Transversal index theorem for foliated filtered manifolds

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Chengdu, Algebraic, analytic and geometric structures emerging from QFT

Plan for the talk :

- I - Usual theory of transversal index
- II - Calculus on filtered manifolds
- III - Foliated filtered manifolds and transversal index

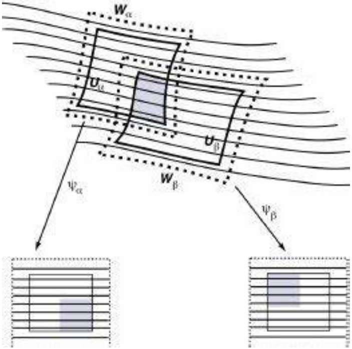
Origin : Atiyah ('71). Action of a compact Lie group and invariant differential operators (e.g. equivariant Dirac operator). The ellipticity condition can only be checked on

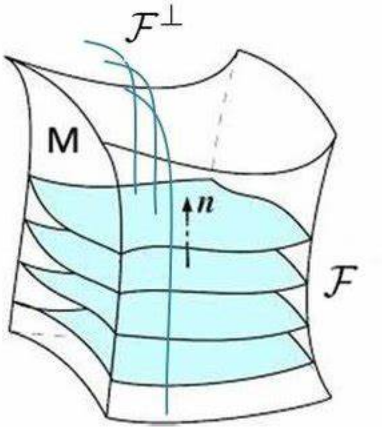
$$T_G^*M := \{(x,\ell), \forall \xi \in \mathfrak{g}, \ell(X_\xi(x)) = 0\}$$

where $\xi \to X_\xi$ denotes the associated infinitesimal action of the Lie algebra $\mathfrak{g}$.

Connes ('85) Replace group actions by (regular) foliations. Index in $K^0(C^*(\mathrm{Hol}(F))$ (K-homology group). Various extensions :

- Hilsum, Skandalis ('87) wrong way maps between two foliations
- Connes, Moscovici ('98) computation of a Chern character in cyclic homology
- Baldare, Benameur ('20) group acting on a foliation
- Kasparov ('16), Hochs, Wang ('20) extension of Atiyah's ideas to KK-theory





Images : S. Hurder ; E. Babalic

(M, F) foliated manifold, M/F often has a bad topology.

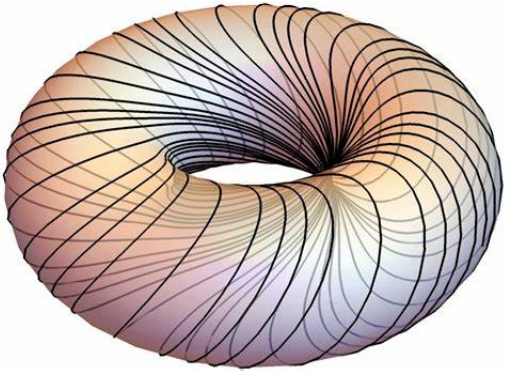


Image : D. Cannarsa

Definition

The holonomy groupoid of (M, F) is $\mathrm{Hol}(F) \rightrightarrows M$. Two elements are composable if they are on the same leaf and arrows between x and y are germs of leafwise diffeomorphisms on transversal manifolds around x and y

$\mathrm{Hol}(F)$ is a Lie groupoid.
Convolution algebra $\mathcal{C}_c^\infty(\mathrm{Hol}(F))$, (maximal) C^*-algebraic completion $C^*(\mathrm{Hol}(F))$ instead of $\mathcal{C}_c^\infty({}^{M}\!/_{F})$ and $\mathcal{C}_0({}^{M}\!/_{F})$.

Exemple

If F is given by a fibration $M \to B$ (with connected fibers) then $C^*(\mathrm{Hol}(F))$ is Morita equivalent to $\mathcal{C}_0(B)$

$\mathrm{Hol}(F)$ acts on the normal bundle ${}^{TM}\!/_{F}$. Identify $\left({}^{TM}\!/_{F}\right)^* = F^\perp$:

Definition (Connes)

Let $P \in \Psi^m(M)$ a pseudodifferential operator, its transverse principal symbol $\sigma^\perp(P)$ is the restriction of its principal symbol to $F^\perp$.
P is transversally elliptic if $\forall \xi \in F^\perp \setminus 0, \sigma^\perp(P)(x,\xi) \neq 0$ and $\sigma^\perp(P)$ is $\mathrm{Hol}(F)$-invariant.

The invariance means that if $(x,y,\gamma) \in \mathrm{Hol}(F)$ then $\sigma^\perp(y,\xi) = \sigma^\perp(x, \mathrm{d}\gamma^T \xi)$.

Theorem (Connes ('85))

Let $P\colon \Gamma(M, E_+) \to \Gamma(M, E_-)$ *be an order* 0 *pseudodifferential operator. Let* $Q\colon \Gamma(M, E_-) \to \Gamma(M, E_+)$ *be such that* $\sigma^\perp(Q) = \sigma^\perp(P)^{-1}$ *then*

$$\left(L^2(M, E_+ \oplus E_-), \begin{pmatrix} 0 & Q \\ P & 0 \end{pmatrix}\right)$$

defines a K-homology class of $C^*(\mathrm{Hol}(F))$ *that only depends on the homotopy class of* $\sigma^\perp(P)$ *among transversally elliptic symbols.*

Definition

A filtered manifold is a manifold M with the data of subbundles $H^1 \subset \cdots H^r = TM$ such that

$$\forall i, j \left[\Gamma(H^i), \Gamma(H^j)\right] \subset \Gamma(H^{i+j})$$

Exemple

Contact manifolds, CR manifolds...
$\Delta = X^2 + Y^2 + f\partial_z$ on $\mathbb{R}^3$ with $X, Y \in \mathfrak{X}(\mathbb{R}^3), [X, Y] = \partial_z$.

Pseudodifferential calculus developped to have $X \in \Gamma(H^i)$ of order i. Gives a noncommutative symbolic calculus.

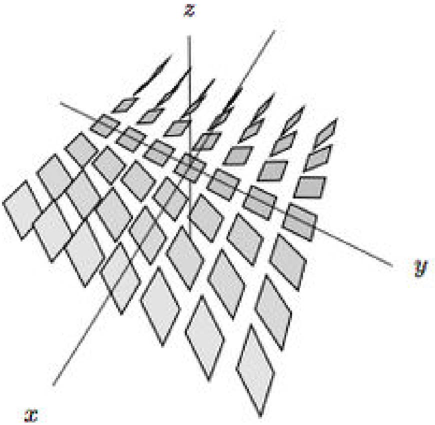


Image : Manifold Atlas Project

$$\mathfrak{t}_H M = H^1 \oplus {}^{H^2}\!/_{H^1} \oplus \cdots {}^{TM}\!/_{H^{r-1}}$$

The Lie bracket of vector fields induces a Lie bracket on $\mathfrak{t}_H M$ which restricts to the fiber. $\mathfrak{t}_H M$ is a family of nilpotent Lie algebras. $T_H M$ the corresponding family of Lie groups. Inhomogeneous dilations $(\delta_\lambda)_{\lambda>0}$ on $T_H M$ given on $\mathfrak{t}_H M$ via

$$\delta_\lambda(x^1, \cdots, x^r) = (\lambda x^1, \cdots, \lambda^r x^r).$$

Definition

A symbol of order $m \in \mathbb{Z}$ in $T_H M$ is a distribution $u \in \mathcal{D}'(T_H M)$ satisfying

- u is properly supported i.e. $\pi\colon \operatorname{supp}(u) \to M$ is a proper map
- u is transversal to π
- $\forall \lambda \in \mathbb{R}_+^*, \delta_{\lambda *} u - \lambda^m u \in \mathcal{C}_p^\infty(G)$

$S^m(G)$ denotes the set of these distributions and $S^*(G) = \bigcup_{m \in \mathbb{Z}} S^m(G)$.

Same symbolic calculus property as in the usual case with $H^1 = TM$ (product, continuity, compacity...)

Definition

A symbol $\sigma \in S^m(T_H M)$ is Rockland at $x \in M$ if for every $\pi \in \widehat{T_{H,x}M} \setminus \{triv\}$, $\pi(\sigma)$ is injective on $\mathcal{H}_\pi^\infty$. σ is Rockland if it is Rockland at every point $x \in M$.

Remark

In the non-filtered case Rockland condition is the usual ellipticity.

Theorem (Rockland (...))

$\sigma \in S^m(T_H M)$ is Rockland iif there exists $\rho \in S^{-m}(T_H M)$ such that $\sigma * \rho - 1, \rho * \sigma - 1 \in S^{-\infty}(T_H M) = \mathcal{C}_p^\infty(T_H M)$

By "quantizing" (tangent groupoid) we can obtain pseudodifferential operators from symbols, denote by $\Psi_H^m(M)$ the associated space of pseudodifferential operators of order m in the filtered calculus.

What we need :

- Noncommutative analog of ${}^{TM}\!/\!_{F}$
- Replacement for transversal ellipticity

The setting is the following :

Definition

A foliated filtered manifold is a manifold M with the data of subbundles $H^0 \subset H^1 \subset \cdots H^r = TM$ with the condition on the Lie brackets :

$$\forall i, j \left[\Gamma(H^i), \Gamma(H^j)\right] \subset \Gamma(H^{i+j})$$

Consequences of the definition :

- H^0 is a foliation
- The action $\mathrm{Hol}(H^0) \curvearrowright {}^{TM}\!/_{H^0}$ preserves each ${}^{H^i}\!/_{H^0}$
- $\mathfrak{t}_{H/H^0}M = {}^{\mathfrak{t}_H M}\!/_{H^0} = {}^{H^1}\!/_{H^0} \oplus {}^{H^2}\!/_{H^1} \oplus \cdots \oplus {}^{TM}\!/_{H^{r-1}}$ is a bundle of nilpotent Lie algebras
- $\mathrm{Hol}(H^0) \curvearrowright \mathfrak{t}_{H/H^0}M$ preserving the bundle of Lie algebras structure
- $\mathrm{Hol}(H^0) \curvearrowright T_{H/H^0}M$ preserving the bundle of Lie groups structure

There is a quotient map $T_H M \to T_{H/H^0} M$ compatible with the inhomogeneous dilations. Space of transverse symbols $S^m(T_{H/H^0} M)$ are defined analogously as for $T_H M$. There is a canonical push-forward map

$$S^m(T_H M) \to S^m(T_{H/H^0} M)$$

Definition

A transverse symbol is transversally Rockland if it satisfies the Rockland condition for $T_{H/H^0} M$ and if it is $\mathrm{Hol}(H^0)$ equivariant.

Crash-Course in KK-theory :

A, B : C^*-algebras. $KK(A, B)$ abelian group of "generalized morphisms" from A to B.

$KK(\mathbb{C}, A) = K^0(A)$ K-theory group (projections/abstract vector bundles)
$KK(A, \mathbb{C}) = K_0(A)$ K-homology group (abstract elliptic operators)

Kasparov product $\otimes_B \colon KK(A, B) \times KK(B, C) \to KK(A, C)$. For $A = C(X), E \to X$ vector bundle, $D \colon \mathcal{C}^\infty(X) \to \mathcal{C}^\infty(X)$ we have $[E] \otimes [D] \in KK(\mathbb{C}, \mathbb{C}) \cong \mathbb{Z}$ is equal to $Ind(D_E)$.

Invariance under Morita equivalence : we can replace $C^*(M \times M) = \mathcal{K}(L^2(M))$ by $\mathbb{C}$.

Theorem (C. ('22))

If $\sigma \in S^0(T_{H/H^0}M)$ is transversally Rockland then it defines a class

$$[\sigma] \in KK^{\mathrm{Hol}(H^0)}(\mathcal{C}_0(M), C^*(T_{H/H^0}M))$$

Theorem (C. ('22))

If $P \in \Psi_H^0(M)$ has its transverse principal symbol transversally Rockland then it defines a class

$$[P] \in KK(C^*(\mathrm{Hol}(H^0)), \mathbb{C}) = K^0(C^*(\mathrm{Hol}(H^0)))$$

Exemple

For $H^0 = 0$ we would get $[\sigma] \in KK(\mathcal{C}_0(M), C^*(TM))$ (on a compact manifold we can refine to $KK(\mathbb{C}, C^*(TM)) = K^0(\mathcal{C}_0(T^*M))$), $[P] \in KK(\mathcal{C}_0(M), \mathbb{C})$

Exemple

If H^0 corresponds to a fibration $\pi\colon M \to B$ with B filtered. P is transversally Rockland iif $\pi_*(P)$ is Rockland and $[P]$ is the usual class in $K^0(\mathcal{C}(B))$.

Exemple

If $\Gamma \curvearrowright M$ is a smooth action of a discrete group we can recover a Γ-equivariant index from the transversal index results.

Exemple

(C. '23) Transverse Bernstein-Gelfand-Gelfand sequences of operators on foliated manifolds with transverse parabolic geometry (e.g. $\Gamma \backslash G/P$ for space of leaves).

Denote by

$$j_{\mathrm{Hol}(H^0)} \colon KK^{\mathrm{Hol}(H^0)}(A, B) \to KK(\mathrm{Hol}(H^0) \ltimes A, \mathrm{Hol}(H^0) \ltimes B)$$

the descent homomorphism.
Let $\mathbb{T}_H^{hol} M = \mathrm{Hol}(H^0) \ltimes T_{H/H^0} M \times \{0\} \bigsqcup M \times M \times \mathbb{R}^*$. The groupoid $\mathbb{T}_H^{hol} M$ induces a canonical E-theory class

$$\mathrm{Ind}_H^{hol} \in E(C^*(\mathrm{Hol}(H^0) \ltimes T_{H/H^0} M), \mathbb{C})$$

Under amenability assumptions it can be obtained as a class in KK-theory.

Theorem (C. ('22))

If $P \in \Psi^0_H(M)$ *has its transverse principal symbol* $\sigma^{\perp}_H(P)$ *transversally Rockland then it defines a class*

$$[\mathbb{P}] \in KK(C^*(\mathrm{Hol}(H^0)), C^*(\mathbb{T}^{hol}_H M_{|[0;1]}))$$

and we have the relations

$$[\mathbb{P}] \otimes [ev_0] = j_{\mathrm{Hol}(H^0)}([\sigma^{\perp}_H(P)]), [\mathbb{P}] \otimes [ev_1] = [P]$$

In particular if $\mathrm{Hol}(H^0)$ *is amenable then we get*

$$[P] = j_{\mathrm{Hol}(H^0)}([\sigma^{\perp}_H(P)]) \otimes \mathrm{Ind}^{hol}_H$$

Idea : Extend P to a family of operators $\mathbb{P} \in \mathcal{E}'_r(\mathbb{T}_H M)$.

Use morphism $\varphi\colon \mathbb{T}_H^M \to \mathbb{T}_H^{\mathrm{Hol}} M$ to get $\varphi_*(\mathbb{P}) \in \mathcal{E}'_r(\mathbb{T}_H^{\mathrm{Hol}} M)$.

$\varphi_*(\mathbb{P})$ has non-trivial singularities, analysis in local coordinate charts (local computation of a Kasparov product).