Resurgence illustrated on partial theta series

Algebraic, analytic and geometric structures emerging from QFT

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David SAUZIN (Capital Normal University, Beijing, on leave from CNRS - IMCCE, Paris Observatory - PSL University)

Based on

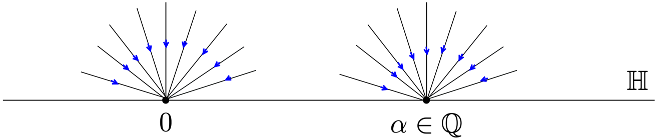
– https://arxiv.org/abs/2112.15223 with Li HAN, Yong LI, Shanzhong SUN (19 p., published in Functional Analysis and Applications)

– https://arxiv.org/abs/2310.15029 (13 p., review of Resurgence Theory prepared for the 2nd edition of the Encyclopedia of Mathematical Physics)

$$\Theta(\tau;\nu,f,M) := \sum_{n\geqslant 1} a_n\, \mathrm{e}^{\mathrm{i}\pi n^2\tau/M} \quad \text{with } a_n = n^\nu f(n)$$

where the function $f:\mathbb{Z}\to\mathbb{C}$ is M-periodic, and $\nu = 0, 1, \ldots$

$\Theta(\tau;\nu,f,M)$ is holo & $2M$-periodic in $\mathbb{H} := \{\Im m\,\tau > 0\}$. We are interested in asymptotics as τ tends non-tangentially to 0 or a rational α



Limit fcn $\alpha \in Q_{f,M} \mapsto \Theta^{\mathrm{nt}}(\alpha) := \lim_{\tau\to\alpha} \Theta(\tau;\nu,\alpha,M)$ is $2M$-periodic.

We'll see $0 \in Q_{f,M} \Leftrightarrow \langle f\rangle = 0$ and appearance of *two resurgent series* (depending on odd/even part of f) in relation to $\tau \to 0$. Since $\Theta(\alpha+\tau;\nu,f,M) = \Theta\big(\frac{M_\alpha}{M}\tau;\nu,f_{\frac{\alpha}{M}},M_\alpha\big)$ with $f_\beta(n) := f(n)\,\mathrm{e}^{\mathrm{i}\pi n^2\beta}$, we get

$$Q_{f,M} = \{\,\alpha\in\mathbb{Q} \mid \langle f_{\frac{\alpha}{M}}\rangle = 0\,\}$$

and asympt behaviour around α for f deduced from that around 0 for $f_{\frac{\alpha}{M}}$.

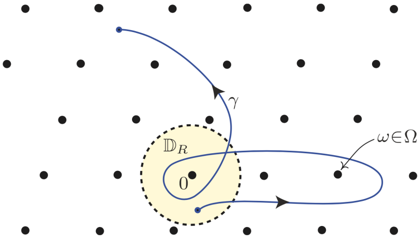
CRASH COURSE IN RESURGENCE THEORY

DEFINITION (Ecalle/[CNP]) A resurgent series is any formal series whose formal Borel transform is an endlessly continuable germ.

In this talk, we use $\tau = 1/z$ rather than the usual variable z:

$$\mathscr{B}\colon\ \tau^{n+1} \mapsto \xi^n/n! \qquad \mathscr{B}\colon\ \tau^{\nu} \mapsto \xi^{\nu-1}/\Gamma(\nu) \quad (\Re e\, \nu > 0).$$

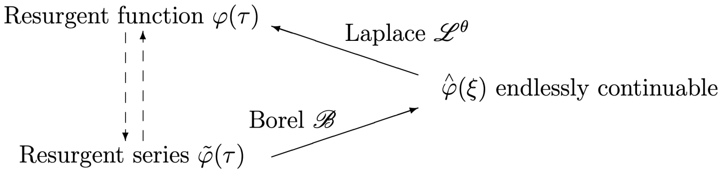
$\widehat{\varphi}(\xi) \in \xi^c\mathbb{C}\{\xi\}$ is called endlessly continuable if one can follow its analytic continuation along any finite path starting near 0 and avoiding a finite subset of the Riemann surface of the log (no natural barrier, only isolated singularities...)



Elementary examples: Meromorphic functions, algebraic functions.

In this talk: $\Omega \subset 2\pi\mathrm{i}M^{-1}\mathbb{Z}_{>0}$.

DEFINITION A resurgent function is any function which can be obtained from a resurgent series by Borel-Laplace summation:



Recall that we use $\tau = 1/z$ rather than the usual variable z:

$$\mathscr{B}\colon\ \tau^{n+1} \mapsto \xi^n/n! \qquad \mathscr{B}\colon\ \tau^\nu \mapsto \xi^{\nu-1}/\Gamma(\nu) \quad (\Re e\, \nu > 0)$$

$$\varphi(\tau) = \mathscr{L}^\theta \widehat{\varphi}(\tau) := \int_0^{e^{i\theta}\infty} e^{-\xi/\tau}\,\widehat{\varphi}(\xi)\,d\xi \sim \tilde{\varphi} := \mathscr{B}^{-1}\widehat{\varphi} \text{ as } \tau \to 0$$

$$\mathscr{S}^\theta := \mathscr{L}^\theta \circ \mathscr{B} \qquad \mathscr{S}^\theta_{\text{med}} := \mathscr{L}^\theta_{\text{med}} \circ \mathscr{B}$$

$\mathscr{S}^\theta_{\text{med}}$ is one of Écalle's average Borel-Laplace summation operators, which all map convergent series to their usual sums, and which all map products to products.

Example with $\theta = 0$: The Borel transform of the Stirling series is

$$\widehat{\mu}(\xi) := \xi^{-2}\Big(\frac{\xi}{2}\coth\frac{\xi}{2} - 1\Big) = \tfrac{1}{12} - \tfrac{1}{360}\tfrac{\xi^2}{2!} + \tfrac{1}{1260}\tfrac{\xi^4}{4!} - \ldots \in \mathbb{C}\{\xi\}$$

meromorphic, poles on $2\pi i\mathbb{Z}^*$ $\rightsquigarrow$ $2\pi i\mathbb{Z}^*$-continuable.
Its Laplace trsf is the log of the normalized Gamma function at $z = 1/\tau$:

$$\mu(\tau) = \log\left(\frac{\Gamma(z)}{\sqrt{2\pi}z^{z-\frac{1}{2}}e^{-z}}\right) \sim \tilde{\mu}(\tau) = \frac{1}{12}\tau - \frac{1}{360}\tau^3 + \frac{1}{1260}\tau^5 + \ldots$$

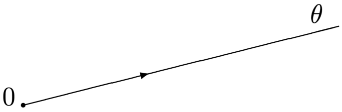
A less elementary example: Denote by $W_0(x) > W_{-1}(x)$ the real branches of the Lambert W fcn for $x \in (-e^{-1}, 0)$ (solving $w\,e^w = x$).

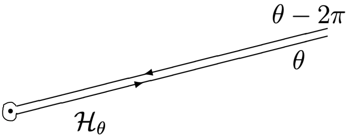
$$\widehat{\lambda}(\xi) := \frac{1}{\sqrt{2\pi}}\big(W_0 - W_{-1}\big)(-e^{-1-\xi}) = \frac{\xi^{1/2}}{\Gamma(3/2)} + \frac{\xi^{3/2}}{12\Gamma(5/2)} + \frac{\xi^{5/2}}{288\Gamma(7/2)} + \ldots \in \xi^{1/2}\mathbb{C}\{\xi\}$$

is $2\pi i\mathbb{Z}$-continble, its Laplace transform is (hal.archives-ouvertes.fr/hal-03502909)

$$\tau^{3/2}e^{\mu(\tau)} = \frac{\Gamma(z)}{\sqrt{2\pi}z^{z+1}e^{-z}} \sim \tau^{3/2}e^{\tilde{\mu}(\tau)} = \tau^{3/2} + \frac{1}{12}\tau^{5/2} + \frac{1}{288}\tau^{7/2} + \ldots$$

“Hankel-Laplace” trsf of singular germs: from “minors” to “majors”





$\widehat{\varphi}(\xi)$ integrable at 0, $\check{\varphi}(\xi) = o(1/|\xi|)$

$$\widehat{\varphi}(\xi) = \check{\varphi}(\xi) - \check{\varphi}(\mathrm{e}^{-2\pi\mathrm{i}}\xi) \implies \mathscr{L}^\theta\widehat{\varphi}(\tau) = \int_{\mathcal{H}_\theta} \mathrm{e}^{-\xi/\tau}\check{\varphi}(\xi)\,\mathrm{d}\xi =: \overset{\triangledown}{\mathscr{L}}{}^\theta\check{\varphi}(\tau)$$

Examples:
- $\widehat{\varphi}(\xi) \in \mathbb{C}\{\xi\}$ is the minor of $\check{\varphi}(\xi) = \widehat{\varphi}(\xi)(\log\xi)/2\pi\mathrm{i}$
- $\widehat{\varphi}(\xi) \in \xi^c\mathbb{C}\{\xi\}$ is the min of $\check{\varphi}(\xi) = \widehat{\varphi}(\xi)/(1-\mathrm{e}^{-2\pi\mathrm{i}c})$ $(c \notin \mathbb{Z})$

We can apply $\overset{\triangledown}{\mathscr{L}}{}^\theta$ to more general singular germs $\check{\varphi}$: for example

$\overset{\triangledown}{\mathscr{L}}{}^\theta\Big[\frac{\xi^c}{(1-\mathrm{e}^{-2\pi\mathrm{i}c})\Gamma(c+1)}\Big] = \tau^{c+1}$ even if $\Re e\, c \leqslant -1$,

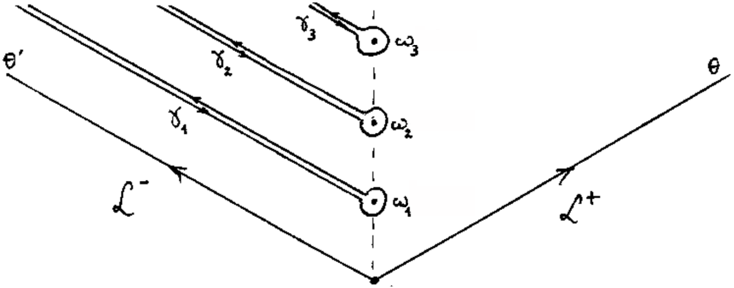
$\overset{\triangledown}{\mathscr{L}}{}^\theta\Big[\frac{1}{2\pi\mathrm{i}\xi}\Big] = 1, \quad \overset{\triangledown}{\mathscr{L}}{}^\theta\Big[\frac{(-1)^n n!}{2\pi\mathrm{i}\xi^{n+1}}\Big] = \tau^{-n}.$

- *Extension of Borel-Laplace summation by* $\mathscr{S}^\theta = \overset{\triangledown}{\mathscr{L}}{}^\theta \circ \overset{\triangledown}{\mathscr{B}}$
- *Adequate formalism to encode singularities via alien operators...*

DEFINITION Alien operator Δ^+_ω (coincides with the alien derivation Δ_ω in meromorphic case)

$$\Delta^+_\omega \tilde\varphi = \overset{\triangledown}{\mathscr{B}}^{-1}\big[\mathrm{cont}_{\Gamma^+_\omega}\big(\min(\overset{\triangledown}{\mathscr{B}}\tilde\varphi)\big)(\omega+\xi) \mod \mathbb{C}\{\xi\}\big]$$

(Γ^+_ω starts near 0, ends near ω, circumvents intermediar sing *to the right*)



$$\mathscr{S}^\theta\tilde\varphi = \mathscr{S}^{\theta'}\tilde\varphi + \sum_{m\geqslant 1} \mathrm{e}^{-\omega_m/\tau}\mathscr{S}^{\theta'}\Delta^+_{\omega_m}\tilde\varphi$$

$$\mathscr{S}^\theta = \mathscr{S}^{\theta'} \circ \Big(\mathrm{Id} + \sum_{m\geqslant 1} \mathrm{e}^{-\omega_m/\tau}\Delta^+_{\omega_m}\Big)$$ (Stokes automorphism).

RESURGENCE IN PARTIAL THETA SERIES

For our partial theta series, we'll get

$$\Theta(\tau;\nu,f,M) = \mathrm{cst}\,\langle f\rangle\left(\tfrac{\tau}{\mathrm{i}}\right)^{-\frac{\nu+1}{2}} + \mathscr{S}^{\frac{\pi}{2}}_{\mathrm{med}}\tilde{\Theta}^{+}(\tau) + \left(\mathscr{S}^{\frac{\pi}{2}-\varepsilon} - \mathscr{S}^{\frac{\pi}{2}+\varepsilon}\right)\tilde{\Theta}^{-}(\tau)$$

• with difference of Borel-Laplace sums of a series $\tilde{\Theta}^{-}$ depending only on the even part of $n \mapsto a_n = n^{\nu} f(n)$, dictating modularity properties

• and median Borel-Laplace sum of a series $\tilde{\Theta}^{+}$ depending only on the odd part of $n \mapsto a_n$, dictating quantum-modularity properties.

Example: $\eta(\tau) = \Theta(\tau;0,\chi,12)$, $\tilde{\eta}(\tau) = \Theta(\tau;1,\chi,12)$

n	1	5	7	11
$\chi(n)$	1	-1	-1	1

Dedekind eta function and its Eichler integral

χ even, $\eta(\tau) = \left(\mathscr{S}^{\frac{\pi}{2}-\varepsilon} - \mathscr{S}^{\frac{\pi}{2}+\varepsilon}\right)\tilde{\Theta}^{-}(\tau)$ modularity! (wt.1/2)

$$= \textstyle\sum_{m\geqslant 1} \left(\tfrac{\tau}{\mathrm{i}}\right)^{-1/2}\chi(m)\,\mathrm{e}^{-\mathrm{i}\pi m^2/12\tau} = \left(\tfrac{\tau}{\mathrm{i}}\right)^{-1/2}\eta(-1/\tau).$$

For $\tilde{\eta}$: now $n \mapsto a_n$ is odd,

$$\tilde{\eta}(\tau) = \mathscr{S}^{\frac{\pi}{2}}_{\mathrm{med}}\tilde{\Theta}^{+}(\tau) = \mathscr{S}^{\frac{\pi}{2}-\varepsilon}\tilde{\Theta}^{+}(\tau) - \tfrac{1}{2}\left(\mathscr{S}^{\frac{\pi}{2}-\varepsilon} - \mathscr{S}^{\frac{\pi}{2}+\varepsilon}\right)\tilde{\Theta}^{+}(\tau)$$

and sth similar happens with $\tilde{\Theta}^{+}$, leading to quantum-modularity (wt.3/2)

Quantum-modularity was discovered by D. Zagier when considering $\tilde{\eta}(\tau)$, "strange identity" *Topology* 2001 (see also "resurgence of the Kontsevich-Zagier series" [Costin-Garoufalidis 2011]),

and $\Theta(\tau; 0, f_+, 60)$, f_+ odd

n	1	11	19	29	31	41	49	59
$f_+(n)$	1	1	1	1	-1	-1	-1	-1

Lawrence-Zagier "Modular forms & quantum invariants of 3-mflds" 1999

$\Theta(\tau; 0, f_+, 60)$ occurs as the GPPV = Gukov-Pei-Putrov-Vafa invariant for the Poincaré homology sphere $\Sigma(2,3,5)$, in connection with $\mathrm{SU}(2,\mathbb{C})$ Chern-Simons theory.

Link with resurgence observed in [Gukov-Mariño-Putrov 2016], in the case of $\Sigma(2,3,5)$ and $\Sigma(2,3,7)$.

[Andersen-Mistegård 2022], using [Gukov-Manolescu 2021], have established this for any fibred Seifert homology sphere $\Sigma(p_1,\ldots,p_r)$.

(For the quantum modularity of partial theta series, see also [Goswami-Osburn 2021].)

Treating [GPPV]/[GM]/[AM] as a black box:

$$\Sigma(p_1,\dots,p_r) \rightsquigarrow \hat{Z}(\tau) = \sum_{\nu=0}^{r-3} \Theta(\tau;\nu,f_\nu,2p_1\dots p_r) \text{ with each } n^\nu f_\nu(n) \text{ odd.}$$

Fact: one can construct vector-valued strong quantum modular forms on $\mathrm{SL}(2,\mathbb{Z})$ by considering f_ν and their DFT $\hat{f}_\nu$ (higher depth for $\nu \geqslant 2$).

Interesting property of $\Theta^{\mathrm{nt}}{}_{|\mathbb{Z}}$: the Fourier numbers of the $2M$-periodic function $k \mapsto \Theta^{\mathrm{nt}}(-k)$ are related to $\mathrm{SL}(2,\mathbb{C})$ Chern-Simons actions,

$\Theta^{\mathrm{nt}}(1/k) \leftrightsquigarrow WRT(k)$, $\quad \tilde{\Theta}^{+}(\tau) \leftrightsquigarrow$ Ohtsuki series at $\tau \to 0$.

To get the Fourier numbers, consider $n \in \mathbb{Z} \mapsto \mathscr{G}(n) := \left[-\frac{n^2}{2M} \right] \in \mathbb{Q}/\mathbb{Z}$

If $\mathrm{supp}(f) \subset \mathscr{G}^{-1}(\zeta)$, then $\Theta(\tau - k;\nu,f,M) = \mathrm{e}^{2\pi i k\zeta}\Theta(\tau;\nu,f,M)$,

hence $\Theta^{\mathrm{nt}}(k) = \mathrm{e}^{2\pi i k\zeta}\Theta^{\mathrm{nt}}(0)$: only one Fourier mode per $\mathscr{G}$-fibre.

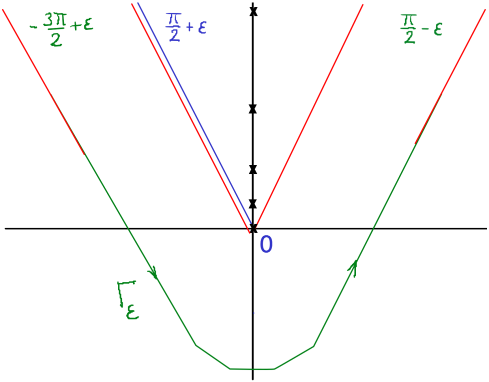
For Dedekind η and $\tilde{\eta}$: $\chi = \hat{\chi}$, $\mathrm{supp}(\chi) \subset \mathscr{G}^{-1}(1/24)$.

For $\Sigma(2,3,5)$: $\mathrm{supp}(f_+) \subset \mathscr{G}^{-1}(1/60)$, $\mathrm{supp}(\hat{f}_+) \subset \mathscr{G}^{-1}(\{1/60, 49/60\})$.

For general $\Sigma(p_1,\dots,p_r)$, [AM] has identified the $\mathrm{SL}(2,\mathbb{C})$ Chern-Simons actions: a certain subset of the range of $\mathscr{G}$, which contains $\mathrm{supp}(\hat{f}_\nu)$.

1st result (implicit in [GMP], [AM]—cf. [Flajolet-Noy FPSAC 2000])

Let $F(t) := \sum_{n\geqslant 1} a_n \mathrm{e}^{-nt}$, holo & bounded in $\{\Re e\, t \geqslant c\}$ (for any $c > 0$) and $C := \left(\frac{4\pi}{M}\right)^{1/2} \mathrm{e}^{\mathrm{i}\pi/4}$. Then $\hat{\phi}(\xi) := \pi^{-1/2}\xi^{-1/2} F(C\,\xi^{1/2})$ is holo for $-\frac{3\pi}{2} < \arg\xi < \frac{\pi}{2}$ and $\Theta(\tau) = \frac{1}{2}\tau^{-1/2}\int_{\Gamma_\varepsilon} \mathrm{e}^{-\xi/\tau}\hat{\phi}(\xi)\,\mathrm{d}\xi$ for all $\tau \in \mathbb{H}$.



Proof just with Borel-Laplace: $\tau^{1/2}\mathrm{e}^{\sigma^2\tau} = \sum \frac{\sigma^{2p}}{p!}\tau^{p+\frac{1}{2}} \in \tau^{1/2}\mathbb{C}\{\tau\}$ is the Laplace trsf of its Borel trsf in any direction, apply it with $\sigma^2 = \mathrm{i}\pi n^2/M$.

$\mathscr{B}(\tau^{1/2}\mathrm{e}^{\sigma^2\tau}) = \sum \frac{\sigma^{2p}}{p!\Gamma(p+\frac{1}{2})}\xi^{p-\frac{1}{2}} = \pi^{-1/2}\xi^{-1/2}\sum \frac{(2\sigma\xi^{1/2})^{2p}}{(2p)!}$ = odd part of

$$\Psi(\xi^{1/2}) := \pi^{-1/2}\xi^{-1/2}\,\mathrm{e}^{-2\sigma\xi^{1/2}}$$

hence $\tau^{1/2}\mathrm{e}^{\sigma^2\tau} = \frac{1}{2}\mathscr{L}^{\theta^-}\big[\Psi(\xi^{1/2})\big] - \frac{1}{2}\mathscr{L}^{\theta^+}\big[\Psi(-\xi^{1/2})\big]$ if $\theta^+ \simeq \theta^-$.

Choosing $\theta^\pm = \frac{\pi}{2} \pm \varepsilon$ and using $\mathscr{L}^{\theta^+}\big[\Psi(-\xi^{1/2})\big] = \mathscr{L}^{\theta^+-2\pi}\big[\Psi(\xi^{1/2})\big]$:

$$\tau^{1/2}\,\mathrm{e}^{\sigma^2\tau} = \frac{1}{2}\pi^{-1/2}\Big(\int_0^{\mathrm{e}^{\mathrm{i}(\frac{\pi}{2}-\varepsilon)}\infty} - \int_0^{\mathrm{e}^{\mathrm{i}(-\frac{3\pi}{2}+\varepsilon)}\infty}\Big)\mathrm{e}^{-\xi/\tau}\,\xi^{-1/2}\,\mathrm{e}^{-2\sigma\xi^{1/2}}\,\mathrm{d}\xi$$

$$= \frac{1}{2}\pi^{-1/2}\int_{\Gamma_\varepsilon}\mathrm{e}^{-\xi/\tau}\,\xi^{-1/2}\,\mathrm{e}^{-2\sigma\xi^{1/2}}\,\mathrm{d}\xi.$$

Apply it with $\sigma^2 = \mathrm{i}\pi n^2/M$, *i.e.* $2\sigma = Cn$, multiply by a_n and check uniform convergence (OK because Γ_ε away from singular half-line)...

□

$\Theta(\tau;\nu,f,M) = \sum_{n\geqslant 1} a_n\, \mathrm{e}^{\mathrm{i}\pi n^2\tau/M}$ $\qquad F(t) := \sum_{n\geqslant 1} a_n\, \mathrm{e}^{-nt}$

$\hat\phi(\xi) = \pi^{-1/2}\xi^{-1/2}F(C\,\xi^{1/2})$ yields $\Theta(\tau) = \frac{1}{2}\tau^{-1/2}\int_{\Gamma_\varepsilon} \mathrm{e}^{-\xi/\tau}\hat\phi(\xi)\,\mathrm{d}\xi$

With $a_n = n^\nu f(n)$ and f M-periodic, we find

$$F(t) = \left(-\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^\nu F_0(t),\ \ F_0(t) = \sum_{n\geqslant 1} f(n)\,\mathrm{e}^{-nt} = \frac{1}{1-\mathrm{e}^{-Mt}} \sum_{1\leqslant \ell\leqslant M} f(\ell)\, e^{-\ell t}$$

Decompose $F(t) = \frac{\nu!\langle f\rangle}{t^{\nu+1}} + F^{\mathrm{od}}(t) + F^{\mathrm{ev}}(t)$, $\ F^{\mathrm{od/ev}}(t) \in \mathbb{C}\{t\}$ odd/even.

Define $\hat\phi^\pm(\xi) \in \mathbb{C}\{\xi\}$ by $F^{\mathrm{od}}(t) = \pi^{1/2}\frac{t}{C}\,\hat\phi^-\left(\frac{t^2}{C^2}\right)$, $F^{\mathrm{ev}}(t) = \pi^{1/2}\,\hat\phi^+\left(\frac{t^2}{C^2}\right)$.

$\hat\phi(\xi) = \frac{\nu!\langle f\rangle}{\pi^{1/2}C^{\nu+1}}\xi^{-\frac{\nu}{2}-1} + \hat\phi^-(\xi) + \xi^{-1/2}\hat\phi^+(\xi)$, $\ \hat\phi^\pm$ meromorphic on $\mathbb{C}$,

all poles among $\xi_n := \frac{\mathrm{i}\pi n^2}{M} \in \mathrm{i}\mathbb{R}_{>0}$. We end up with

$$\Theta(\tau;\nu,f,M) = \tfrac{1}{2}\Gamma\left(\tfrac{\nu+1}{2}\right)\langle f\rangle\left(\tfrac{\pi}{M}\cdot\tfrac{\tau}{\mathrm{i}}\right)^{-\frac{\nu+1}{2}} + \Theta^-(\tau;\nu,f,M) + \Theta^+(\tau;\nu,f,M)$$

$$\Theta^- := \frac{\tau^{-1/2}}{2}\int_{\Gamma_\varepsilon} \mathrm{e}^{-\xi/\tau}\,\hat\phi^-(\xi)\,\mathrm{d}\xi, \quad \Theta^+ := \frac{\tau^{-1/2}}{2}\int_{\Gamma_\varepsilon} \mathrm{e}^{-\xi/\tau}\,\xi^{-1/2}\hat\phi^+(\xi)\,\mathrm{d}\xi$$

Moving Γ_ε upward, we get

$$\Theta^- = \tau^{-1/2} \times \left(\mathscr{L}^{\frac{\pi}{2}-\varepsilon} - \mathscr{L}^{\frac{\pi}{2}+\varepsilon}\right)\left[\tfrac{1}{2}\hat{\phi}^-\right] = \left(\mathscr{S}^{\frac{\pi}{2}-\varepsilon} - \mathscr{S}^{\frac{\pi}{2}+\varepsilon}\right)\tilde{\Theta}^-(\tau)$$

with $\tilde{\Theta}^-(\tau) := \tau^{-1/2} \times \mathscr{B}^{-1}\left[\tfrac{1}{2}\hat{\phi}^-\right]$ and, in the case of Θ^+,

due to the change of branch of $\xi^{-1/2}$ from $\mathrm{e}^{\mathrm{i}(-\frac{3\pi}{2}+\varepsilon)}\mathbb{R}_{\geqslant 0}$ to $\mathrm{e}^{\mathrm{i}(\frac{\pi}{2}+\varepsilon)}\mathbb{R}_{\geqslant 0}$,

$$\Theta^+ = \tau^{-1/2} \times \frac{1}{2}\left(\mathscr{L}^{\frac{\pi}{2}-\varepsilon} + \mathscr{L}^{\frac{\pi}{2}+\varepsilon}\right)\left[\xi^{-1/2}\hat{\phi}^+(\xi)\right] = \mathscr{S}^{\frac{\pi}{2}}_{\mathrm{med}}\tilde{\Theta}^+(\tau)$$

$$\tilde{\Theta}^+(\tau) := \tau^{-1/2} \times \mathscr{B}^{-1}\left[\xi^{-1/2}\hat{\phi}^+(\xi)\right] = \sum_{p\geqslant 0}\frac{1}{p!}L(-2p-\nu,f)\left(\frac{\pi\mathrm{i}}{M}\right)^p\tau^p$$

$$\boxed{\Theta(\tau;\nu,f,M) = \mathrm{cst}\,\langle f\rangle\left(\tfrac{\tau}{\mathrm{i}}\right)^{-\frac{\nu+1}{2}} + \left(\mathscr{S}^{\frac{\pi}{2}-\varepsilon} - \mathscr{S}^{\frac{\pi}{2}+\varepsilon}\right)\tilde{\Theta}^-(\tau) + \mathscr{S}^{\frac{\pi}{2}}_{\mathrm{med}}\tilde{\Theta}^+(\tau)}$$

(Moreover, $F_0^{\mathrm{ev}}(t) = -\frac{1}{2}f^{\mathrm{ev}}(0) + \frac{1}{1-\mathrm{e}^{-Mt}}\sum_{\ell=0}^{M-1} f^{\mathrm{od}}(\ell)\,e^{-\ell t}$

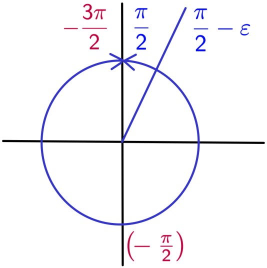
$$F_0^{\mathrm{od}}(t) = \frac{1}{1-\mathrm{e}^{-Mt}}\sum_{\ell=0}^{M-1} f^{\mathrm{ev}}(\ell)\,e^{-\ell t}$$

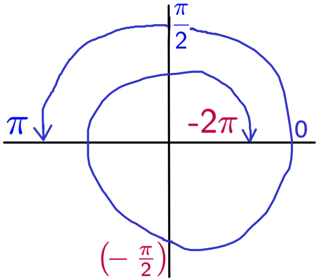
yields the decomposition $F(t) = \frac{\nu!\langle f\rangle}{t^{\nu+1}} + F^{\mathrm{od}}(t) + F^{\mathrm{ev}}(t)$.)

Each $\mathscr{S}^{\theta}\tilde{\Theta}^{\pm}$ is holomorphic in a domain containing $\mathbb{H}$ but much larger...

By varying θ in the ξ-plane, $\mathscr{S}^{\frac{\pi}{2}-\varepsilon}\tilde{\Theta}^{\pm}(\tau)$ extends analytically from

$0 < \arg\tau < \pi$ to negative values of $\arg\tau$:





In the end, $\mathscr{S}^{\frac{\pi}{2}-\varepsilon}\tilde{\Theta}^{\pm}(\tau)$ extends through $\mathbb{R}_{>0}$ to $-2\pi < \arg\tau < \pi$.

Similarly, $\mathscr{S}^{\frac{\pi}{2}-\varepsilon}\tilde{\Theta}^{\pm}(\tau)$ extends through $\mathbb{R}_{<0}$ to $0 < \arg\tau < 3\pi$.

But $\left(\mathscr{S}^{\frac{\pi}{2}-\varepsilon} - \mathscr{S}^{\frac{\pi}{2}+\varepsilon}\right)\tilde{\Theta}^{-}(\tau)$ and $\mathscr{S}^{\frac{\pi}{2}}_{\text{med}}\tilde{\Theta}^{+}(\tau)$ are holo only in $\mathbb{H}$!

Alien derivatives

DFT operator $U_M\colon\ f \mapsto \hat f,\ \ \hat f(n) := \frac{1}{\sqrt{M}} \sum_{\ell \bmod M} f(\ell) \mathrm{e}^{-2\pi \mathrm{i}\ell n/M}$ for $n \in \mathbb{Z}$.

$\mathrm{Res}\big(F_0^{\mathrm{od}}(t), t = \frac{2\pi \mathrm{i} n}{M}\big) = M^{-\frac12} \hat f^{\mathrm{ev}}(n),\ \ \mathrm{Res}\big(F_0^{\mathrm{ev}}(t), t = \frac{2\pi \mathrm{i} n}{M}\big) = M^{-\frac12} \hat f^{\mathrm{od}}(n)$

whence we get the polar parts of F^{od} and F^{ev}, and those of $\hat\phi^-$ and $\hat\phi^+$.

From $\tilde\Theta^+ = \tau^{-1/2} \mathscr{B}^{-1}\big[\xi^{-1/2}\hat\phi^+(\xi)\big]$ and $\tilde\Theta^- = \tau^{-1/2}\mathscr{B}^{-1}\big[\frac12 \hat\phi^-\big]$, we get

$$\nu = 0 \ \Rightarrow\ \Delta_{\xi_n}\tilde\Theta^+ = 2\,\mathrm{e}^{\frac{\mathrm{i}\pi}{4}} \hat f^{\mathrm{od}}(n)\tau^{-\frac12},\ \nu = 1 \ \Rightarrow\ \Delta_{\xi_n}\tilde\Theta^+ = 2\mathrm{i}\,\mathrm{e}^{\frac{3\mathrm{i}\pi}{4}} n \hat f^{\mathrm{ev}}(n)\tau^{-\frac32}$$

$$\nu = 0 \ \Rightarrow\ \Delta_{\xi_n}\tilde\Theta^- = \mathrm{e}^{\frac{\mathrm{i}\pi}{4}} \hat f^{\mathrm{ev}}(n)\tau^{-\frac12},\ \nu = 1 \ \Rightarrow\ \Delta_{\xi_n}\tilde\Theta^- = \mathrm{i}\,\mathrm{e}^{\frac{3\mathrm{i}\pi}{4}} n \hat f^{\mathrm{od}}(n)\tau^{-\frac32}$$

and also, using $\Delta_\omega \frac{\mathrm{d}}{\mathrm{d}\tau} = \big(\frac{\mathrm{d}}{\mathrm{d}\tau} + \omega\tau^{-2}\big)\Delta_\omega$ and

$$\Theta(\tau; 2\mu, f) = \big(\tfrac{M}{\mathrm{i}\pi}\tfrac{\mathrm{d}}{\mathrm{d}\tau}\big)^\mu\,\Theta(\tau; 0, f),\quad \Theta(\tau; 2\mu+1, f) = \big(\tfrac{M}{\mathrm{i}\pi}\tfrac{\mathrm{d}}{\mathrm{d}\tau}\big)^\mu\,\Theta(\tau; 1, f),$$

$$\nu = 2 \ \Rightarrow\ \Delta_{\xi_n}\tilde\Theta^+ = -2\,n^2 \hat f^{\mathrm{od}}(n)\big(\tfrac{\tau}{\mathrm{i}}\big)^{-\frac52} + \tfrac{M}{\pi}\hat f^{\mathrm{od}}(n)\big(\tfrac{\tau}{\mathrm{i}}\big)^{-\frac32}$$

and so on.

Bridge Equations: The directional alien derivative is

$$\mathbb{\Delta}_{\frac{\pi}{2}} = \sum_{\arg\omega=\frac{\pi}{2}} \mathrm{e}^{-\omega/\tau}\Delta_\omega = \sum_{m\geqslant 1} \mathrm{e}^{-\xi_m/\tau}\Delta_{\xi_m}$$

(here, Stokes automorphism $= \mathrm{Id} + \mathbb{\Delta}_{\frac{\pi}{2}}$).

For $\nu = 0$:

$$\mathbb{\Delta}_{\frac{\pi}{2}}\tilde{\Theta}^+ = 2\left(\tfrac{\tau}{\mathrm{i}}\right)^{-\frac{1}{2}}\Theta(-\tau^{-1};0,\hat{f}^{\mathrm{od}}) \quad \mathbb{\Delta}_{\frac{\pi}{2}}\tilde{\Theta}^- = \left(\tfrac{\tau}{\mathrm{i}}\right)^{-\frac{1}{2}}\Theta(-\tau^{-1};0,\hat{f}^{\mathrm{ev}})$$

For $\nu = 1$:

$$\mathbb{\Delta}_{\frac{\pi}{2}}\tilde{\Theta}^+ = 2\mathrm{i}\left(\tfrac{\tau}{\mathrm{i}}\right)^{-\frac{3}{2}}\Theta(-\tau^{-1};1,\hat{f}^{\mathrm{ev}}) \quad \mathbb{\Delta}_{\frac{\pi}{2}}\tilde{\Theta}^- = \mathrm{i}\left(\tfrac{\tau}{\mathrm{i}}\right)^{-\frac{3}{2}}\Theta(-\tau^{-1};1,\hat{f}^{\mathrm{od}})$$

Modularity (here with $S(\tau) = \tau^{-1}$ and both f and $\hat{f}$, but one can also get true modularity with f alone on $\Gamma(2M)$ = principal congruence subgroup):

$$f \text{ even}, \quad \theta(\tau;f) := \tfrac{1}{2}f(0) + \Theta(\tau;0,f) \Rightarrow \theta(\tau;f) = \left(\tfrac{\tau}{\mathrm{i}}\right)^{-\frac{1}{2}}\theta(-\tau^{-1};\hat{f})$$

$$f \text{ odd} \;\Rightarrow\; \Theta(\tau;1,f) = \mathrm{i}\left(\tfrac{\tau}{\mathrm{i}}\right)^{-\frac{3}{2}}\Theta(-\tau^{-1};1,\hat{f}).$$

Suppose $\langle f \rangle = 0$ and $n \mapsto a_n$ odd, so $\Theta(\tau;\nu,f) = \mathscr{S}^{\frac{\pi}{2}}_{\mathrm{med}}\tilde{\Theta}^+(\tau)$ is a half-sum of Borel-Laplace lateral sums.

What about the difference $D(\tau) := \left(\mathscr{S}^{\frac{\pi}{2}-\varepsilon} - \mathscr{S}^{\frac{\pi}{2}+\varepsilon}\right)\tilde{\Theta}^+$? Interest:

$$\Theta(\tau;\nu,f) = \mathscr{S}^{\frac{\pi}{2}-\varepsilon}\tilde{\Theta}^+ - \tfrac{1}{2}D(\tau) = \mathscr{S}^{\frac{\pi}{2}+\varepsilon}\tilde{\Theta}^+ + \tfrac{1}{2}D(\tau).$$

The Bridge Equation gives

$$\{\nu = 0 \text{ and } f \text{ odd}\} \;\Rightarrow\; D(\tau) = 2\left(\tfrac{\tau}{\mathrm{i}}\right)^{-\frac{1}{2}}\Theta(-\tau^{-1};0,\hat{f})$$

$$\{\nu = 1 \text{ and } f \text{ even}\} \;\Rightarrow\; D(\tau) = 2\mathrm{i}\left(\tfrac{\tau}{\mathrm{i}}\right)^{-\frac{3}{2}}\Theta(-\tau^{-1};1,\hat{f}).$$

Rephrasing in terms of modular obstruction, with weight 1/2 or 3/2:

$$f \text{ odd} \;\Rightarrow\; G_\pm(\tau) := \Theta(\tau;0,f) \pm \left(\tfrac{\tau}{\mathrm{i}}\right)^{-\frac{1}{2}}\Theta(-\tau^{-1};0,\hat{f}) = \mathscr{S}^{\frac{\pi}{2}\mp\varepsilon}\,\tilde{\Theta}(\tau;0,f)$$

$$f \text{ even} \;\Rightarrow\; G_\pm(\tau) := \Theta(\tau;1,f) \pm \mathrm{i}\left(\tfrac{\tau}{\mathrm{i}}\right)^{-\frac{3}{2}}\Theta(-\tau^{-1};1,\hat{f}) = \mathscr{S}^{\frac{\pi}{2}\mp\varepsilon}\,\tilde{\Theta}(\tau;1,f)$$

right-hand sides clearly have analytic continuation through $\mathbb{R}$ to the right or to the left of 0 and are asymptotic to $\tilde{\Theta}^+(\tau)$ as $\tau \to 0$.

A consequence for the boundary function:

$\Theta^{\mathrm{nt}}(\alpha;\nu,f)$ exists if and only if $\Theta^{\mathrm{nt}}(-\alpha^{-1};\nu,\hat{f})$ exists and

$$f \text{ odd} \Rightarrow \Theta^{\mathrm{nt}}\big(\tfrac{1}{k};0,f\big) = -\mathrm{i}\,\mathrm{e}^{-\frac{\mathrm{i}\pi}{4}}\,k^{1/2}\,\Theta^{\mathrm{nt}}(-k;0,\hat{f}) + G_+\big(\tfrac{1}{k}\big),$$

$$f \text{ even} \Rightarrow \Theta^{\mathrm{nt}}\big(\tfrac{1}{k};1,f\big) = \mathrm{e}^{\frac{\mathrm{i}\pi}{4}}\,k^{3/2}\,\Theta^{\mathrm{nt}}(-k;1,\hat{f}) - \tfrac{M\langle f\rangle}{2\pi\mathrm{i}}\,k + G_+\big(\tfrac{1}{k}\big),$$

with $G_+\big(\frac{1}{k}\big) \sim_1 \tilde{\Theta}^+\big(\frac{1}{k};\nu,f\big) = \sum_{p\geqslant 0}(-1)^p\frac{L(-2p-\nu,f)}{p!}\big(\frac{\pi\mathrm{i}}{M}\big)^p\big(\frac{1}{k}\big)^p$ as $k\to+\infty$, while the first terms of the right-hand sides contain periodic functions of k with Fourier numbers depending on $\mathrm{supp}(\hat{f})$ (& similar statement at $-\frac{1}{k}$ using G_-).

Work in progress with Li HAN, Yong LI, Shanzhong SUN + Jørgen ANDERSEN, William MISTEGÅRD:

For the GPPV invariant of $\Sigma(p_1, \ldots, p_r)$, we know by [AM2022] the values of the $\mathrm{SL}(2,\mathbb{C})$ Chern-Simons actions and

$\Theta^{\mathrm{nt}}(1/k; f) \leftrightsquigarrow WRT(k)$, $\tilde{\Theta}^{+}(\tau; f) \leftrightsquigarrow$ Ohtsuki series at $\tau \to 0$

with f finite sum of functions $n \mapsto n^{\nu} f_{\nu}(n)$.

Yong LI's talk:

Analyse the DFTs $\hat{f}_{\nu}$.

Their supports (= the Fourier numbers of the $2M$-periodic function $k \mapsto \Theta^{\mathrm{nt}}(-k; \hat{f})$) are related to the $\mathrm{SL}(2,\mathbb{C})$ Chern-Simons actions — more precisely to the $\mathrm{SU}(2)$ *Chern-Simons actions...*

谢谢！