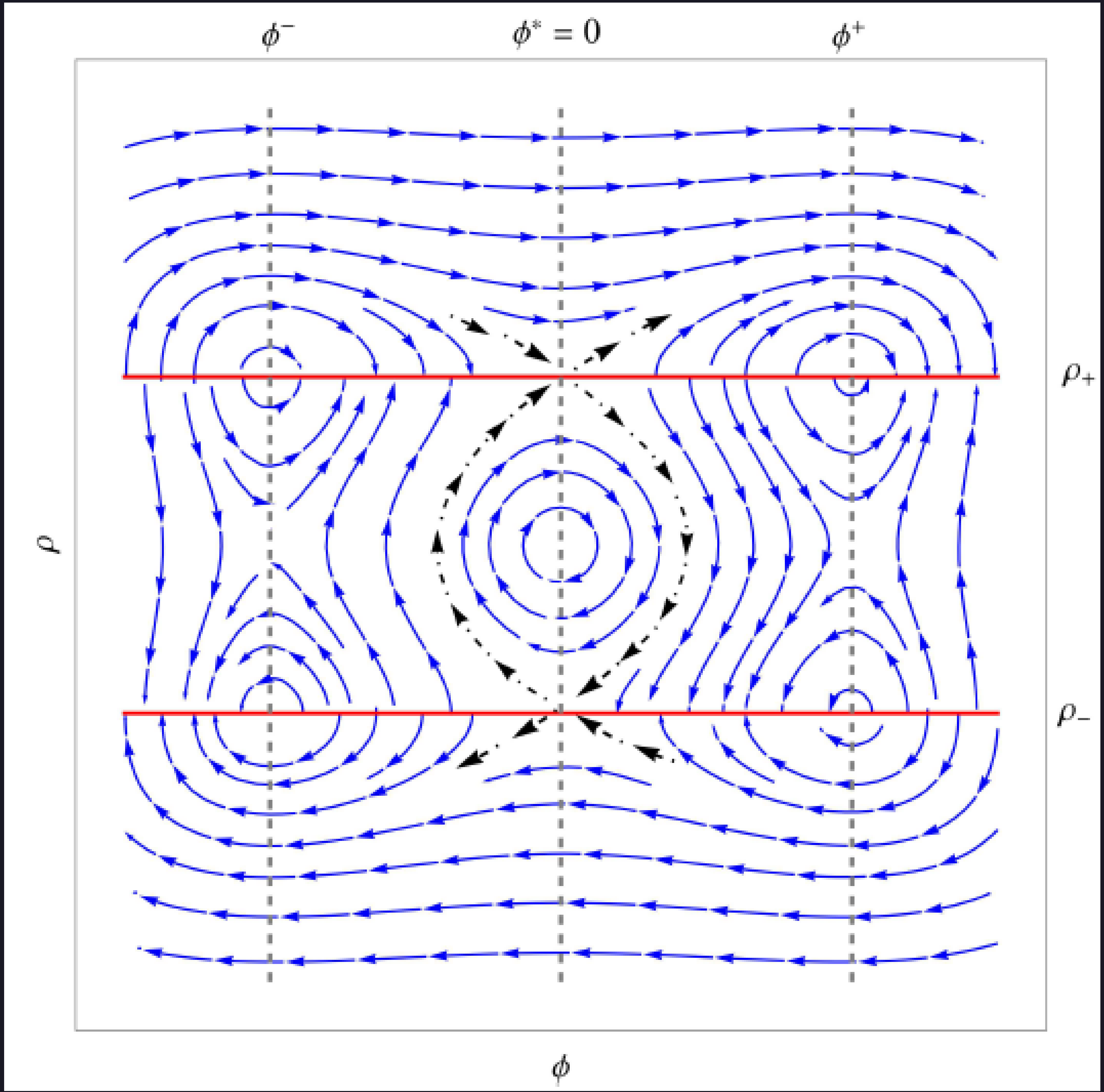
$$\Delta = (3\beta\rho^2 - \kappa).$$

$$\rightarrow \Phi = -6\beta\rho V'(\phi_d),$$



$$\rho_\pm := \pm\sqrt{\kappa/3\beta}.$$

$$V'(\phi_d) > 0$$



$$\rho_\pm := \pm\sqrt{\kappa/3\beta}.$$

$$V(\phi) = \frac{\lambda}{4}\phi^4 - \frac{\omega}{2}\phi^2,$$

Dynamical dimensional reduction in multivalued Hamiltonians

Alexsandre L. Ferreira Junior, Nelson Pinto-Neto, and Jorge Zanelli
Phys. Rev. D **105**, 084064 – Published 29 April 2022

arXiv:2203.07099v2 [hep-th]

Inflation and late-time accelerated expansion driven by k-essence degenerate dynamics

Alexsandre L. Ferreira, Jr., Nelson Pinto-Neto, and Jorge Zanelli
Phys. Rev. D **109**, 023515 – Published 12 January 2024

arXiv:2311.01456v2 [hep-th]

Investigate the quantisation of the toy model:

$$\begin{aligned} L &= xy\dot{x} - \nu y \,, \\ H &= \nu y = -A_0 \,, \end{aligned}$$

F. de Micheli and J. Zanelli,
J. Math. Phys. **53** (2012), 102112.

To quantise the system we define de Dirac bracket

$$\{x, y\}^* = \frac{1}{f(x, y)}$$

Through which we have the commutation

$$[\hat{x}, \hat{y}] = i\hbar \frac{1}{x}$$

So that we define the operators

$$\hat{x} \quad : \quad = x$$

$$\hat{y} \quad : \quad = -i\hbar\frac{1}{x}\partial_x$$

$$\hat{H} = \nu\hat{y} = -i\hbar\nu\frac{1}{x}\partial_x$$

Moreover, we define the weight $w(x)$

$$\|\varphi(x)\| = \left(\int |\varphi(x)|^2 w(x)\, \mathrm{d}x\right)^{\frac{1}{2}}$$

$$\int \varphi_1^*(x) \left[\hat{H}\varphi_2(x)\right] w(x)dx = \int \left[\hat{H}\varphi_1(x)\right]^* \varphi_2(x) w(x) dx$$

A natural choice

$$w(x) = |F_{ij}| = |x|$$

$$< \varphi_1, \varphi_2 >= \int \varphi_1^* |x| \varphi_2 \, \mathrm{d}x$$

$$-i\hbar\nu \frac{1}{x} \frac{\partial}{\partial x} \Psi(x,t) = i\hbar \frac{\partial}{\partial t} \Psi(x,t)$$

First, let $x \in (0, a)$

$$\Psi_{\mathrm{E}}(x,t) = \frac{\sqrt{2}}{a} \exp\left[\frac{iE}{2\hbar\nu}(x^2 - 2\nu t)\right]$$

$$< \hat{H}\psi, \phi >=< \psi, \hat{H}\phi > \quad \forall \psi, \phi \in \mathcal{D}(\hat{H})$$

$$\psi^*(a)\phi(a) - \psi^*(0^+)\phi(0^+) = 0$$

$$\mathcal{D}_{(0,a)}(\hat{H}) = \{L^2((0,a), |x|dx) : \psi(a) = e^{i\theta}\psi(0^+) \neq 0\}.$$

The boundary conditions gives us a discrete energy

$$E_n := \frac{2\nu\hbar}{a^2}(2n\pi + \theta) = \frac{4\pi\nu\hbar n}{a^2} + \frac{2\nu\hbar}{a^2}\theta,$$

$$E_n - E_m := \Delta E = \frac{4\pi\nu\hbar}{a^2}(n - m)$$

Now, if $x \in (a^-, a^+)$

$$\psi_E(x) = \sqrt{\frac{2}{(a^-)^2 + (a^+)^2}} \exp\left[i \frac{E}{2\hbar\nu} x^2\right]$$

With the boundary condition

$$\psi^*(a^-)\phi(a^-) - \psi^*(0^-)\phi(0^-) - \psi^*(0^+)\phi(0^+) + \psi^*(a^+)\phi(a^+) = 0$$

As the Hamiltonian in singular at the origin: $(a^-, 0) \cup (0, a^+)$

So that
$$\psi^*(a^-)\phi(a^-) = \psi^*(0^-)\phi(0^-)$$
$$\psi^*(a^+)\phi(a^+) = \psi^*(0^+)\phi(0^+)$$

The Hilbert is space is then $\mathcal{H} = \mathcal{H}_- \bigoplus \mathcal{H}_+$

$$\mathcal{H}_- = \{\phi(x) \in L^2((a^-, 0), |x|dx) : \phi(a^-) = e^{i\theta^-}\phi(0^-)\}$$
$$\mathcal{H}_+ = \{\phi(x) \in L^2((0, a^+), |x|dx) : \phi(a^+) = e^{i\theta^+}\phi(0^+)\}$$

So that $$\psi = \begin{pmatrix} \psi^+(x) \\ \psi^-(x) \end{pmatrix} = \psi^+(x) \oplus \psi^-(x)$$

$$\psi^-(x) = \begin{cases} \frac{\sqrt{2}}{a^-} \exp\left[\frac{i}{2\hbar\nu} E_n^- x^2\right], & a^- < x < 0 \\ 0, & 0 < x < a^+ \end{cases}$$

$$\psi^+(x) = \begin{cases} 0, & a^- < x < 0 \\ \frac{\sqrt{2}}{a^+} \exp\left[\frac{i}{2\hbar\nu} E_n^+ x^2\right], & 0 < x < a^+ \end{cases}$$

The energy are the same $\hat{H}\psi^{\pm} = E^{\pm}\psi^{\pm}$

$$E_n^{\pm} = (2n\pi + \theta^{\pm})\frac{2\hbar\nu}{(a^{\pm})^2}.$$

If on try to match them: $E_n^{+} = E_m^{-}$

$$n = \left(\frac{a^{+}}{a^{-}}\right)^2 m + \kappa$$

The full time dependent solution

$$\Psi(x,t) = \begin{cases} \sum c_m^- \frac{\sqrt{2}}{a^-} \exp\left[2\pi i m (x^2 - 2\nu t)(a^-)^{-2}\right], & x \in (a^-, 0) \\ \sum c_n^+ \frac{\sqrt{2}}{a^+} \exp\left[2\pi i n (x^2 - 2\nu t)(a^+)^{-2}\right], & x \in (0, a^+) \end{cases}$$

$$\rho(x,t) = |\Psi(x,t)|^2 |x|$$

The degeneracy acts as a sink or a source of probability:

$$\partial_t \rho + \partial_x J = \sigma$$

$$\begin{aligned} J &= \nu \, \mathrm{sgn}(x) \, |\Psi(x,t)|^2 \\ \sigma &= 2\nu \, \delta(x) \, |\Psi(x,t)|^2 . \end{aligned}$$

At last

We gave a new, consistent, and interesting interpretation to systems with multi-valued Hamiltonians.

Investigate the quantum mechanics of the degeneracy.

Investigate systems with infinite degrees of freedom: local degeneracies, caustics...

Many possible degenerate systems: Non-linear electrodynamics, Bose-Einstein condensates, alternative gravities...

Muito obrigado

Thank you

谢谢