(A diagrammatic calculus for) Interacting scalar QFT on a causal set

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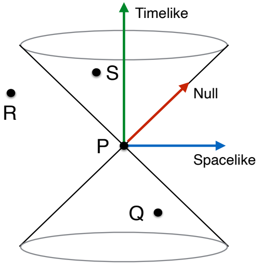
- arxiv:0909.0944 S. Johnston
- arxiv:1703.00610 R.D. Sorkin
- arxiv:1107.0698 R.D. Sorkin
- arXiv:1908.01973 E. Dable-Heath, C.J. Fewster, K. Rejzner, N. Woods
- arXiv:2306.12484 I. Jubb
- arXiv:2402.08555 E. Albertini, FD, A. Nasiri, S. Zalel

Plan

- Lorentzian spacetime: a measure space with an order relation.
- A causal set: a discrete space (counting measure) with an order relation
- Free scalar QFT on a fixed causal set
- Interacting QFT e.g. $\lambda\phi^3$
- Diagrams for time ordered correlators in the in-in formalism: perturbation series expansion terminates if causal set is past-finite
- Towards a comparison with the continuum

Lorentzian spacetime



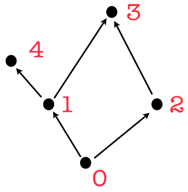
- $Q \prec P, \; P \prec S, \; Q \prec S \ldots$ etc.
- The causal relation is a mathematical order: continuum spacetime is a poset
- Continuum spacetime is a measure space (volume measure)

Theorem (Kronheimer & Penrose, 1967; Hawking, 2014; Malament, 1977)

For every distinguishing Lorentzian spacetime ($d > 2$) the topology, differentiable structure and metric is encoded in the order and the measure.

Causal set = Discrete spacetime

A causal set (causet) $(C, \prec)$ is a **discrete** ordered measure space



- The measure is the counting measure
- **Hasse diagram**: no arrows needed
- $0 \prec 1$, $0 \prec 2$, $1 \prec 3$, $2 \prec 3$, $1 \prec 4$, $0 \prec 3$, $0 \prec 4$.
- Can always think of the *ground set* as $C \subseteq \mathbb{N}$.
- Terminology: minimal, maximal, past, future, up-set, down-set

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- Quantum gravity: Physically, continuum spacetime is an approximation to a causal set and the gravitational path integral is a sum over causal sets
- Regularizing continuum theories: Physically, the causal set is an approximation to continuum.

By analogy with QFT in curved spacetime, we'll look at QFT on a fixed causal set.

Free real scalar QFT on finite globally hyperbolic spacetime M

- The retarded Green function G is unique and $(\Box - m^2)G = \delta$.
- The Pauli-Jordan function is $\Delta := G - G^T$ and G^T equals the advanced Green function.

$$[\phi^x, \phi^y] = i\Delta^{xy} = i(G^{xy} - G^{yx}) \qquad \text{Peierls form of CCR.}$$

- Lemma (Sorkin, 2017): $Kernel(\Box - m^2) = Image(\Delta)$

Proof (one way): Think of $i\Delta$ as an integral operator on functions on M:

$$i\Delta f^x := \int_M dV_y \, i\,\Delta^{xy} \, f^y$$

$i\Delta$ is Hermitian, $\overline{(i\Delta)} = -i\Delta$ and so the nonzero eigenvalues come in pairs $\pm\lambda$:

$$i\Delta\, u := \lambda\, u\,, \quad i\Delta\, \overline{u} := -\lambda\, \overline{u}\,.$$

If u satisfies $\qquad i\Delta\, u = \lambda\, u\,, \quad \lambda \neq 0\,,$

then, $\qquad (\Box - m^2)\, i\Delta\, u = \lambda\, (\Box - m^2)\, u$

$$i\,(\Box - m^2)\,(G - G^T)\, u = \lambda\, (\Box - m^2)\, u$$

$0 = \lambda(\Box - m^2)\, u\,,$ so u is a solution.

Free real scalar QFT on finite C (Johnston, 2009)

- Let $|C| = N$. A scalar field history is a vector in $\mathbb{R}^N$.
- Assume we **do**, somehow, have the retarded Green function G^{xy}, an $N \times N$ retarded (lower triangular) matrix on C (more later).
- Then the Pauli-Jordan function is still $\Delta^{xy} = G^{xy} - G^{T\,xy}$ and we require

$$[\phi^x, \phi^y] = i\Delta^{xy} \qquad \text{CCR}$$

- Motivated by Lemma: Hilbert space of solutions $\rightarrow$ image of $i\Delta$. i.e. span of eigenvectors

$$i\Delta\; u_k := \lambda_k\; u_k\,, \quad i\Delta\; \overline{u}_k := -\lambda_k\; \overline{u}_k\,, \quad \lambda_k > 0\,.$$

- $$\phi = \sum_k \sqrt{\lambda_k}\;(a_k\, u + a_k^\dagger\, \overline{u}_k)\,, \qquad [a_k, a_l^\dagger] = \delta_{kl}.$$
- Define $|SJ\rangle$, the *Sorkin-Johnston* state, to be the state annihilated by the a_k's. Hilbert space is then the usual Fock space.
- $|SJ\rangle$ is a distinguished "ground state". No need for the existence of timelike Killing vector. In contrast to the usual philosophy in Algebraic QFT.
- The free theory is a finite set of oscillator modes.

(Sorkin, 2011; Sorkin, 2017): $G \longrightarrow i\Delta \longrightarrow W \longrightarrow \phi$

$\lambda\phi^3$ theory on $C = \{1, 2, \dots N\}$ (in-in formalism)

The interacting theory was first defined in path integral form (Sorkin, 2011).

Let the interaction region be a subset of C, excluding (at least) the minimal elements of C. The Heisenberg and interaction pictures coincide on the minimal elements of C where the state is $|SJ\rangle$ which is the in-state.
Let ϕ denote the interaction picture field. The local interaction Hamiltonian operator is $H^x = \frac{g}{3!}(\phi^x)^3$.

The Heisenberg picture fields are defined by (Albertini, 2021; Jubb, 2023)

$$\Phi^x = e^{iH^1} e^{iH^2} \dots e^{iH^{x-1}} \phi^x e^{-iH^{x-1}} \dots e^{-iH^2} e^{-iH^1}$$

where $H^y = \dfrac{g}{3!}(\phi^y)^3$. Then, $\Phi^x = \phi^x +$ a polynomial in the ϕ's in the past of x.

Sketch of proof: use spacelike commutativity to reorder the unitaries so that the ones spacelike to x are next to x. Commute the spacelike unitaries through ϕ^x and cancel them off. Then use $e^{-B} A e^B = A + [A, B] + \dfrac{1}{2!}[[A, B], B] + \dots$ iteratively on the rest.

An interacting QFT on a causal set was also defined in (Dable-Heath *et al.*, 2020) using deformation quantization. I believe they are the same theory!

Algebras

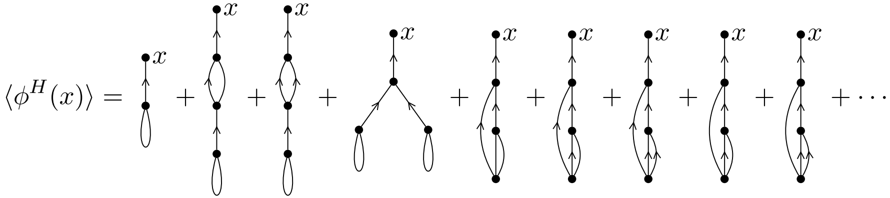
The algebra $\mathfrak{A}$ generated by all the Φ's equals the algebra generated by all the ϕ's.

Proof by induction: The Heisenberg fields are finite polynomials in the interaction picture fields. For inclusion the other way, argue level by level. For all minimal elements, $\phi = \Phi$. Now consider an x that has only minimal elements in its past. $\Phi^x = \phi^x$ + a polynomial in ϕ's at minimal elements $= \phi^x$ + a polynomial in Φ's. So Φ^x is a polynomial in ϕ's. And so on.

- Result holds for subalgebras associated to subcausets that are down sets.
- All algebras are finitely generated.
- Haag duality does not hold in general: depends on C and what subcausets one allows for local algebras.
- Heisenberg operators at spacelike elements commute. However, the expected relationships between algebras in subcausets (for example domains of dependence) have to be examined.

Causal Diagrams ((Albertini *et al.*, 2024) after (Dickinson *et al.*, 2014))

Simplest example: the VEV in the in-state to order 3 (assuming $<\phi>=0$)



- n vertices (excluding x) at order n,
- each vertex is connected to x by at least one directed path and no closed directed cycles: explicitly causal.

- each vertex z equals $i\frac{g}{3!}$ and is summed over $Past(x)$.
- each arrowed edge $a \to b$ equals $-iG_{ba}$, the retarded Green fn.,
- each undirected edge a —— b equals Δ^F_{ab}, the Feyman propagator,
- Symmetry factors from ways of connecting the half-legs,
- Divide by $|Aut(diagram)|$, the number of automorphisms.

On C, $\Delta^F_{zz} < \infty$: the perturbative series terminates and is finite.

Exploring the Relationship with Continuum QFT

- Consider a causet Poisson sprinkled into some finite globally hyperbolic spacetime.
- Do simulations to approximate the sum over diagrams for examples of in-in time ordered correlators. And compare to the continuum.
- To do this in practice, need the causet analogue of G for Klein Gordon operator.

Examples:

- 2d Minkowski space, $m = 0$:
  $G^{xy} \propto R^{xy}$ where R is the graph adjacency matrix: $R^{xy} = 1$ if $y \prec x$ and 0 otherwise.
- 4d Minkowski space, $m = 0$:
  $G^{xy} \propto L^{xy}$ where L is the nearest-neighbour adjacency matrix: $K^{xy} = 1$ if $y \prec x$ is a link, and 0 otherwise.
- Can use fact that deSitter space is conformally flat to deduce causet G.
- If one knows the massless G one can write down the massive G as a finite expansion in convolutions.

Discussion

- All goes through for any polynomial self-interaction – can also vary.
- Termination of perturbation series comes from past-finiteness: cosmology
- There exists a path integral (sum-over-histories) formulation, generating functionals.
- Compare to continuum and confront the "non-continuum" modes

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- Renormalisation and the continuum limit of the theory as the density of sprinkling $\to \infty$.
- G for other spacetimes. Other sorts of fields.
- Back reaction on the causal set: towards quantum gravity

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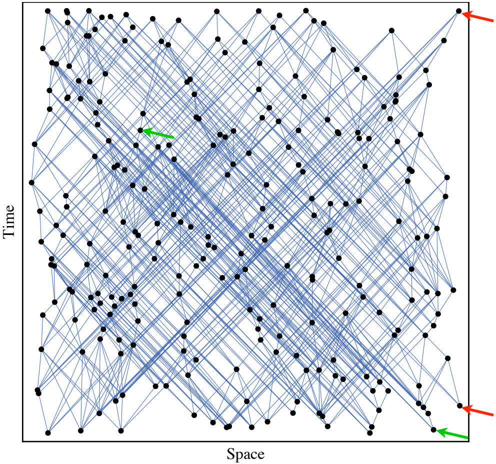
Typical sprinkled causet in 2D Minkowski space



- 2D Minkowski
- Hasse diagram (of covering relations)
- highly non-isotropic: null directions
- highly nonlocal [BB]
- essential structure of Lorentzian geometry **revealed**

- Not a cuddly low valance graph but a nonlocal beast!
- Continuum Lorentzian spacetimes are, if not actually nonlocal themselves, then teetering on the edge of being nonlocal.