Singular vectors and associated varieties of vertex algebras of A type

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Outline:

1. vertex algebras

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3. Varieties of affine VAs

4. Varieties of affine W-algebras

5. Singular vectors of affine VAs

1. Vertex algebras

A vertex algebra (VA): $V = (V, Y, \mathbb{1})$

(Borcherds 86, I. Frenkel – Lepowsky – Meurman 89, Lepowsky – Li 04]) is a vector space s.t

$\exists\, Y: \quad V \to (\mathrm{End}\, V)[[z, z^{-1}]]$

$$v \longmapsto Y(v, z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-1} \quad (v_n \in \mathrm{End}\, V)$$

- for $u, v \in V$, $u_n v = 0$ for $n >> 0$.
- $Y(\mathbb{1}, z) = \mathrm{id}$, $\lim_{z \to 0} Y(u, z) = u$
- Jacobi identity:

$$z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)Y(u,z_1)Y(v,z_2)-z_0^{-1}\delta\left(\frac{-z_2+z_1}{z_0}\right)Y(v,z_1)Y(u,z_2)$$

$$=z_1^{-1}\delta\left(\frac{z_2+z_0}{z_1}\right)Y(Y(u,z_0)v,z_2).$$

A VA V is called a VOA, if

$V=\bigoplus_{n\in\mathbb{Z}} V_n$ is graded

and $\exists\ \omega\in V_2$ (conformal vector)

s.t.

$$[L(m),L(n)]=(m-n)L(m+n)+\delta_{m+n,0}\frac{m^3-m}{12}c,$$

$$[L(-1),Y(u,z)]=\frac{d}{dz}Y(u,z)$$

$$L(0)\big|_{V_n}=n\cdot id,$$

where $c \in \mathbb{C}$ and $Y(w, z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2}$.

Ex. Heisenberg VOA of rank one

$$M(1) = L(z_1, z_2, \cdots)$$

with

$$Y(z_m, z) = \frac{1}{(m-1)!} \left(\frac{d}{dz} \right)^m .$$

Ex. The famous moonshine VOA $V^\natural$

Ex. Affine VAS

Given a VOA, there are three module categories:

$\{$ ordinary modules $\} \subseteq \{$ admissible modules $\} \subseteq \{$ weak modules $\}$

Several directions in VOA theory:

- classification of rational VOAS
- Coset and Orbifold construction
- Tensor categories arising from VOAS
- W - algebras
- Geometric objects in VOAS

2. Varieties of VAS

Let V be a vertex algebra, and $S \subset V$. V is called strongly generated by S, if V is linearly spanned by elements of the form:

$$\{ x^1_{-m_1} \cdots x^s_{-m_s} \mathbb{1} \mid x^i \in S,\ m_i \in \mathbb{Z}_{\geq 1} \}.$$

If further, $|S| < \infty$, V is called finitely strongly generated.

Recall [Zhu]

$$C_2(V) = \mathrm{span}\{ u_{-2}v \mid u, v \in V \}$$

and

$$R_V = V / C_2(V).$$

Then R_V is a Poisson algebra with relations

$$\bar{u} \cdot \bar{v} = \overline{u_{-1}v}, \quad \{\bar{u}, \bar{v}\} = \overline{u_0 v}.$$

R_V is usually called the C_2-Zhu alg of V.

It is easy to see that V is finitely strongly generated if and only if R_V is finitely generated.

Let

$$X_V = \mathrm{Specm}(R_V).$$

X_V is called the assoc. variety of V. [Arakawa, 10].

V is called lisse if $X_V = \{0\}$

[Arakawa, 10]

Fact: Let V be a finitely generated vertex algebra, then V is Lisse if and only if R_V is finite-dimensional.

Since R_V is a f.g. Poisson alg, we have [Brown - Gordon, 03]

$$X_V = \bigsqcup_{i=1}^{r} X_i ,$$

X_i are smooth analytic Poisson varieties. So X_V is a union of symplectic leaves.

V is called quasi-lisse if X_V has only finitely many symplectic leaves [Arakawa-Kawasetsu, 18].

Theorem [Arakawa-Kawasetsu, 18]

If V is quasi-lisse, then V has finitely many irreducible ordinary modules.

Given a VA V, $X_V = ?$

(fairly open)

3. Varieties of affine VAS.

Let $\mathfrak{g}$ be a f.d. simple Lie alg $/\mathbb{C}$ with $(\cdot|\cdot) = \frac{1}{2h^\vee}$ Killing form.

Let $\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K$

be the affinization of $\mathfrak{g}$ with

$$[x \otimes t^m, y \otimes t^n] = [x, y] \otimes t^{m+n} + m\delta_{m+n,0}(x|y)K.$$

$\hat{\mathfrak{g}}$ has the following triangular decomp.

$$\hat{\mathfrak{g}}^+ = \mathfrak{g} \otimes t\mathbb{C}[t], \quad \hat{\mathfrak{g}}^- = \mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}],$$

$$\hat{\mathfrak{g}}^0 = \mathfrak{g} \oplus \mathbb{C}K.$$

For $k \in \mathbb{C}$, let $\mathbb{C}_k$ be the one-dimensional module of $\hat{\mathfrak{g}}^0 \oplus \hat{\mathfrak{g}}^+$ s.t.

$$\mathfrak{g} \cdot \mathbb{1} = 0, \quad K \cdot \mathbb{1} = k \cdot \mathbb{1}$$

Then we have the induced $\hat{\mathfrak{g}}$-module

$$V^k(\mathfrak{g}) = U(\hat{\mathfrak{g}}) \otimes_{U(\hat{\mathfrak{g}}^+ \oplus \hat{\mathfrak{g}}^0)} \mathbb{C}_k$$

$V^k(\mathfrak{g})$ has the vertex algebra structure given by

$$Y(a, z) = \sum_{n \in \mathbb{Z}} a(n) z^{-n-1},$$

for $a \in \mathfrak{g}$, where $a(n) = a \otimes t^n$.

By PBW theorem, the C_2-zhu alg

$$R_{V^k(\mathfrak{g})} \cong \mathbb{C}[\mathfrak{g}^*].$$

So

$$X_{V^k(\mathfrak{g})} \cong \mathfrak{g}^* \cong \mathfrak{g}.$$

Let V be a quotient of $V^k(\mathfrak{g})$.

Then X_V is a closed G-invariant subvariety of $\mathfrak{g}^*$.

Q: $X_V = ?$

In particular, $X_{L_k(\mathfrak{g})} = ?$

In general, we have

Theorem [Arakawa – J – Moreau, 20]

For any k and f.d. simple Lie algebra $\mathfrak{g}$, we have

$$X_{V^k(\mathfrak{g})} = X_{L_k(\mathfrak{g})}$$

if and only if $V^k(\mathfrak{g}) = L_k(\mathfrak{g})$.

Theorem [Gorelik – Kac, 07]

For $k \in \mathbb{C}$, $V^k(\mathfrak{g}) \neq L_k(\mathfrak{g})$ if and only if

$$r(k+h^\vee) \in \mathbb{Q}_{\geq 0} \setminus \{\tfrac{1}{m} \mid m \in \mathbb{Z}_{\geq 1}\}$$

where r is the lacing number of $\mathfrak{g}$.

Theorem [I. Frenkel – Zhu, 92], [Li, 96], [Dong – Mason 04], [Arakawa, 10]

$L_k(\mathfrak{g})$ is C_2-cofinite $\iff k \in \mathbb{Z}_{\geq 0}$

$\iff X_{L_k(\mathfrak{g})} = \{0\}$.

Theorem [E. Frenkel – Gaitsgory 07], [Arakawa, 11]

If $k + h^\vee = 0$, then

$X_{L_k(\mathfrak{g})} = \mathcal{N}$ (the nilpotent cone).

Recall that $k \in \mathbb{Q}$ is called admissible [Kac–Wakimoto 03, ⋯], if

$$k + h^\vee = \frac{p}{q}, \quad p, q \in \mathbb{Z}_{\geq 0}, \quad (p, q) = 1$$

and

$$p \geq \begin{cases} h^\vee & (r^\vee, q) = 1 \\ h & (r^\vee, q) = r^\vee \end{cases}$$

If further, $q \geq h$ for $(q, r^\vee) = 1$

or

$$q \geq r^\vee h^\vee_{{}^L\mathfrak{g}} \text{ for } (q, r^\vee) = r^\vee,$$

then k is called a non-degenerate admissible number.

Theorem [Arakawa. 11] Let k be an admissible number, then

$$X_{L_k(\mathfrak{g})} = \overline{\mathbb{O}_q} \subset \mathcal{N}.$$

Furthermore, $X_{L_k(\mathfrak{g})} = \mathcal{N}$ if and only if k is a non-degenerate admissible number.

Q: What about $X_{L_k(\mathfrak{g})}$ when k is not admissible?

Up to now, complete results only for $L_k(sl_2)$.

Following [Collingwood - McGovern, 93]

Let $\mathfrak{g}/\mathbb{C}$ be a f.d. simple Lie alg.

Let $x \in \mathfrak{g}$, and $x = x_s + x_n$ be the Jordan - Chevalley decomposition of x.

The Jordan class of x is defined as

$$J_G(x) = G.\left(z(\mathfrak{g}^{x_s})^{reg} + x_n \right),$$

where for $X \subset \mathfrak{g}$,

$$X^{reg} = \{ y \in X \mid \dim O_y \text{ is maximal} \}.$$

Let $\mathfrak{l} = \mathfrak{g}^{x_s}$, then $\mathfrak{l}$ is a Levi subalg of $\mathfrak{g}$. Let $\mathcal{O}_{\mathfrak{l}}$ be the nilpotent orbit in $\mathfrak{l}$ of x_n.

- Jordan classes $\overset{\text{one to one}}{\longleftrightarrow}$ $(\mathfrak{l}, \mathcal{O}_{\mathfrak{l}})$.

For $m \in \mathbb{Z}_{\geq 0}$, denote

$$\mathfrak{g}^{(m)} = \{ x \in \mathfrak{g} \mid \dim \mathfrak{g}^x = m \}.$$

An irreducible component of $\mathfrak{g}^{(m)}$ is called a sheet of $\mathfrak{g}$.

- A sheet is a finite disjoint union of Jordan classes. So a sheet S contains a unique dense open Jordan class J, and

$$\bar{S} = \bar{J}, \quad S = (\bar{J})^{reg}.$$

So we have

$$S \longleftrightarrow (\mathfrak{l}, \mathcal{O}_{\mathfrak{l}})$$

- A sheet with datum $(\mathfrak{l}, \mathcal{O}_{\mathfrak{l}})$ is called Dixmier if $\mathcal{O}_{\mathfrak{l}} = \{0\}$.

Theorem [Arakawa - Moreau, 16]

- For $n \geq 4$, $X_{L_{-1}(\mathfrak{sl}_n)} \cong \overline{S_{1_1}}$
- For $m \geq 2$, $X_{L_{-m}(\mathfrak{sl}_{2m})} \cong \overline{S_{1_m}}$
- $r \in 2\mathbb{Z}_{\geq 4}$, $X_{L_{2-r}(\mathfrak{so}_{2r})} \cong \overline{S_r}$
- $r \in 2\mathbb{Z}$, $r \geq 6$, $\overline{O_{min}} \subsetneq X_{L_{2-r}(\mathfrak{so}_{2r})} \subseteq \overline{O_{(2^{r-2}, 1^4)}}$

We will consider $X_{L_k(\mathfrak{sl}_3)}$, for non-admissible k, that is, $k+3 = \frac{2}{2m+1}$, $m \geq 0$.

Note that if $m=0$, then $k=-1$.

Theorem [J-Song, 23]

Let $k+3=\frac{2}{2m+1}$, $m\geq 1$. Then

1) $$X_{L_k(\mathfrak{sl}_3)}=\overline{G\cdot(\mathbb{C}^*\lambda+f_\theta)}\supsetneq\mathcal{N},$$

where $\lambda=\alpha_2-\alpha_1$.

2) There is no k such that

$$X_{L_k(\mathfrak{sl}_3)}=\overline{G\cdot\mathbb{C}^*h},$$

for a semi-simple principal $h\in\mathfrak{h}$.

Corollary [J–Song, 23]

For $k \in \mathbb{C}$, $X_{L_k(\mathfrak{sl}_3)}$ is one of the following:

$$\mathfrak{g}^*, \quad \{0\}, \quad \mathcal{N}, \quad \overline{\mathbb{O}}_{min}, \quad \overline{G\cdot(\mathbb{C}^*(\alpha_1-\alpha_2))},$$

$$\overline{G\cdot(\mathbb{C}^*(\alpha_1-\alpha_2)+f_\theta)},$$

where $\mathfrak{g} = \mathfrak{sl}_3$.

In particular, $X_{L_k(\mathfrak{sl}_3)}$ for all $k+3 = \frac{p}{q}$ is irreducible

We conjecture that for $k = -n + \frac{p}{q}$, $X_{L_k(\mathfrak{sl}_n)}$ is irreducible.

4. Varieties of assoc. affine W-algebras

Recall from [Kac – Roan – Wakimoto, 03], [Kac – Wakimoto, 04], [E. Frenkel – Ben-Zvi, 04], ...

Let $\mathfrak{g}/\mathbb{C}$ be a simple Lie algebra and f a nilpotent element of $\mathfrak{g}$. By the Jacobson – Morozov theorem, there exists an $\mathfrak{sl}_2$-triple $\{e, f, h\}$.

Set

$$\mathfrak{g}_j = \{ x \in \mathfrak{g} \mid [h, x] = 2jx \}, \text{ for } j \in \tfrac{1}{2}\mathbb{Z}.$$

Then

$$\mathfrak{g} = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}} \mathfrak{g}_j .$$

The element f defines a non-degenerate skew-symmetric bilinear form on $\mathfrak{g}_{\frac{1}{2}}$ by

$$\langle a, b \rangle = (f | [a, b])$$

Let $\{x_\alpha \mid \alpha \in I_{\frac{1}{2}}\}$ be a basis of $\mathfrak{g}_{\frac{1}{2}}$.

We have the Clifford affinization Cl_{ne} associated with $\mathfrak{g}_{\frac{1}{2}}$, with even generators $\Phi_\alpha(n)$, $\alpha \in I_{\frac{1}{2}}$, $n \in \mathbb{Z}$ such that

$$[\Phi_\alpha(m), \Phi_\beta(n)] = \langle x_\alpha, x_\beta \rangle \delta_{m+n+1, 0}.$$

Let F_{ne} be the associated vertex alg

with fields:

$$\phi_\alpha(z) = \sum_{n \in \mathbb{Z}} \phi_\alpha(n) z^{-n-1}$$

Let $\{x_\alpha \mid \alpha \in I_+\}$ be a basis of $\mathfrak{g}_+ = \bigoplus_{j>0} \mathfrak{g}_j$

and $\{x_{-\alpha} \mid \alpha \in I_+\}$ a basis of $\mathfrak{g}_- = \bigoplus_{j>0} \mathfrak{g}_{-j}$

such that $(x_\alpha \mid x_{-\beta}) = \delta_{\alpha,\beta}$.

We have the Clifford affinization Cl_{ch}

of $\mathfrak{g}_+ \oplus \mathfrak{g}_-$, with odd generators $\psi_\alpha(n)$,

$\psi^*_\alpha(n)$, $\alpha \in I$, $n \in \mathbb{Z}$, and non-trivial relations:

$$[\psi_\alpha(m), \psi^*_\beta(n)]_+ = \delta_{\alpha,\beta}\,\delta_{m+n,0}$$

Let F_{ch} be the irreducible module of Cl_{ch} generated by the vacuum $\mathbb{1}$ s.t.

$$\psi_\alpha(n)\mathbb{1} = 0, \quad \psi^*_\alpha(m)\mathbb{1} = 0, \quad n \geq 0, \ m \geq 1.$$

Then F_{ch} is a vertex superalgebra with fields:

$$\psi_\alpha(z) = \sum_{n\in\mathbb{Z}} \psi_\alpha(n)\, z^{-n-1}$$

$$\psi_\alpha^*(z) = \sum_{n\in\mathbb{Z}} \psi_\alpha(n) z^{-n}$$

Consider the vertex superalgebra

$$C_k(\mathfrak{g}, f) = V^k(\mathfrak{g}) \otimes F_{ch} \otimes F_{ne}$$

Let $F = F_{ch} \otimes F_{ne} = \bigoplus_{i\in\mathbb{Z}} F^i$

be an additional $\mathbb{Z}$-gradation defined by

$$\deg \mathbf{1} = 0, \quad \deg \psi_\alpha^*(z) = 1, \quad \deg \psi_\alpha(z) = -1$$

Set

$$C^i(\mathfrak{g}, f) = V^k(\mathfrak{g}) \otimes F^i.$$

This gives a $\mathbb{Z}$-gradation of

$$C_k(\mathfrak{g}, f) = \bigoplus_{i \in \mathbb{Z}} C_k^i(\mathfrak{g}, f)$$

Introduce the following field:

$$d(z) = d^{st}(z) + \sum_{\alpha \in I_+} (f|x_\alpha)\, \varphi_\alpha^*(z) + \sum_{\alpha \in I_{\frac{1}{2}}} \varphi_\alpha^*(z)\, \Phi_\alpha(z)$$

where

$$d^{st}(z) = \sum_{\alpha \in I_+} :x_\alpha(z) \otimes \varphi_\alpha^*(z): - \frac{1}{2} \sum_{\alpha,\beta,\gamma \in I_+} c_{\alpha\beta}^{\gamma} :\varphi_\alpha^*(z)\, \varphi_\beta^*(z)\, \varphi_\gamma(z):$$

Let $d_0 = \mathrm{Res}_z\, d(z)$. Then $d_0^2 = 0$, and

$$d_0(C_k^i(\mathfrak{g}, f)) \subseteq C_k^{i+1}(\mathfrak{g}, f)$$

Thus $(C_k(\mathfrak{g},f), d_0)$ is a complex of vertex algebras, which is called the BRST complex of the generalized quantized Drinfeld - Sokolov reduction.

Theorem (Feigin - E. Frenkel 90, de Boer - Tjin 94, Kac - Roan - Wakimoto 03, E. Frenkel - Ben - Zvi 04, Arakawa 15)

(1) $H^i(C_k(\mathfrak{g},f)) = 0$ for $i \neq 0$

(2) $W^k(\mathfrak{g},f) = H^0(C_k(\mathfrak{g},f))$ is a vertex algebra called the universal affine W-algebra

associated with $V^k(\mathfrak{g})$ and f.

Denote by $W_k(\mathfrak{g}, f)$ the simple quotient of $W^k(\mathfrak{g}, f)$

Remark. $V^k(\mathfrak{g})$ can be replaced by any module M of $V^k(\mathfrak{g})$ to get the BRST cohomology, denoted by $H_f^{\frac{\infty}{2}+0}(M)$.

Then $H_f^{\frac{\infty}{2}+0}(V^k(\mathfrak{g})) = W^k(\mathfrak{g}, f)$.

Theorem [De Sole – Kac, 06. Arakawa, 15]

(1) $X_{H_f^{\frac{\infty}{2}+0}(V)} \cong X_V \cap \mathcal{S}_f$,

and

$H_f^{\frac{\infty}{2}+0}(V) \neq 0$ iff $\overline{G\cdot f} \subset X_V$.

(2)

$$X_{W^k(\mathfrak{g},f)} \cong \mathcal{S}_f = f \oplus \mathfrak{g}^e$$

the Slodowy slice.

Remark. It may happen that $W_k(\mathfrak{g},f) \neq 0$

But $H_f^{\frac{\infty}{2}+0}(L_k(\mathfrak{g})) = 0$.

Ex. Let $k + h^\vee = \frac{p}{q}$, $q < h$ or $r^\vee h^\vee_{L_g}$.

Then

$$X_{L_k(\mathfrak{g})} = \overline{\mathbb{O}_q}$$

But for a regular nilpotent vector $f \in \mathfrak{g}$,

$$\overline{G \cdot f} = \mathcal{N}.$$

So $$\overline{G \cdot f} \not\subset X_{L_k(\mathfrak{g})}.$$

Then $$H^{\frac{\infty}{2}+0}(L_k(\mathfrak{g})) = 0.$$

But $$W_k(\mathfrak{g}, f) \neq 0.$$

Theorem [Arakawa - J - Moreau, 20]

Let f be a nilpotent element of $\mathfrak{g}$

The following are equivalent

(1) $V^k(\mathfrak{g})$ is simple;

(2) $W^k(\mathfrak{g}, f) = H_f^{\frac{\infty}{2}+0}(L_k(\mathfrak{g}))$;

(3) $X_{H_f^{\frac{\infty}{2}+0}(L_k(\mathfrak{g}))} = S_f$.

Remark. Results on varieties of affine W-algebras are very limited.

Complete results just for the sl_2 case.

For $\mathfrak{sl}_3$ case, we have

Theorem [Arakawa, 10]

(1) Let $f \in \mathbb{O}_{min}$ be a minimal nilpotent element of $\mathfrak{sl}_3$.

(a) If $k = -3 + \frac{2m+1}{2}$, $m \geq 1$, then

$$X_{W_k(\mathfrak{sl}_3, f)} = 0.$$

(b) If $k = -3 + \frac{q}{p}$, $(p, q) = 1$, $p, q \geq 3$, then

$$\dim X_{W_k(\mathfrak{sl}_3, f)} = 2.$$

(2) Let f be principal nilpotent, then

$$X_{W_k(\mathfrak{sl}_3, f)} = \{0\}.$$

Theorem [J - Song, 23]

(1) Let $f \in \mathbb{O}_{min}$ be a minimal nilpotent element of $\mathfrak{sl}_3$.

(a) If $k = -1$, then

$$\dim X_{W_k(\mathfrak{sl}_3, f)} = 1.$$

(b) If $k = -3 + \frac{2}{2m+1}$, $m \geq 1$, then

$$\dim X_{W_k(\mathfrak{sl}_3, f)} = 3.$$

(2) Let f be principal nilpotent, then for $k = -3 + \frac{2m+1}{2}$ or $k = -3 + \frac{2}{2m+1}$,

$$\dim X_{W_k(\mathfrak{sl}_3, f)} = 1.$$

Corollary [J-Song, 23].

For all non-admissible numbers k or dedengerate admissible k, the simple principal W-algebras $W_k(sl_3)$ are not Lisse.

Further aim: determine varieties for simple affine VAS and associated affine W algebras for the sl_n case.

5. Weights of singular vectors.

Let $\mathfrak{g}$ be a f.d. simple Lie alg / $\mathbb{C}$.

Theorem [Gorelik–Kac, 07]

For $k \in \mathbb{C}$, $V^k(\mathfrak{g}) \neq L_k(\mathfrak{g})$ if and only if

$$r(k+h^\vee) \in \mathbb{Q}_{\geq 0} \setminus \{ \tfrac{1}{m} \mid m \in \mathbb{Z}_{\geq 1} \}$$

where r is the lacing number of $\mathfrak{g}$.

In particular, for $\mathfrak{g} = \mathfrak{sl}_n$, $V^k(\mathfrak{sl}_n) \neq L_k(\mathfrak{g})$ iff $k+n = \frac{p}{q}$, $(p,q)=1$, $p \geq 2$.

Theorem (J - Song, 23)

Assume $\mathfrak{g} = \mathfrak{sl}_n$. For $k = -n + \frac{p}{q}$, $(p,q)=1$, $p \geqslant 2$, Let v_p be a singular vector with minimal weight $k\Lambda_0 - \Gamma$. Then we have

1. For $p \geq n$, $\Gamma = (p-n+1)q\delta - (p-n+1)(\alpha_1 + \cdots + \alpha_n)$

2. For $n > p = 2$, $\Gamma = \frac{n}{2} q\delta - (\alpha_1 + \cdots + \alpha_{n-1})$ if $2 | n$, and $\Gamma = nq\delta - (2\alpha_1 + \cdots + 2\alpha_{n-2} + \alpha_{n-1})$ or $nq\delta - (\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_n)$, if $2 \nmid n$.

3. If $n=5$ and $p=3$, then

$$\Gamma = 48\delta - 2(\alpha_1+\alpha_2+\alpha_3+\alpha_4),\quad 48\delta - (2\alpha_1+3\alpha_2+2\alpha_3+\alpha_4)$$

or $48\delta - (\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4)$.

4. If $n=8$ and $p=3$, then

$$\Gamma = 68\delta - 2(\alpha_1+\alpha_2+\cdots+\alpha_7).$$

5. For other cases, set $s_1 = \lfloor n/p \rfloor$, $s_2 = \lceil n/p \rceil$, and define

$$M(s) = (|sp-n|+1)s,$$

and

$$M = \min\{M(s_1), M(s_2)\}$$

Then

$$\Gamma = M g \delta - \Gamma_i \quad , \qquad i=1,2$$

with $M(s_i) = M$, and

$$\Gamma_1 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + \cdots + r_1 \alpha_{r_1} + r_1 \alpha_{r_1+1} + \cdots + r_1 \alpha_{n-r_1}$$
$$+ (r_1 - 1)\alpha_{n-(r_1-1)} + \cdots + \alpha_{n-1} ,$$

where $r_i = |s_i p - n| + 1$.

Thanks !