A calculus for magnetic pseudodifferential super operators

Gihyun Lee

Ghent University

Gihyun.Lee@ugent.be

Algebraic, Analytic, Geometric Structures Emerging from Quantum Field Theory

@Sichuan University, Chengdu, China

March 12, 2024

Outline

This talk is based on the joint works with M. Lein [LL22, LL24].

- Von Neumann algebras
- The (magnetic) Weyl calculus
- Motivation
- The (magnetic) super Weyl calculus
- Future Projects

Von Neumann algebras

Let $\mathscr{H}$ be a separable Hilbert space and $\mathcal{S} \subseteq \mathscr{L}(\mathscr{H})$. The commutant of $\mathcal{S}$ is

$$\mathcal{S}' := \{T \in \mathscr{L}(\mathscr{H});\ ST = TS \text{ for all } S \in \mathcal{S}\}.$$

We denote the bicommutant $(\mathcal{S}')'$ by $\mathcal{S}''$.

Definition

A von Neumann algebra is a unital $*$-subalgebra $\mathcal{N}$ of $\mathscr{L}(\mathscr{H})$ such that $\mathcal{N}'' = \mathcal{N}$.

The two most basic examples:

- $\mathcal{N} = \mathscr{L}(\mathscr{H})$.
- $\mathcal{N} = L_\infty(X)$, where $(X, \mathfrak{M}, \mu)$ is a σ-finite measure space.

Traces

$\mathcal{N}_+ := \mathcal{N} \cap \mathscr{L}(\mathscr{H})_+.$

Definition

A trace on $\mathcal{N}$ is a map $\tau : \mathcal{N}_+ \to [0, +\infty]$ satisfying

$$\tau(c_1 S + c_2 T) = c_1 \tau(S) + c_2 \tau(T) \qquad \forall S, T \in \mathcal{N}_+ \ \forall c_1, c_2 \in \mathbb{R}_+,$$
$$\tau(USU^*) = \tau(S) \qquad \forall S \in \mathcal{N}_+ \ \forall \text{unitary elements } U \in \mathcal{N}.$$

Traces

Definition

- τ is called faithful if $S \in \mathcal{N}_+$ and $\tau(S) = 0$ implies $S = 0$.
- τ is called normal if for each increasing bounded net $(S_\alpha) \subset \mathcal{N}_+$ such that $S := \sup_\alpha S_\alpha \in \mathcal{N}_+$, we have $\sup_\alpha \tau(S_\alpha) = \tau(S)$.
- τ is called semifinite if for each projection $P \in \mathcal{N}$, there exists an increasing net of projections $(P_\alpha) \subset \mathcal{N}$ such that $\tau(P_\alpha) < \infty \ \forall \alpha$ and $P = \inf\{Q \in \mathcal{N};\ Q \text{ is a projection such that } P_\alpha \leq Q \ \forall \alpha\}$.

Noncommutative L_p-spaces

Let τ be a faithful, normal, semifinite (f.n.s.) trace on $\mathcal{N}$. We denote the set of τ-measurable operators affiliated with $\mathcal{N}$ by $\mathfrak{M}(\mathcal{N},\tau)$. For each $1 \leq p < \infty$ we define

$$\mathscr{L}_p(\mathcal{N}) = \mathscr{L}_p(\mathcal{N},\tau) := \left\{ T \in \mathfrak{M}(\mathcal{N},\tau);\ \tau\big(|T|^p\big)^{\frac{1}{p}} < \infty \right\}.$$

The two most basic examples:

- If $\mathcal{N} = \mathscr{L}(\mathscr{H})$, then the standard operator trace Tr is an f.n.s. trace on $\mathscr{L}(\mathscr{H})$. We have $\mathscr{L}_p(\mathcal{N}) = \mathscr{L}_p$, where $\mathscr{L}_p$ is the ideal of Schatten p-class operators.
- If $\mathcal{N} = L_\infty(X)$, then $\int d\mu$ is an f.n.s. trace on $L_\infty(X)$. We have $\mathscr{L}_p(\mathcal{N}) = L_p(X)$.

Canonical commutation relations

Phase space notations

- $\Xi = T^*\mathbb{R}^d = \mathbb{R}^d \times \mathbb{R}^{d^*}$ (the phase space)
- Elements of Ξ will be denoted by $X = (x, \xi)$, $Y = (y, \eta)$, $Z = (z, \zeta)$ with space components $x, y, z \in \mathbb{R}^d$ and momentum components $\xi, \eta, \zeta \in \mathbb{R}^d$.
- Ξ is endowed with the symplectic form $\sigma(X, Y) := \xi \cdot y - x \cdot \eta$.

Canonical commutation relations

The canonical commutation relations (CCR),

$$-\mathrm{i}[Q_j, Q_k] = 0, \qquad -\mathrm{i}[P_j, P_k] = 0, \qquad -\mathrm{i}[Q_j, P_k] = \varepsilon\delta_{jk}.$$

can be equivalently reformulated in terms of the family of unitary operators $\{w(X)\}_{X\in\Xi}$, called the Weyl system.

$$w(X) := \mathrm{e}^{-\mathrm{i}\sigma(X,(Q,P))} = \mathrm{e}^{-\mathrm{i}(\xi\cdot Q - x\cdot P)}.$$

We have

- $w(X)w(Y) = \mathrm{e}^{+\mathrm{i}\frac{\varepsilon}{2}\sigma(X,Y)}w(X+Y)$.
- $\{w(X);\, X\in\Xi\}'' = \mathscr{L}(L_2(\mathbb{R}^d))$.

The magnetic setting

- Consider a charged particle moving in $\mathbb{R}^d$ subjected to the magnetic field B. Pick a vector potential A so that $B = \mathrm{d}A$.
- Assume one of the following two assumptions on the magnetic field B and associated vector potential A.
  - (PB) All components of B belong to $\mathcal{C}^\infty_{\mathrm{u,pol}}(\mathbb{R}^d)$ and all components of A belong to $\mathcal{C}^\infty_{\mathrm{pol}}(\mathbb{R}^d)$.
  - (B) All components of B belong to $\mathcal{C}^\infty_{\mathrm{b}}(\mathbb{R}^d)$ and all components of A belong to $\mathcal{C}^\infty_{\mathrm{pol}}(\mathbb{R}^d)$.
- We introduce two small parameters [Le10], which are crucial for asymptotic expansions arise in the study of semiclassical limits.:
  - The coupling of the charge to the magnetic field λ.
  - A semiclassical parameter ε.

The magnetic Weyl system

- In the magnetic setting, the building block observables are the position operators $Q = (Q_1, \dots, Q_d)$ and the kinetic momentum operators $P^A = (P_1^A, \dots, P_d^A)$ defined by

$$Q_j f(x) := \varepsilon x_j f(x), \qquad P_j^A f(x) := -\mathrm{i}\frac{\partial f}{\partial x_j}(x) - \lambda A_j(\varepsilon x) f(x).$$

- Q and P^A satisfy the commutation relations:

$$-\mathrm{i}[Q_j, Q_k] = 0, \qquad -\mathrm{i}[P_j^A, P_k^A] = \varepsilon\lambda B_{jk}(Q), \qquad -\mathrm{i}[Q_j, P_k^A] = \varepsilon\delta_{jk}.$$

The magnetic Weyl system

These relations can be encoded into the magnetic Weyl system $\{w^A(X)\}_{X\in\Xi}$ defined by

$$w^A(X) := \mathrm{e}^{-\mathrm{i}\sigma(X,(Q,P^A))} = \mathrm{e}^{-\mathrm{i}(\xi\cdot Q - x\cdot P^A)}.$$

We have

- $w^A(X)w^A(Y) = \mathrm{e}^{+\mathrm{i}\frac{\varepsilon}{2}\sigma(X,Y)}\omega^B(Q;x,y)w^A(X+Y)$.
- $\{w^A(X);\, X\in\Xi\}'' = \mathscr{L}(L_2(\mathbb{R}^d))$.

Magnetic Weyl quantization

Definition (Magnetic Weyl quantization [MP04])

For all $f \in \mathcal{S}(\Xi)$ and magnetic fields satisfying (PB), we define

$$\mathrm{op}^A(f) := \frac{1}{(2\pi)^d} \int_\Xi \mathrm{d}X \, (\mathcal{F}_\sigma f)(X) \, w^A(X).$$

Here $\mathcal{F}_\sigma f$ is the symplectic Fourier transform of f defined by

$$(\mathcal{F}_\sigma f)(X) := \frac{1}{(2\pi)^d} \int_\Xi \mathrm{d}X' \mathrm{e}^{\mathrm{i}\sigma(X,X')} f(X').$$

Magnetic Weyl quantization inherits gauge-covariance from the magnetic Weyl system.

$$\mathrm{op}^{A+\mathrm{d}\chi}(f) = \mathrm{e}^{+\mathrm{i}\lambda\chi(Q)} \, \mathrm{op}^A(f) \, \mathrm{e}^{-\mathrm{i}\lambda\chi(Q)}.$$

The magnetic Weyl product

Given $f, g \in \mathcal{S}(\Xi)$, we define

$$(f \star^B g)(X) := \frac{1}{(2\pi)^{2d}} \int_\Xi \mathrm{d}Y \int_\Xi \mathrm{d}Z \, \mathrm{e}^{+\mathrm{i}\sigma(X,Y+Z)} \, e^{+\mathrm{i}\frac{\varepsilon}{2}\sigma(Y,Z)}.$$

$$\mathrm{e}^{-\mathrm{i}\frac{\lambda}{\varepsilon}\Gamma^B(\langle x-\frac{\varepsilon}{2}(y+z), x+\frac{\varepsilon}{2}(y-z), x+\frac{\varepsilon}{2}(y+z)\rangle)} \, (\mathcal{F}_\sigma f)(Y) \, (\mathcal{F}_\sigma g)(Z).$$

Then we have $\mathrm{op}^A(f) \, \mathrm{op}^A(g) = \mathrm{op}^A(f \star^B g)$.

The magnetic Weyl product

The class of Hörmander symbols of order $m \in \mathbb{R}$ and type (ρ, δ) with $0 \leq \rho \leq \delta \leq 1$ is the Fréchet space defined by

$$S^m_{\rho,\delta}(\Xi) := \left\{ f \in \mathcal{C}^\infty(\Xi) \mid \sup_{X \in \Xi} \langle \xi \rangle^{-m-|a|\delta+|\alpha|\rho} \left| \partial_x^a \partial_\xi^\alpha f(X) \right| < \infty \ \forall a, \alpha \in \mathbb{N}_0^d \right\}.$$

Theorem ([IMP07])

Let $m_1, m_2 \in \mathbb{R}$. For magnetic fields B satisfying (B) and $0 \leq \delta \leq \rho \leq 1$, the magnetic Weyl product $\star^B$ gives rise to a continuous bilinear map,

$$\star^B : S^{m_1}_{\rho,\delta}(\Xi) \times S^{m_2}_{\rho,\delta}(\Xi) \longrightarrow S^{m_1+m_2}_{\rho,\delta}(\Xi).$$

L_2-boundedness

Theorem ([IMP07])

Suppose that B satisfies (B) and $f \in S^0_{\rho,\delta}(\Xi)$, $0 \leq \delta < \rho \leq 1$. Then $\mathrm{op}^A(f)$ gives rise to a bounded linear operator from $L_2(\mathbb{R}^d)$ to itself.

Asymptotic expansion

Furthermore, we can asymptotically expand the product $f \star^B g$ in ε and λ:

Theorem ([Le10])

Let $m_1, m_2 \in \mathbb{R}$ and assume B satisfies (B), $f \in S^{m_1}_{\rho,0}(\Xi)$ and $g \in S^{m_2}_{\rho,0}(\Xi)$. Then there is $N \in \mathbb{N}_0$ such that

$$f \star^B g = \sum_{n=0}^{N} \sum_{k=0}^{n} \varepsilon^n \lambda^k (f \star^B g)_{(n,k)} + \tilde{R}_N,$$

where $(f \star^B g)_{(n,k)} \in S^{m_1+m_2-(n+k)\rho}_{\rho,0}(\Xi)$, $\tilde{R}_N \in S^{m_1+m_2-(N+1)\rho}_{\rho,0}(\Xi)$ and the semi-norms of $\tilde{R}_N$ is sufficiently small.

Motivation

- Let H be a Hamiltonian affiliated to a von Neumann algebra $\mathcal{N}$ endowed with a f.n.s. trace τ. In quantum mechanics, density operators evolve according to the Liouville equation,

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(t) = L_H\big(\rho(t)\big) := -\mathrm{i}[H, \rho(t)], \qquad \rho(t_0) \in \mathscr{L}_1(\mathcal{N}).$$

- L_H maps linear operators to linear operators; physicists refer to those as super operators.

Motivation

- This kind of algebraic approach was useful in many cases, e.g.,
  - systems from statistical mechanics in the thermodynamic limit [BR17].
  - linear response theory [DL17].
- Algebraic approach
  - Makes the mathematical descriptions for various systems rigorous.
  - But it involves technical assumptions on operators which seem difficult to verify for concrete models.

Motivation

- Pseudodifferential theory
  - The way of assigning functions (called symbols) to operators.
  - Translates properties of symbols to associated pseudodifferential operators.
- We can put $\rho(t)$ into the framework of pseudodifferential theory (the magnetic Weyl calculus). But what about L_H or other super operators?

Goal

The goal of this talk is to introduce the newly constructed pseudodifferential calculus for super operators, which is a natural receptacle of L_H and other relevant super operators.

Magnetic pseudodifferential super operators

- $\Xi^2 := \Xi \times \Xi$ (the doubled phase space).
- Elements of Ξ^2 will be denoted by $\mathbf{X} = (X_L, X_R), \mathbf{Y} = (Y_L, Y_R), \mathbf{Z} = (Z_L, Z_R)$. Here we follow the same convention as before for variables in Ξ, e.g., $X_L = (x_L, \xi_L)$ and $Y_R = (y_R, \eta_R)$ with $x_L, y_R \in \mathbb{R}^d$ and $\xi_L, \eta_R \in {\mathbb{R}^d}^*$.
- We endow Ξ^2 with the symplectic form Σ defined by

$$\Sigma(\mathbf{X}, \mathbf{Y}) := \sigma(X_L, Y_L) + \sigma(X_R, Y_R).$$

- The symplectic Fourier transform $\mathcal{F}_\Sigma F$ of a function F on Ξ^2 is defined by

$$(\mathcal{F}_\Sigma)(\mathbf{X}) := \frac{1}{(2\pi)^{2d}} \int_{\Xi^2} \mathrm{d}\mathbf{X}' \, \mathrm{e}^{+\mathrm{i}\Sigma(\mathbf{X},\mathbf{X}')} \, F(\mathbf{X}').$$

Given a function $g \in \mathcal{S}(\Xi)$, set $\hat{g}^A := \mathrm{op}^A(g)$.

Definition (Magnetic Super Weyl System)

For $\mathbf{X} \in \Xi^2$, we define

$$W^A(\mathbf{X})\, \hat{g}^A \equiv W^A(X_L, X_R)\, \hat{g}^A := w^A(X_L)\; \hat{g}^A\, w^A(X_R).$$

Definition (Magnetic Pseudodifferential Super Operator)

The magnetic pseudodifferential super operator $\mathrm{Op}^A(F)$ associated with a function $F \in \mathcal{S}(\Xi^2)$ is defined by

$$\mathrm{Op}^A(F)\, \hat{g}^A := \frac{1}{(2\pi)^{2d}} \int_{\Xi^2} \mathrm{d}\mathbf{X}\, (\mathcal{F}_\Sigma F)(\mathbf{X})\, W^A(\mathbf{X})\, \hat{g}^A.$$

Magnetic pseudodifferential super operators

If F is a function of the form $F(\mathbf{X}) = f_L(X_L) f_R(X_R)$, $f_L, f_R \in \mathcal{S}(\Xi)$, then we get

$$\begin{aligned}\mathrm{Op}^A(F)\,\hat{g}^A &= \frac{1}{(2\pi)^{2d}} \int_\Xi \mathrm{d}X_L \int_\Xi \mathrm{d}X_R\, (\mathcal{F}_\sigma f_L)(X_L)\, (\mathcal{F}_\sigma f_R)(X_R) \cdot \\ &\qquad w^A(X_L)\, \hat{g}^A\, w^A(X_R) \\ &= \mathrm{op}^A(f_L)\, \hat{g}^A\, \mathrm{op}^A(f_R).\end{aligned}$$

- Example: $L_h(X_L, X_R) := -\mathrm{i}h(X_L) + \mathrm{i}h(X_R)$ is the symbol of the Liouville super operator $\hat{g}^A \mapsto -\mathrm{i}[\hat{h}^A, \hat{g}^A] = -\mathrm{i}\hat{h}^A \hat{g}^A + \mathrm{i}\hat{g}^A \hat{h}^A$.

The magnetic semi-super Weyl product

Given $F \in \mathcal{S}(\Xi^2)$ and $g \in \mathcal{S}(\Xi)$, can $\mathrm{Op}^A(F)\,\hat{g}^A = \mathrm{Op}^A(F)\,\mathrm{op}^A(g)$ be seen as a magnetic Weyl pseudodifferential operator? If so, then what is its symbol? Answer:

$$F \bullet^B g(X) := \frac{1}{(2\pi)^{3d}} \int_{\Xi^2} \mathrm{d}\mathbf{Y} \int_{\Xi} \mathrm{d}Z\, \mathrm{e}^{+\mathrm{i}\sigma(X, Y_L + Y_R + Z)}\, \mathrm{e}^{+\mathrm{i}\frac{\varepsilon}{2}\sigma(Y_L + Z, Y_R + Z)}\, \mathrm{e}^{-\mathrm{i}\lambda\Omega^B(x, y_L, y_R, z)}\, (\mathcal{F}_\Sigma F)(\mathbf{Y})\, (\mathcal{F}_\sigma g)(Z).$$

Proposition (L.-Lein)

Assume B satisfies (PB). Then the following holds.

(1) $F \bullet^B g \in \mathcal{S}(\Xi)$ *and* $\mathrm{Op}^A(F)\,\mathrm{op}^A(g) = \mathrm{op}^A(F \bullet^B g)$.

(2) *For the special case* $F(\mathbf{X}) = f_L(X_L) f_R(X_R)$, $f_L, f_R \in \mathcal{S}(\Xi)$, *the semi-super product reduces to*

$$F \bullet^B g = f_L \star^B g \star^B f_R.$$

The magnetic super Weyl product

Given $F, G \in \mathcal{S}(\Xi^2)$, can $\mathrm{Op}^A(F)\,\mathrm{Op}^A(G)$ be seen as a magnetic Weyl pseudodifferential super operator? If so, then what is its symbol? Answer:

$$F \sharp^B G(\mathbf{X}) := \frac{1}{(2\pi)^{4d}} \int_{\Xi^2} \mathrm{d}\mathbf{Y} \int_{\Xi^2} \mathrm{d}\mathbf{Z}\, \mathrm{e}^{+\mathrm{i}\Sigma(\mathbf{X},\mathbf{Y}+\mathbf{Z})}\, \mathrm{e}^{+\mathrm{i}\frac{\varepsilon}{2}\Sigma(r(\mathbf{Y}),\mathbf{Z})}.$$

$$\mathrm{e}^{-\mathrm{i}\lambda\gamma^B(x_L,y_L,z_L)}\, \mathrm{e}^{-\mathrm{i}\lambda\gamma^B(x_R,z_R,y_R)}\, (\mathcal{F}_\Sigma F)(\mathbf{Y})\, (\mathcal{F}_\Sigma G)(\mathbf{Z}).$$

Here we have set $r(\mathbf{Y}) \equiv r(Y_L, Y_R) := (-Y_L, Y_R)$.

Proposition (L.-Lein)

Assume B satisfies (PB). Then we have $F \sharp^B G \in \mathcal{S}(\Xi^2)$ and $\mathrm{Op}^A(F)\,\mathrm{Op}^A(G) = \mathrm{Op}^A(F \sharp^B G)$.

Hörmander Double Symbol Classes

Definition (Hörmander super symbol classes $S^{m_L,m_R}_{\rho,\delta}(\Xi^2)$)

Let $m_L, m_R \in \mathbb{R}$, $0 \le \delta \le \rho \le 1$ and $\delta < 1$. $S^{m_L,m_R}_{\rho,\delta}(\Xi^2)$ is the Fréchet space consists of functions $F \in \mathcal{C}^\infty(\Xi^2)$ such that, for all $a_L, a_R, \alpha_L, \alpha_R \in \mathbb{N}_0^d$, there exists $C_{a_L a_R \alpha_L \alpha_R} > 0$ such that, for all $\mathbf{X} = (X_L, X_R) \in \Xi^2$, we have

$$\left| \partial^{a_L}_{x_L} \partial^{\alpha_L}_{\xi_L} \partial^{a_R}_{x_R} \partial^{\alpha_R}_{\xi_R} F(\mathbf{X}) \right| \le C_{a_L a_R \alpha_L \alpha_R} \langle \xi_L \rangle^{m_L - |\alpha_L|\rho + |a_L|\delta} \langle \xi_R \rangle^{m_R - |\alpha_R|\rho + |a_R|\delta}.$$

Hörmander double symbol classes

Definition (Hörmander super symbol classes $S^m_{\rho,\delta}(\Xi^2)$)

Let $m \in \mathbb{R}$, $0 \leq \delta \leq \rho \leq 1$ and $\delta < 1$. $S^m_{\rho,\delta}(\Xi^2)$ is the Fréchet space consists of functions $F \in \mathcal{C}^\infty(\Xi^2)$ such that, for all $a_L, a_R, \alpha_L, \alpha_R \in \mathbb{N}_0^d$, there exists $C_{a_L a_R \alpha_L \alpha_R} > 0$ such that, for all $\mathbf{X} = (X_L, X_R) \in \Xi^2$, we have

$$\left| \partial_{x_L}^{a_L} \partial_{\xi_L}^{\alpha_L} \partial_{x_R}^{a_R} \partial_{\xi_R}^{\alpha_R} F(\mathbf{X}) \right| \leq C_{a_L a_R \alpha_L \alpha_R} \langle (\xi_L, \xi_R) \rangle^{m - (|\alpha_L| + |\alpha_R|)\rho + (|a_L| + |a_R|)\delta}.$$

The semi-super product of Hörmander symbols

Using oscillatory integral techniques, we can prove the following result. Here we assume B satisfies (B), $0 \leq \rho \leq 1$ and $0 < \varepsilon, \lambda \leq 1$.

Proposition (L.-Lein)

The map $(F, g) \mapsto F \bullet^B g$ gives rise to continuous bilinear maps,

$$\bullet^B : S^{m}_{\rho,0}(\Xi^2) \times S^{m'}_{\rho,0}(\Xi) \longrightarrow S^{m+m'}_{\rho,0}(\Xi)$$
$$\bullet^B : S^{m_L, m_R}_{\rho,0}(\Xi^2) \times S^{m}_{\rho,0}(\Xi) \longrightarrow S^{m+m_L+m_R}_{\rho,0}(\Xi).$$

Again, by applying oscillatory integral techniques, we get the following results. We assume B satisfies (B), $0 \leq \rho \leq 1$ and $0 < \varepsilon, \lambda \leq 1$.

Proposition (L.-Lein)

The map $(F, G) \mapsto F \sharp^B G$ gives rise to continuous bilinear maps,

$$\sharp^B : S^{m}_{\rho,0}(\Xi^2) \times S^{m'}_{\rho,0}(\Xi^2) \longrightarrow S^{m+m'}_{\rho,0}(\Xi^2)$$

$$\sharp^B : S^{m_L,m_R}_{\rho,0}(\Xi^2) \times S^{m'_L,m'_R}_{\rho,0}(\Xi^2) \longrightarrow S^{m_L+m'_L,m_R+m'_R}_{\rho,0}(\Xi^2).$$

Asymptotic expansion

Both $\bullet^B$ and $\sharp^B$ can be asymptotically expanded in ε and λ:

Theorem (L.-Lein)

Assume B satisfies (B). Then the following holds.

(1) *Let $F \in S^m_{\rho,0}(\Xi^2)$ and $g \in S^{m'}_{\rho,0}(\Xi)$. Then there is $N \in \mathbb{N}_0$ such that*

$$F \bullet^B g = \sum_{n=0}^{N} \sum_{k=0}^{n} \varepsilon^n \lambda^k (F \bullet^B g)_{(n,k)} + \tilde{R}_N, \qquad (*)$$

where $(F \bullet^B g)_{(n,k)} \in S^{m+m'-\rho(n+k)}_{\rho,0}(\Xi)$, $\tilde{R}_N \in S^{m+m'-\rho(N+1)}_{\rho,0}(\Xi)$ and the semi-norms of $\tilde{R}_N$ is sufficiently small.

(2) *If $F \in S^{m_L,m_R}_{\rho,0}(\Xi^2)$ and $g \in S^m_{\rho,0}(\Xi)$, then we also have an asymptotic expansion of $F \bullet^B g$ as in (*). In this case, $(F \bullet^B g)_{(n,k)} \in S^{m+m_L+m_R-\rho(n+k)}_{\rho,0}(\Xi)$, $\tilde{R}_N \in S^{m+m_L+m_R-\rho(N+1)}_{\rho,0}(\Xi)$ and the semi-norms of $\tilde{R}_N$ is sufficiently small.*

Asymptotic expansion

Theorem (L.-Lein)

Assume B satisfies (B) and let $F \in S^{m}_{\rho,0}(\Xi^2)$ and $G \in S^{m'}_{\rho,0}(\Xi^2)$. Then there is $N \in \mathbb{N}_0$ such that

$$F \sharp^B G = \sum_{n=0}^{N} \sum_{k=0}^{n} \varepsilon^n \lambda^k (F \sharp^B G)_{(n,k)} + \tilde{R}_N,$$

where $(F \sharp^B G)_{(n,k)} \in S^{m+m'-\rho(n+k)}_{\rho,0}(\Xi^2)$, $\tilde{R}_N \in S^{m+m'-\rho(N+1)}_{\rho,0}(\Xi^2)$ and the semi-norm of $\tilde{R}_N$ is sufficiently small.

$\mathscr{L}_2$-boundedness

Theorem (L.-Lein)

Suppose B satisfies (B) and let $F \in S^0_{\rho,0}(\Xi^2)$, $0 \leq \rho \leq 1$. Then $\mathrm{Op}^A(F)$ gives rise to a bounded linear operator from $\mathscr{L}_2$ to itself.

Our proof is based on the Parseval frame method which can be found in [CHP24].

Noncommutative Euclidean spaces

- $\theta :=$ a real skew-symmetric $n \times n$ matrix.
- For each $t \in \mathbb{R}^d$, we define the unitary operator $\lambda_\theta(t)$ on $L_2(\mathbb{R}^d)$ by letting
$$\big(\lambda_\theta(t)f\big)(x) = \mathrm{e}^{\mathrm{i}t\cdot x} f\big(x - \tfrac{1}{2}\theta t\big).$$
- We have
$$\lambda_\theta(t)\lambda_\theta(s) = \mathrm{e}^{\frac{\mathrm{i}}{2}t\cdot\theta s}\lambda_\theta(t+s) \qquad \forall t, s \in \mathbb{R}^d.$$

Definition (NC Euclidean spaces)

$L_\infty(\mathbb{R}^d_\theta) := \{\lambda_\theta(t);\ t \in \mathbb{R}^d\}''$.

The structure theorem

Suppose that $k := \mathrm{rk}\theta/2$ and $l = \dim\ker\theta$. In particular, we have $n = 2k + l$. By using the spectral theorem we can find an orthogonal matrix Q satisfying

$$Q^{-1}\theta Q = \begin{pmatrix} & & & & -\theta_1 & & & & \\ & & & & & -\theta_2 & & & \\ & & & & & & \ddots & & \\ & & & & & & & -\theta_k & \\ \theta_1 & & & & & & & & \\ & \theta_2 & & & & & & & \\ & & \ddots & & & & & & \\ & & & \theta_k & & & & & \\ & & & & & & & & 0_l \end{pmatrix}.$$

The structure theorem

Theorem (see, e.g., [GIV06, Ri93])

There is an isomorphism between von Neumann algebras,

$$L_\infty(\mathbb{R}^d_\theta) \simeq \mathscr{L}(L_2(\mathbb{R}^k))\bar{\otimes} L_\infty(\mathbb{R}^l).$$

- In particular, if θ has full rank, then $L_\infty(\mathbb{R}^d_\theta) \simeq \mathscr{L}(L_2(\mathbb{R}^{d/2}))$.
- On the other hand, if $\theta = 0$, then $L_\infty(\mathbb{R}^d_\theta) \simeq L_\infty(\mathbb{R}^d)$.

Future Projects

- The boundedness of $\mathrm{Op}^A(F)$ on $\mathscr{L}_p$ for general p?
- Generalization to the case of more general von Neumann algebras?
- Generalization to the case of noncommutative Euclidean spaces (i.e., hybrid quantum-classical setting)?

Thank you for your attention!

References

[BR97] Bratteli, O.; Robonsin, D.W.: *Operator algebras and quantum statistical mechanics. II. Equlibrium states. Models in quantum statistical mechanics*. 2nd edition. Texts and Monographs in Physics. Springer-Verlag, Berlin. 1997.

[CHP24] Cornean, H.D.; Helffer, B.; Purice, R.: *Matrix representation of Magnetic pseudo-differential operators via tight Gabor frames*. To appear in J. Fourier Anal. Appl..

[DL17] De Nittis, G.; Lein, M.: *Linear response theory. An analytic-algebraic approach*. Spinger Briefs in Mathematical Physics. Springer, 2017.

[GIV06] Gayral, V.; Iochum, B.; Várilly, J.: *Dixmier traces on noncompact isospectral deformations*. J. Funct. Anal. **237** (2006), 507–539.

References

[IMP07] Iftimie, V.; Măntoiu, M.; Purice, R.: *Magnetic pseudodifferential operators*. Publ. Res. Inst. Math. Sci. **44** (2007), 585–623.

[LL22] Lee, G.; Lein, M.: *A calculus for magnetic pseudodifferential super operators*. J. Math. Phys. **63**, 103506, (2022).

[LL24] Lee, G.; Lein, M.: *A proof of $\mathfrak{L}^2$-boundedness for magnetic pseudodifferential super operators via matrix representations with respect to Parseval frames*. In preparation.

[Le10] Lein, M.: *Two-parameter asymptotics in magnetic Weyl calculus*. J. Math. Phys. **51**, 123519 (2010).

[MP04] Măntoiu, M.; Purice, R.: *The magnetic Weyl calculus*. J. Math. Phys. **45** (2004), no. 4, 1394–1417.

[Ri93] Rieffel, M.: *Deformation quantization for actions of R^d*. Mem. Amer. Math. Soc. **106** (1993), no. 506, 93 pp..