Minimal Models in Algebra and Operad Theory

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Algebraic, analytic and geometric structures emerging from quantum field theory

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Plan

- Koszul duality and minimal models for associative algebras
- Koszul duality and minimal models for operads
- Deformation theory and minimal models
- Deformations of associative algebras and Hochschild cohomology
- Koszul duality and minimal models for operated algebras

Part I: Computing $\mathrm{Tor}_*^A(\mathbf{k},\mathbf{k})$ and $\mathrm{Ext}_A^*(\mathbf{k},\mathbf{k})$ via projective resolutions

Let $\mathbf{k}$ be a field.
Let $A = T(V)/(R)$ be an algebra, where V is a vector space,

$$T(V) = \mathbf{k} \oplus V \oplus \cdots \oplus V^{\otimes n} \otimes \cdots$$

is the tensor algebra generated by V. Then $\mathbf{k}$ is a simple A-module.

Problem

Compute $\mathrm{Tor}_*^A(\mathbf{k},\mathbf{k})$ *and* $\mathrm{Ext}_A^*(\mathbf{k},\mathbf{k})$.

Answer

Find a (minimal) projective resolution $P_* \to \mathbf{k} \to 0$ *of* $\mathbf{k}$ *as* A*-modules then*

$$\mathrm{Tor}_*^A(\mathbf{k},\mathbf{k}) = H_*(\mathbf{k} \otimes_A P_*), \ \mathrm{Ext}_A^*(\mathbf{k},\mathbf{k}) = H^*(\mathrm{Hom}_A(P_*,\mathbf{k})).$$

Part I: Reconstruction problem

Problem (Reconstruction problem)

Can we reconstruct A from the $\mathrm{Ext}^*_A(\mathbf{k},\mathbf{k})$ or $\mathrm{Tor}^A_*(\mathbf{k},\mathbf{k})$?

Answer

Yes, but with the A_∞-algebra structure on $\mathrm{Ext}^*_A(\mathbf{k},\mathbf{k})$ or the A_∞-coalgebra structure on $\mathrm{Tor}^A_*(\mathbf{k},\mathbf{k})$

Part I: From associative algebras to A_∞-algebras

Definition

- *A (nonunital) associative* $\mathbb{k}$*-algebra is a pair* (A, μ)*, where A is a* $\mathbb{k}$*-vector space,*

$$\mu : A \otimes A \to A, \ a \otimes b \mapsto a \cdot b = ab$$

is a linear map, called multiplication, which satisfy the associativity axiom:

$$(ab)c = a(bc)$$

or equivalently,

$$(\mu \otimes \mathrm{id}_A)\mu = (\mathrm{id}_A \otimes \mu)\mu$$

- *A graded algebra is a graded space* $A = \oplus_{n \in \mathbb{Z}} A_n$ *endowed with an associative product such that* $A_p \cdot A_q \subseteq A_{p+q}, \forall p, q \in \mathbb{Z}$*.*
- *A differential graded algebra is a graded algebra* $A = \oplus_{n \in \mathbb{Z}} A_n$ *equipped with a square zero linear map* $d : A \to A$ *of degree* -1 *subject to*

$$d(ab) = d(a)b + (-1)^{|a|} a d(b),$$

for any $a, b \in A$ *homogeneous, where* $|a|$ *denotes the degree of a.*

Part I: From associative algebras to A_∞-algebras

Definition (Stasheff 1963)

An A_∞-algebra structure on a graded space V consists a family of operators $\{m_n\}_{n\geqslant 1}$ with $m_n : V^{\otimes n} \to V, |m_n| = n-2$, and the family $\{m_n\}_{n\geqslant 1}$ satisfies the following **Stasheff identities***:*

$$\sum_{i+j+k=n, i,k\geq 0, j\geq 1} (-1)^{i+jk} m_{i+1+k} \circ (\mathrm{Id}^{\otimes i} \otimes m_j \otimes \mathrm{Id}^{\otimes k}) = 0, \forall n \geqslant 1.$$

Example

- $n=1$, $m_1 \circ m_1 = 0$ *with* $|m_1| = -1$, *i.e.* m_1 *is a differential;*
- $n=2$, $m_1 \circ m_2 = m_2 \circ (\mathrm{Id} \otimes m_1 + m_1 \otimes \mathrm{Id})$, *i.e.* m_1 *is a derivation with respect to the multiplication* m_2*;*
- $n=3$, $m_2 \circ (m_2 \otimes \mathrm{Id}) - m_2 \circ (\mathrm{Id} \otimes m_2) = -(m_1 \circ m_3 + m_3 \circ (m_1 \otimes \mathrm{Id}^{\otimes 2} + \mathrm{Id} \otimes m_1 \otimes \mathrm{Id} + m_1 \otimes \mathrm{Id}^{\otimes 2}))$, *i.e.* m_2 *is associative up to homotopy.*

J. D. Stasheff, *Homotopy associativity of H-spaces. I, II.* Trans. Amer. Math. Soc. **108** (1963), 275-292; ibid. **108** (1963) 293-312.

Part I: A_∞-algebras via bar construction

Let $A = \Bbbk 1_A \oplus \overline{A}$ be an augmented differential graded algebra. Its bar construction $\mathrm{B}(A)$ is defined to be the tensor coalgebra

$$T^c(s\overline{A}) = \Bbbk \oplus s\overline{A} \oplus (s\overline{A})^{\otimes 2} \oplus \cdots$$

via the deconcartenation coproduct

$$\Delta(sa_1 \otimes \cdots \otimes sa_n) = \sum_{i=0}^{n} (sa_1 \otimes \cdots \otimes sa_i) \otimes (sa_{i+1} \otimes \cdots \otimes sa_n)$$

The differential d_A of A induces a differential d_1 on $\mathrm{B}(A)$ via

$$d_1([a_1|a_2|\cdots|a_n]) = \sum_{i=1}^{n} \pm [a_1|\cdots|d_A(a_i)|\cdots|a_n].$$

The product $\mu_A : A^{\otimes}2 \to A$ on A also induces a differential d_2 on $\mathrm{B}(A)$ via

$$d_2([a_1|a_2|\cdots|a_n]) = \sum_{i=1}^{n} \pm [a_1|\cdots|\mu_A(a_i \otimes a_{i+1})|\cdots|a_n].$$

Now $(\mathrm{B}(A), d = d_1 + d_2)$ is a (coaugmented) differential graded (conilpotent cofree) coalgebra.

Part I: A_∞-algebras via bar construction

Theorem

A graded vector space A is an A_∞-algebra iff $\mathrm{B}(A)$ is a (coaugmented) differential graded (conilpotent cofree) coalgebra via its canonical coalgebra structure.

$$m_n : A^{\otimes n} \to A \leftrightarrow d_n : (s\overline{A})^{\otimes n} \to s\overline{A}$$

Part I: From coalgebras to A_∞-coalgebras

Definition

A noncounital coalgebra is a pair (C, Δ) where C is a vector space, $\Delta : C \to C \otimes C$ is a linear map, called the comultiplication which satisfies the coassociativity axion

$$(\Delta \otimes \mathrm{id}_C) \circ \Delta = (\mathrm{id}_C \otimes \Delta) \circ \Delta$$

or equivalently there exists a commutative diagram

$$\begin{array}{ccc} C \otimes C \otimes C & \xleftarrow{\mathrm{Id}_C \otimes \Delta} & C \otimes C \\ \uparrow {\scriptstyle \Delta \otimes \mathrm{Id}_C} & & \uparrow {\scriptstyle \Delta} \\ C \otimes C & \xleftarrow[\Delta]{} & C \end{array} .$$

We can also define graded coalgebras, differential graded coalgebras etc. An A_∞-coalgebra structure on a graded space V consists a family of higher comultiplications $\Delta_n : V \to V^{\otimes n}, |\Delta_n| = n - 2$ subject to **coStasheff identities**.

Let $C = \mathbb{k}1_C \oplus \overline{C}$ be a coaugmented differential graded coalgebra. Its cobar construction $\Omega(C)$ is defined to be the tensor algebra $T(s^{-1}\overline{C})$ with concatenation product.
We shall write an element of $\Omega(A)$ as $\langle a_1|a_2|\cdots|a_n\rangle \in (s^{-1}\overline{A})^{\otimes n}$ for $a_1, \cdots, a_n \in \overline{A}$.
The differential d_C of C induces a differential d_1 on $\Omega(C)$ via

$$d_1(\langle c_1|\cdots|c_n\rangle) = \sum_{i=1}^{n}(-1)^{i+|c_1|+\cdots+|c_{i-1}|}\langle c_1|\cdots|d_C(c_i)|\cdots|c_n\rangle.$$

The coproduct Δ_C induced induces a differential d_2 on $\Omega(C)$ via

$$d_2(\langle c_1|\cdots|c_n\rangle) = \sum_{i=1}^{n}(-1)^{i+|c_1|+\cdots+|c_{i-1}|+|c_{i(1)}|}\langle c_1|\cdots|c_{i(1)}|c_{i(2)}|\cdots|c_n\rangle.$$

The augmented differential graded free algebra $(\Omega(C), d = d_1 + d_2)$ is called the *cobar construction* of the coaugmented differential graded coalgebra C.

Part I: A_∞-coalgebras via cobar construction

Theorem

A graded vector space C is an A_∞-coalgebra iff $\Omega(C)$ is an (augmented) differential graded (free) algebra via its canonical algebra structure.

$$\Delta_n : V \to V^{\otimes n} \leftrightarrow d_n : s^{-1}V \to (s^{-1}V)^{\otimes n}$$

Part I: Computing A_∞-structures via homotopy transfer theory

Theorem (Kadeishivili)

Let A be a a differential graded algebra. Then its homology $H_*(A)$ has an A_∞-algebra structure which can be computed via homotopy transfer theory

$$H_*(A) \rightleftarrows A \quad \circlearrowleft$$

Theorem

Let C be a a differential graded algebra. Then its homology $H_*(C)$ has an A_∞-coalgebra structure which can be computed via homotopy transfer theory

$$H_*(C) \rightleftarrows C \quad \circlearrowleft$$

Part I: Computing A_∞-structures via homotopy transfer theory

Problem

Compute the A_∞-structure on $\mathrm{Ext}_A^*(\mathbf{k}, \mathbf{k})$ and $\mathrm{Tor}_*^A(\mathbf{k}, \mathbf{k})$.

Answer

Form the bicomplex $\mathrm{Hom}_A(P_*, P_*)$ and take its total complex, still denoted by $\mathrm{Hom}_A(P_*, P_*)$. We have

$$\mathrm{Ext}_A^*(\mathbf{k}, \mathbf{k}) = H^*(\mathrm{Hom}_A(P_*, P_*)).$$

Moreover, we have a deformation retract:

$$\mathrm{Ext}_A^*(\mathbf{k}, \mathbf{k}) \underset{p}{\overset{i}{\rightleftarrows}} \mathrm{Hom}_A(P_*, P_*) \quad \circlearrowleft h$$

Observe that $\mathrm{Hom}_A(P_*, P_*)$ is a differential graded algebra. Via homotopy transfer theory, one gets an A_∞-algebra structure on $\mathrm{Ext}_A^*(\mathbf{k}, \mathbf{k})$.
Similarly, one gets an A_∞-coalgebra structure on $\mathrm{Tor}_*^A(\mathbf{k}, \mathbf{k})$.

Part I: Computing A_∞-structures via homotopy transfer theory

Problem (Reconstruction problem)

Can we reconstruct A from the $\mathrm{Ext}^*_A(\mathbf{k}, \mathbf{k})$ or $\mathrm{Tor}^A_*(\mathbf{k}, \mathbf{k})$?

Answer

Under certain conditions of finiteness and minimality, there is a sequence of quasi-isomorphism of differential graded algebras

$$\Omega(\mathrm{Tor}^A_*(\mathbf{k}, \mathbf{k})) \to \Omega(B(A)) \to A.$$

In this case, $\Omega(\mathrm{Tor}^A_*(\mathbf{k}, \mathbf{k}))$ is a minimal model of A.

Part I: Minimal models in Algebra

Definition

A cofibrant resolution (resp. a minimal model) of A is a quasi-isomorphism of differential graded algebras

$$(T(V'), d) \to A$$

(resp. subject to a certain minimality condition).
Equivalently, A cofibrant resolution (resp. a minimal model) of A is an A_∞-coalgebra $A^{¡} = s^{-1}V'$ such that $\Omega(A^{¡})$ is quasi-isomorphic to A (resp. subject to a certain minimality condition).

Definition

When the minimal model exists, $A^{¡}$ will be called the Koszul dual A_∞-coalgebra of A.

Fact

From a cofibrant resolution (resp. a minimal model), one can construct the (minimial) projective resolution $P_* \to A \to 0$ via

$$P_* = A \otimes_\tau A^{¡},$$

where $\tau : A^{¡} \to A$ is a twisting cochain and $A \otimes_\tau A^{¡}$ is the associated twisted tensor product.

Corollary

Cofibrant resolutions (resp. the minimal model) of an algebra are equivalent to (resp. minimal) projective resolutions of the trivial module.

Part I: Koszul case

If $A = T(V)/(R)$ with $R \subset V \otimes V$, can define its Koszul dual algebra

$$A^! = T(V^*)/(R^\perp)$$

and its Koszul dual coalgebra

$$C(sV, s^2R) = \mathbb{k} \oplus V \oplus R \oplus \cdots \oplus \Big(\bigcap_{i+2+j=n} V^{\otimes i} \otimes R \otimes V^{\otimes j} \Big) \oplus \cdots .$$

Theorem

A is a Koszul algebra iff $A^{¡}$ is a graded coalgebra iff $A^{¡} = C(sV, s^2R)$. In this case, the cobar construction of its Koszul dual coalgebra is a minimal model A.

Example

- *$T(V)$ is Koszul*
- *The polynomial algebra $S(V) = T(V)/(v \otimes w - w \otimes v, v, w \in V)$ is Koszul*
- *The algebra $A = T(V)/(v \otimes w, v, w \in V) = \mathbf{k} \oplus V$ is Koszul*

Part I: Quadratic-Linear (QL) Koszul algebras

Let $A = T(V)/(R)$ with $R \subset V \otimes V \oplus V$, let $q : T(V) \to V^{\otimes 2}$ be the natural projection. Then $R = \{r - \varphi(r) \mid r \in q(R)\}$ for some linear map $\varphi : qR \to V$. Define $qA = T(V)/(qR)$, its Koszul dual algebra

$$(qA)^! = T(V^*)/(qR^\perp)$$

and its Koszul dual coalgebra

$$C(sV, s^2(qR)).$$

Fact

Under certain nondegeneracy conditions, $C(sV, s^2(qR))$ can be endowed with a differential, so that it becomes a differential graded coalgebra.

Theorem

When the nondegeneracy conditions hold and qA is Koszul, we say that A is Koszul. In this case, there is a quasi-isomorphism $\Omega(A^{¡}) \to A$.

Example

The universal enveloping algebra of a finite dimensional Lie algebra is (QL-)Koszul

Part I: Non-Koszul case

Theorem

When A is not Koszul, $A^{\text{¡}} = \operatorname{Tor}_*^A(\mathbf{k},\mathbf{k})$ is a genuine A_∞-coalgebra and there is a quasi-isomorphism $\Omega(A^{\text{¡}}) \to A$.

Remark

Tomaroff found the A_∞-coalgebra structure on $A^{\text{¡}} = \operatorname{Tor}_*^A(\mathbf{k},\mathbf{k})$ for monomial algebras.

Part II: A crash course on operads

Operad theory is a tool to describe algebraic operations.

Roughly speaking, a nonsymmetric operad $\mathcal{P}$ is a sequence of spaces $\mathcal{P}(n), n \geq 1$, where $\mathcal{P}(n)$ is considered as a space of n-ary operations and where it is asked there are compositions of operations subject to associativity axioms etc.

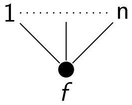
Part II: A crash course on operads

Example

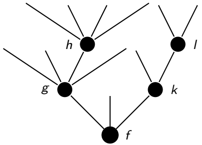
Let V be a vector space. The endomorphism operad End_V is defined to be $\mathrm{End}_V(n) = \mathrm{Hom}(V^{\otimes n}, V)$, the space of n-linear maps on V.
For $f \in \mathrm{End}_V(n), g \in \mathrm{End}_V(m)$ and $1 \leq i \leq n$, define partial composition

$$f \circ_i g = f(\mathrm{id}^{\otimes i-1} \otimes g \otimes \mathrm{id}^{n-i-1}).$$

An n-ary operation $f \in \mathrm{End}_V(m)$ will be presented by



For example, the 10-ary operation $(((f \circ_1 g) \circ_3 h) \circ_9 k) \circ_{10} l$ can be represented by the following tree



Part II: The operad of associative algebras

An associative algebra is a pair (A, μ) such that

$$\mu \circ (\mu \otimes \mathrm{id}_A) - \mu \circ (\mathrm{id}_A \otimes \mu)$$

Definition

The operad of associative algebras is defined to be

$$\mathcal{A}ss = \mathcal{F}(M)/I$$

where $M = \mathbf{k}\mu$, $\mathcal{F}(M)$ is the free operad generated by M, I is the operadic ideal generated by

$$\mu \circ (\mu \otimes \mathrm{id}_A) - \mu \circ (\mathrm{id}_A \otimes \mu).$$

Associative algebras are exactly algebras over $\mathcal{A}ss$.

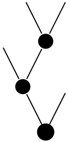
Definition

Let $\mathcal{P}$ be an operad. An $\mathcal{P}$-algebra structure on a vector space V is given by an operad map $\mathcal{P} \to \mathrm{End}_V$.

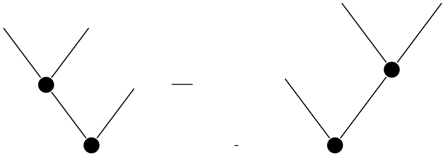
Part II: The operad of associative algebras

Present μ as the corolla with two leaves

Then elements of $\mathcal{F}(M)$ are presented by all plane binary rooted trees, such as



The associativity axiom can be presented as



So each element of $\mathcal{A}ss = \mathcal{F}(M)/I$ can be presented (uniquely) by a "right comb".

Part III: Koszul case

Assume the operad $\mathcal{P}$ is Koszul.

Example

- *associative algebras*
- *commutative associative algebras*
- *Lie algebras*
- *Poisson algebras*
- *pre-Lie algebras*
- *Leibniz algebras*
- *Lie triple systems*
- *etc*

Part II: Minimal models in Koszul cases

Answer

$\mathcal{P}_\infty = \Omega(\mathcal{P}^{\text{i}})$ *is the minimal model of* $\mathcal{P}$, *where* $\mathcal{P}^{\text{i}}$ *is a genuine cooperad.*

V. Ginzburg, M. Kapranov, *Koszul duality for operads.* Duke Math. J. **76** (1994), no. 1, 203-272.

E. Getzler and D. S. J. Jones, *Operads, homotopy algebra and iterated integrals for double loop spaces*, hep-th/9403055 (1994).

Part II: A_∞ operad vs Ass

Theorem (Ginzburg-Kapranov 90)

The operad Ass is a Koszul operad and the cobar construction of $Ass^{¡}$ is the operad governing A_∞-algebras. The latter is a minimal model of Ass.

V. Ginzburg, M. Kapranov, *Koszul duality for operads.* Duke Math. J. **76** (1994), no. 1, 203 – 272.

Part III: History about algebraic deformation theory

- 1945, G. Hochschild introduced a cohomology theory of associative rings

G. Hochschild, *On the cohomology groups of an associative algebra.* Ann. of Math. (2) **46** (1945), 58-67.

Part III: History about algebraic deformation theory

- 1945, G. Hochschild introduced a cohomology theory of associative rings

G. Hochschild, *On the cohomology groups of an associative algebra.* Ann. of Math. (2) **46** (1945), 58-67.

- 1963-64, M. Gerstenhaber discovered a dg Lie algebra structure over Hochschild cochain complex and developed algebraic deformation theory

M. Gerstenhaber, *The cohomology structure of an associative ring*. Ann. Math. (2) **78** (1963) 267-288.

M. Gerstenhaber, *On the deformation of rings and algebras*. Ann. Math. (2) **79** (1964) 59-103.

- 1963, J. Stasheff defined A_∞-algebras

J. Stasheff, *Homotopy associativity of H -spaces. I, II*. Trans. Amer. Math. Soc. **108** (1963), 275-292; ibid. 108 1963 293 – 312.

Part III: History about algebraic deformation theory

- 1966, A. Nijenhuis and R. W. Richardson investigated deformations and cohomologies of graded Lie algebras.

A. Nijenhuis, R. W. Richardson, *Cohomology and deformations in graded Lie algebras*. Bull. Amer. Math. Soc. **72** (1966), 1-29.

- 1990, J. Stasheff introduced L_∞-algebras

J. Stasheff, *Differential graded Lie algebras, quasi-Hopf algebras and higher homotopy algebras*. Quantum groups (Leningrad, 1990), pp. 120-137, Lecture Notes in Mathematics, 1510. Springer, Berlin (1992)

Part III: History about algebraic deformation theory

- 1993, P. Deligne proposed Deligne conjecture via operad theory

P. Deligne, *Letter to Stasheff, Gerstenhaber, May, Schechtman, Drinfeld*, 1993.

- 1997 M. Kontsevich proved his deformation quantization theorem for Poisson manifolds

M. Kontsevich, *Deformation quantization of Poisson manifolds*. Lett. Math. Phys. **66**(2003), 157-216.

Part III: Philosophy of Deformation Theory after Deligne, Drinfeld, Kontsevich,...

- (Deligne 1986): "The deformation theory of any mathematical object, e.g., an associative algebra, a complex manifold, etc., can be discribed starting from a certain dg Lie algebra associated to the mathematical obejct in question."

Part III: Philosophy of Deformation Theory after Deligne, Drinfeld, Kontsevich,...

- (Deligne 1986): `"The deformation theory of any mathematical object, e.g., an associative algebra, a complex manifold, etc., can be discribed starting from a certain dg Lie algebra associated to the mathematical obejct in question."`

Theorem (Lurie, Pridham)

In characteristic 0, there exists an equivalence between the ∞-category of formal moduli problems and the ∞-category of DG Lie algebras (L_∞-algebras).

J. Lurie, DAG X: Formal moduli problems.

J. P. Pridham, *Unifying derived deformation theories*, Adv. Math. **224** (2010), no. 3, 772-826.

Part III: Differential graded Lie algebras

Throughout this talk, let k be a field of characteristic zero.

Definition

A differential graded Lie algebra (aka dg Lie algebra) is a graded space $L = \oplus_{i \in \mathbb{Z}} L_i$ together with two operations:

$$l_1 : L_i \to L_{i-1}$$

of degree -1 and

$$l_2 : L_i \otimes L_j \to L_{i+j}$$

of degree zero such that

(i) $l_1 : L_i \to L_{i-1}$ *is a differential,*

(ii) $l_2 : L_i \otimes L_j \to L_{i+j}$ *is a Lie bracket,*

(IV) l_1 *is a derivation for* l_2 *,i.e.*

$$l_1 l_2(a \otimes b) = l_2(l_1(a) \otimes b) + (-1)^{|a|} l_2(a \otimes l_1(b))$$

for $a, b \in L$ homogeneous.

Part III: Maurer-Cartan elements in dg Lie algebra

Definition

Let L be a dg Lie algebra. An element $\alpha \in L_{-1}$ is a Maurer-Cartan element if

$$l_1(\alpha) - \frac{1}{2} l_2(\alpha \otimes \alpha) = 0.$$

Proposition (Twisting procedure)

Let L be a dg Lie algebra. Given a Maurer-Cartan element $\alpha \in L_{-1}$, one can produce a new dg Lie algebra by imposing

$$l_1^{\alpha}(x) = l_1(x) - l_2(\alpha \otimes x)$$

and

$$l_2^{\alpha}(x \otimes y) = l_2(x \otimes y)$$

Part III: L_∞-algebras

Definition

Let $L = \bigoplus_{i \in \mathbb{Z}} L_i$ be a graded space over k. Assume that L is endowed with a family of linear operators $l_n : L^{\otimes n} \to L$, $n \geq 1$ with $|l_n| = n - 2$ satisfying the following conditions: $\forall \sigma \in S_n, x_1, \ldots, x_n \in L$,

(i) *(Skew-symmetry)*

$$l_n(x_{\sigma(1)} \otimes \ldots \otimes x_{\sigma(n)}) = \chi(\sigma, x_1, \ldots, x_n) l_n(x_1, \ldots, x_n),$$

(ii) *(Higher Jacobi identities)*

$$\sum_{i=1}^{n} \sum_{\sigma \in Sh(i,n-i)} \chi(\sigma, x_1, \ldots, x_n)(-1)^{i(n-i)} \\ l_{n-i+1}(l_i(x_{\sigma(1)} \otimes \ldots \otimes x_{\sigma(i)}) \otimes x_{\sigma(i+1)} \otimes \ldots \otimes x_{\sigma(n)}) = 0,$$

where $Sh(i, n-i)$ is the set of $(i, n-i)$ shuffles, i.e., $Sh(i, n-i) = \{\sigma \in S_n$ such that $\sigma(1) < \sigma(2) < \cdots < \sigma(i)$, and $\sigma(i+1) < \sigma(i+2) < \ldots \sigma(n)\}$.

Then $(L, \{l_n\}_{n \geq 1})$ is called a L_∞-algebra.

Example

Let $(L, \{l_n\}_{n\geq 1})$ *be a* L_∞*-algebra.*

(i) $n = 1$, l_1 *is a differential,*

(ii) $n = 2$, l_1 *is a derivation for* l_2,

(IV) $n = 3$, l_2 *satisfies the Jacobi identity up to homotopy.*

Part III: Maurer-Cartan elements and L_∞-algebras

Definition

Let $(L, \{l_n\}_{n\geq 1})$ be an L_∞-algebra and $\alpha \in L_{-1}$. Then α is called a Maurer-Cartan element if it satisfies equation:

$$\sum_{n=1}^{\infty} \frac{1}{n!}(-1)^{\frac{n(n-1)}{2}} l_n(\alpha^{\otimes n}) = 0,$$

whenever this infinite sum exists.

Proposition

Let $(L, \{l_n\}_{n\geq 1})$ be an L_∞-algebra. Given a Maurer-Cartan element α in L_∞-algebra L, we can define a new L_∞ structure $\{l_n^\alpha\}_{n\geq 1}$ on graded space L, where $l_n^\alpha : L^{\otimes n} \to L$ is defined as :

$$l_n^\alpha(x_1 \otimes \ldots \otimes x_n) = \sum_{i=0}^{\infty} \frac{1}{i!}(-1)^{in+\frac{i(i-1)}{2}} l_{n+i}(\alpha^{\otimes i} \otimes x_1 \otimes \ldots \otimes x_n).$$

Part III: Problems from Deformation Theory

Given an algebraic structure governed by an operad $\mathcal{P}$, two basic problems of deformation theory:

Problem

Find a cofibrant resolution (or minimal model) $\mathcal{P}_\infty$ of $\mathcal{P}$, that is, there exists a quasi-isomorphism

$$\mathcal{P}_\infty = \Omega(\mathcal{P}^{\text{i}}) \to \mathcal{P}.$$

In general, the Koszul dual $\mathcal{P}^{\text{i}}$ is only a homotopy cooperad.

Problem

Define the deformation cohomology of $\mathcal{P}$-algebras and describe the dg Lie algebra (or L_∞ structure) on the deformation complex

Part III: From minimal models to L_∞-structures

Problem

Describe the L_∞ structure on the deformation complex

Answer

Given a cofibrant resolution, or in particular, the minimal model $\mathcal{P}_\infty = \Omega(\mathcal{P}^{\text{i}})$ of $\mathcal{P}$, can introduce the deformation complex $\mathrm{Hom}(\mathcal{P}^{\text{i}}, \mathrm{End}_V)$ and describe the L_∞-algebra structure on the deformation complex.

- M. Kontsevich and Y. Soibelman, *Deformations of algebras over operads and the Deligne conjecture*, Conférence Moshé Flato 1999, Vol. I (Dijon), Math. Phys. Stud. 21 (2000), 255-307.
- P. Van der Laan, *Operads up to Homotopy and Deformations of Operad Maps*, arXiv 0208041.
- P. Van der Laan, *Coloured Koszul duality and strongly homotopy operads*, arXiv 0312147.

Part III: Koszul case

Assume the operad $\mathcal{P}$ is Koszul.

Example

- *associative algebras*
- *commutative associative algebras*
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- *Poisson algebras*
- *pre-Lie algebras*
- *Leibniz algebras*
- *Lie triple systems*
- *etc*

Part II: Koszul case

Let $\mathcal{P}$ be a Koszul operad.

- Can compute Koszul dual cooperad $\mathcal{P}^{\text{i}}$.
- The dg operad $\mathcal{P}_\infty = \Omega(\mathcal{P}^{\text{i}})$ is the minimal model of $\mathcal{P}$.
- The homotopy version of $\mathcal{P}$-algebras are exactly $\mathcal{P}_\infty$-algebras.
- For a vector space, there is a graded Lie algebra

  $$(\mathrm{Hom}(\mathcal{P}^{\text{i}}, \mathrm{End}_V), l_2)$$

  such that its Maurer-Cartan elements are in bijection with $\mathcal{P}$-algebra structures on V.
- Let μ be a $\mathcal{P}$-algebra structures on V. Then the underlying complex of the twisted dg Lie algebra

  $$(\mathrm{Hom}(\mathcal{P}^{\text{i}}, \mathrm{End}_V), l_1^\mu, l_2^\mu)$$

  is the deformation complex of $\mathcal{P}$-algebra (V, μ).

Part III: Non-Koszul case

When $\mathcal{P}$ is NOT Koszul, no general answer so far.

Example

- *Rota-Baxter associative/Lie algebras*
- *differential associative/Lie algebras with nonzero weight*
- *Hom-associative algebras, Hom-Lie algebras,* $\cdots$
- *etc*

Part III: Our method for non-Koszul case

Four steps:

$$\begin{array}{ccc} \text{formal deformations} & & \\ \Downarrow & & \\ \text{deformation complex} & & \\ \Downarrow & & \\ L_\infty\text{−structure} & & \dashleftarrow \text{derived brackets (Bai, Guo, Sheng, Tangetc)} \\ \Downarrow & & \\ \text{minimal model} & & \end{array}$$

Given an associative **k**-algebra $(A, \mu = \cdot)$, consider $k[[t]]$-bilinear associative products on

$$A[[t]] = \{\sum_{i=0}^{\infty} a_i t^i \mid a_i \in A, \forall i \geqslant 0\}.$$

Such a product is determined by

$$\mu_t = \sum_{i=0}^{\infty} \mu_i t^I I : A \otimes A \to A[[t]],$$

where for all $i \geqslant 0$, $\mu_I I : A \otimes A \to A$ are linear maps. When $\mu_0 = \mu$, we say that μ_t is a formal deformation of μ and μ_1 is called the infinitesimal of formal deformation μ_t.

Part IV: Formal deformations

Associativity of μ_t:

$$\mu_t(\mu_t(a\otimes b)\otimes c) = \mu_t(a\otimes\mu(b\otimes c)), \forall a, b, c \in A$$

$\Leftrightarrow$ for each $n \geq 0$,

$$\sum_{i+j=n} \mu_i(\mu_j(a\otimes b)\otimes c) - \mu_i(a\otimes\mu_j(b\otimes c)) = 0, \forall a, b, c \in A$$

Part IV: Formal deformations

Associativity of μ_t:

$$\mu_t(\mu_t(a \otimes b) \otimes c) = \mu_t(a \otimes \mu(b \otimes c)), \forall a, b, c \in A$$

$\Leftrightarrow$ for each $n \geq 0$,

$$\sum_{i+j=n} \mu_i(\mu_j(a \otimes b) \otimes c) - \mu_i(a \otimes \mu_j(b \otimes c) = 0, \forall a, b, c \in A$$

- $n = 0$,

$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$

Part IV: Formal deformations

Associativity of μ_t:

$$\mu_t(\mu_t(a \otimes b) \otimes c) = \mu_t(a \otimes \mu(b \otimes c)), \forall a, b, c \in A$$

$\Leftrightarrow$ for each $n \geq 0$,

$$\sum_{i+j=n} \mu_i(\mu_j(a \otimes b) \otimes c) - \mu_i(a \otimes \mu_j(b \otimes c) = 0, \forall a, b, c \in A$$

- $n = 0$,
$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$
- $n = 1$,
$$a\mu_1(b \otimes c) - \mu_1(ab \otimes c) + \mu_1(a \otimes bc) - \mu_1(a \otimes b)c = 0$$

Part IV: Hochschild cochain complex and Hochschild cohomology

Hochschild cochain complex $C^{\bullet}_{\mathrm{Alg}}(A, M)$ with coefficients in a bimodule M is defined as follows:

For $n \geq 0$, $C^n_{\mathrm{Alg}}(A, M) = \mathrm{Hom}(A^{\otimes n}, M)$
The differential $\partial^n_{\mathrm{Alg}} : C^n_{\mathrm{Alg}}(A, M) \to C^{n+1}_{\mathrm{Alg}}(A, M)$ is given by

$$\begin{aligned} & \partial^n_{\mathrm{Alg}}(f)(a_1 \otimes \cdots \otimes a_{n+1}) \\ = \quad & (-1)^{n-1} a_1 f(a_2 \otimes \cdots \otimes a_{n+1}) \\ & + \sum_{i=1}^{n} (-1)^{n-i+1} f(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}) \\ & + f(a_1 \otimes \cdots \otimes a_n) a_{n+1}. \end{aligned}$$

Definition

For $n \geq 0$, the n-th Hochschild cohomology group of A is defined to be

$$\mathrm{H}^n_{\mathrm{Alg}}(A, M) = \mathrm{H}^n(C^{\bullet}_{\mathrm{Alg}}(A, M))$$

Write $C^{\bullet}_{\mathrm{Alg}}(A) = C^{\bullet}_{\mathrm{Alg}}(A, A)$ and $\mathrm{H}^{\bullet}_{\mathrm{Alg}}(A) = \mathrm{H}^{\bullet}_{\mathrm{Alg}}(A, A)$.

Part IV: Hochschild cochain complex and Hochschild cohomology

For $g \in C^2_{\mathrm{Alg}}(A) = \mathrm{Hom}(A \otimes A, A)$,

$\partial^2_{\mathrm{Alg}}(g) : A \otimes A \otimes A \to A,\ a \otimes b \otimes c \mapsto a g(b \otimes c) - g(ab \otimes c) + g(a \otimes bc) - g(a \otimes b)c.$

$a\mu_1(b \otimes c) - \mu_1(ab \otimes c) + \mu_1(a \otimes bc) - \mu_1(a \otimes b)c = 0 \Leftrightarrow \mu_1$ is a 2-cocyle

Lie bracket $[\ ,\ ]_G : C^n_{\mathrm{Alg}}(A) \times C^m_{\mathrm{Alg}}(A) \to C^{n+m-1}_{\mathrm{Alg}}(A)$
Let $f \in C^n_{\mathrm{Alg}}(A)$ and $g \in C^m_{\mathrm{Alg}}(A)$. Then $[f,g]_G \in C^{n+m-1}_{\mathrm{Alg}}(A)$ is defined as follows:
If $n \geq 1$, for $1 \leq i \leq n$,

$$f \circ_i g(a_1 \otimes \cdots \otimes a_{n+m-1}) = f(a_1 \otimes \cdots \otimes a_{i-1} \otimes g(a_i \otimes \cdots \otimes a_{i+m-1}) \otimes \cdots \otimes a_{n+m-1});$$

if $n = 0$, $f \circ_i g = 0$.
Define

$$f \bar{\circ} g = \sum_{i=1}^{n} (-1)^{(i-1)(m-1)} f \circ_i g$$

$$[f,g]_G = f \bar{\circ} g - (-1)^{(n-1)(m-1)} g \bar{\circ} f$$

Part IV: Gerstenhaber Lie bracket

Example

- $n = 1, m = 0,\ f : A \to A$ *and* $g = a \in A$
  $[f, g]_G = f\bar{\circ}g = f \circ_1 g = f(a) \in A;$
- $n = m = 1,\ f, g : A \to A$
  $f\bar{\circ}g = f \circ_1 g : A \to A, a \mapsto f(g(a))$ *and*

$$[f, g]_G = f \circ g - g \circ f.$$

Example

- $n = m = 2,\ \mu_i, \mu_j : A \otimes A \to A$

$$\begin{aligned}[\mu_i, \mu_j]_G(a\otimes b\otimes c) &= \mu_i(\mu_j(a\otimes b)\otimes c) - \mu_i(a\otimes\mu_j(b\otimes c)) \\ &\quad +\mu_j(\mu_i(a\otimes b)\otimes c) - \mu_j(a\otimes\mu_i(b\otimes c))\end{aligned}$$

For a grade vector space V, the shift sV is defined as $(sV)_p = V_{p-1}, p \in \mathbb{Z}$.

Theorem (Gerstenhaber 1963)

For algebra A, $sC^{\bullet}_{\mathrm{Alg}}(A)$ together with differential $-\partial^{\bullet}_{\mathrm{Alg}}$ and the Gerstenhaber Lie bracket is a dg Lie algebra.

M. Gerstenhaber, *The cohomology structure of an associative ring*. Ann. Math. (2) **78** (1963) 267-288.

Part IV: Gerstenhaber dg Lie algebra controls formal deformations of associative algebras

Associativity of μ_t:

$$\mu_t(\mu_t(a \otimes b) \otimes c) = \mu_t(a \otimes \mu(b \otimes c)), \forall a, b, c \in A$$

$\Leftrightarrow$ for each $n \geq 0$,

$$\sum_{i+j=n} \mu_i(\mu_j(a \otimes b) \otimes c) - \mu_i(a \otimes \mu_j(b \otimes c)) = 0, \forall a, b, c \in A$$

- $n = 1$, $\partial^2_{\mathrm{Alg}}(\mu_1) = 0$ i.e. μ_1 is a Hochschild 2-cocyle
- $n = 2$,

$$\partial^2_{\mathrm{Alg}}(\mu_2) = \frac{1}{2}[\mu_1, \mu_1]_G$$

- for each $n \geq 2$,

$$\partial^2_{\mathrm{Alg}}(\mu_n) = \frac{1}{2} \sum_{i+j=n, i,j<n} [\mu_i, \mu_j]_G$$

Part IV: Return to Gerstenhaber graded Lie algeba

Lemma

- *Even if V is only a vector space, $\mathfrak{C}_{Alg}(V) = \prod_{n=0}^{\infty} \mathrm{Hom}((sV)^{\otimes n}, sV)$ is still a graded Lie algebra endowed with the Gerstenhaber Lie bracket.*
- *The set of Maurer-Cartan elements in graded Lie algebra $\mathfrak{C}_{Alg}(V)$ is in bijection with associative algebra structures on V.*
- *Let (A, μ) be an associative algebra. We obtain thus a new dg Lie algebra by twisting $\mathfrak{C}_{Alg}(A)$ by μ which is exactly the Gerstenhaber dg Lie algebra $sC^{\bullet}_{\mathrm{Alg}}(A)$.*

Let V be a graded vector space. Let

$$T(sV) = k \oplus sV \oplus (sV)^{\otimes 2} \oplus \cdots .$$

Let

$$\mathfrak{C}_{Alg}(V) = \mathrm{Hom}(T(sV), sV).$$

One can generalise the construction of Gerstenhaber and obtain a graded Lie algebra structure on $\mathfrak{C}_{Alg}(V)$.

Definition

Let V be a graded vector space. Let

$$\overline{T}(sV) = sV \oplus (sV)^{\otimes 2} \oplus \cdots .$$

Define $\overline{\mathfrak{C}_{Alg}}(V) = \mathrm{Hom}(\overline{T}(sV), sV)$ and it is naturally a graded Lie subalgebra of $\mathfrak{C}_{Alg}(V)$.
*An A_∞-**algebra structure** on graded space V is defined to be a Maurer-Cartan element in graded Lie algebra $\overline{\mathfrak{C}_{Alg}}(V)$.*

Part IV: Defining A_∞-algebras

Definition (Stasheff 1963)

An A_∞-**algebra structure** *on* V *consists a family of operators* $\{m_n\}_{n\geqslant 1}$ *with* $m_n : V^{\otimes n} \to V, |m_n| = n-2$*, and the family* $\{m_n\}_{n\geqslant 1}$ *satisfies the following* **Stasheff identities***:*

$$\sum_{i+j+k=n, i,k\geq 0, j\geq 1} (-1)^{i+jk} m_{i+1+k} \circ (\mathrm{id}^{\otimes i} \otimes m_j \otimes \mathrm{id}^{\otimes k}) = 0, \forall n \geqslant 1.$$

J. D. Stasheff, *Homotopy associativity of H -spaces. I, II.* Trans. Amer. Math. Soc. **108** (1963), 275-292; ibid. 108 1963 293 – 312.

One can prove directly the following result by hand instead of useing Koszul duality theory for operads.

Theorem (Ginzburg-Kapranov 90)

The operad governing A_∞-algebras is a minimal cofibrant resolution of the operad of associative algebras in the model category of operads.

V. Ginzburg, M. Kapranov, *Koszul duality for operads.* Duke Math. J. **76** (1994), no. 1, 203 – 272.

Part IV: Homotopy theory of associative algebras

$$\begin{array}{ccc} \textit{Gerstenhaber}\ 1963, 1964 & & \textit{Stasheff}\ 1963 \\ \textit{formal deformations} & & \\ \downarrow & & \\ \textit{deformation complex} & & \\ \downarrow & & \\ \textit{dg Lie algebra} & \longleftrightarrow & A_\infty - \textit{algebras} \\ & \textit{Ginzburg}, \textit{Kapranov}\, 1994 & \end{array}$$

Definition

Let $(R, \mu = \cdot)$ be an associative algebra and $\lambda \in k$. A linear operator $T : R \to R$ is said to be a Rota-Baxter operator of weight λ if it satisfies

$$T(a) \cdot T(b) = T(a \cdot T(b)) + T(T(a) \cdot b) + \lambda T(a \cdot b) \tag{1}$$

for any $a, b \in R$, that is,

$$\mu \circ (T \otimes T) = T \circ \mu \circ (Id \otimes T) + T \circ \mu \circ (T \otimes Id) + \lambda T \circ \mu. \tag{2}$$

Then (R, μ, T) is called a Rota-Baxter algebra of weight λ.

Remark

The Rota-Baxter relation does not lie in degree two, so one cannot use the Koszul duality theory for quadratic operads to study its cohomology theory and its minimal model.

Part V.1: Rota-Baxter associative algebras

Theorem

- *Can define cohomology theory of (relative) Rota-Baxter associative algebras*
- *Can found the L_∞-structure on the deformation complex*
- *Can defined homotopy Rota-Baxter associative algebras*
- *Can prove that the operad of homotopy Rota-Baxter algebras is a minimal model of the operad of Rota-Baxter algebras.*
- *Can find the Koszul dual homotopy cooperad of the operad of Rota-Baxter algebras.*

K. Wang (王凯) and G. Zhou (周国栋), *Deformations and homotopy theory of Rota-Baxter algebras of any weight*, arXiv:2108.06744.

K. Wang (王凯) and G. Zhou (周国栋), *The homotopy theory of Rota-Baxter associative algebras of arbitrary weights*, arXiv:2203.02960.

Part V.2: (Relative) Rota-Baxter Lie algebras

Definition

Let $\mathfrak{g}$ be a Lie algebra over a field k and V a representation of $\mathfrak{g}$. A k-linear map $T : V \to \mathfrak{g}$ is said to be a **Rota-Baxter operator of weight** λ *if it satisfies*

$$[T(a), T(b)] = T([a, T(b)] + [T(a), b] + \lambda[a \cdot b])$$

for any $a, b \in \mathfrak{g}$ or

$$\mu \circ (T \otimes T) = T \circ \mu \circ (Id \otimes T + T \otimes Id + \lambda\mu).$$

The triple $(\mathfrak{g}, V, T)$ is called a relative Rota-Baxter Lie algebra. When $V = \mathfrak{g}$ is the adjoint representation, the pair $(\mathfrak{g}, T)$ is called a relative Rota-Baxter Lie algebra.

Theorem

- *Can define cohomology theory of (relative) Rota-Baxter Lie algebras*
- *Can defined homotopy (relative) Rota-Baxter Lie algebras*
- *Can found the L_∞-algebra on the deformation complex of (relative) Rota-Baxter Lie algebras*

R. Tang (唐荣), C. Bai (白承铭), L. Guo (郭锂) and Y. Sheng (生云鹤), *Deformations and their controlling cohomologies of O-operators*. Comm. Math. Phys. **368** (2019), no. 2, 665-700.

A. Lazarev, Y. Sheng (生云鹤), R. Tang (唐荣), *Deformations and Homotopy Theory of Relative Rota-Baxter Lie Algebras*. Comm. Math. Phys. **383** (2021), no. 1, 595-631.

J. Chen (陈骏), Z. Qi (齐子豪), K. Wang (王凯), G. Zhou (周国栋), *(De)colouring in operad theory with applications to homotopy theory of operated algebras*, preprint 2024.

Part V.2: (Relative) Rota-Baxter Lie algebras

Theorem

- *Can show the minimal model for (relative) Rota-Baxter Lie algebras*
- *Can find the Koszul dual homotopy cooperad*
- *Can show that the L¡nfty-algebra found before is deduced from the minimal model*

J. Chen (陈骏), Z. Qi (齐子豪), K. Wang (王凯), G. Zhou (周国栋), *(De)colouring in operad theory with applications to homotopy theory of operated algebras*, preprint 2024.

Part V.3: Differential associative algebras

Definition

Let $\lambda \in \mathbf{k}$ be a fixed element. A differential associative algebra of weight λ is an associative algebra (A, μ_A) together with a linear operator $d_A : A \to A$ such that

$$d_A(ab) = d_A(a)b + a d_A(b) + \lambda\, d_A(a) d_A(b), \quad \forall a, b \in A.$$

Example

- *Let $A = C^\infty(\mathbb{R})$ be the algebra of smooth functions on $\mathbb{R}$. Let d be the classical derivation operation of smooth functions. Then A is a differential algebra of weight 0.*
- *A differential operator of weight 1 is sometimes called an Ito derivation (due to Loday), a crossed homomorphism or a difference operator.*

Part V.3: Zero weight vs nonzero weight

Remark

The defining relation of differential associative operators of any weight is given by

$$d_A \circ \mu_A = \mu_A \circ (d_A \otimes \mathrm{Id} + \mathrm{Id} \otimes \mu_A) + \lambda\ \mu_A \circ (d_A \otimes d_A)$$

expressed in terms of maps. If $\lambda = 0$, the operad of differential associative algebras of weight zero is Koszul, as shown by Loday. However, when $\lambda \neq 0$, this relation is NOT quadratic and the operad of differential associative algebras of nonzero weight is not quadratic and not even homogeneous, so the Koszul duality theory for operads could not be applied directly to develop a cohomology theory of differential associative algebras of any weight.

Problem (Loday 2010)

Develop the homotopy theory of differential associative algebras of nonzero weight.

J.-L. Loday, *On the operad of associative algebras with derivation*, Georgian Math. J. **17** (2010), 347-372.

Part V.3: Differential associative algebras

Theorem

- *Can introduce cohomology theory of differential associative algebras of arbitrary weight*

L. Guo (郭锂), Y. Li (黎允楠), Y. Sheng (生云鹤) and G. Zhou (周国栋), *Cohomologies, extensions and deformations of differential algebras with any weights,* Theory Appl. Categ. Vol. 38, 2022, No. 37, pp 1409-1433.

Theorem

- *Can define homotopy differential associative algebras.*
- *The dg operad ${}_{\lambda}\mathfrak{Dif}_{\infty}$ of homotopy differential algebras is the minimal model of the operad ${}_{\lambda}\mathfrak{Dif}$ governing differential associative algebras.*
- *Can introduce the Koszul dual homotopy cooperad ${}_{\lambda}\mathfrak{Dif}^{¡}$ such that $\Omega({}_{\lambda}\mathfrak{Dif}^{¡)={}_{\lambda}\mathfrak{Dif}_{\infty}}$.*
- *Can show that the $L_{¡}$nfty-algebra found before is deduced from the minimal model*

J. Chen (陈骏), L. Guo (郭锂), K. Wang (王凯) and G. Zhou (周国栋), *The homotopy theory, minimal model and L_{∞}-structure of differential algebras of arbitrary weights*, Adv. Math. **437**, (2024), Paper No. 109438, 41pp.

Definition

Let $\mathfrak{g} = (\mathfrak{g}, \mu = [,])$ *be a Lie algebra over a field* k *and* V *a representation of* $\mathfrak{g}$. *A* k*-linear map* $d : V \to \mathfrak{g}$ *is said to be a* **relative differential operator of weight** λ *if it satisfies*

$$d \circ \mu = \mu \circ (d \otimes \mathrm{Id} + \mathrm{Id} \otimes \mu) + \lambda \, \mu \circ (d \otimes d).$$

The triple $(\mathfrak{g}, V, D)$ *is called a relative differential Lie algebra. When* $V = \mathfrak{g}$ *is the adjoint representation, the pair* $(\mathfrak{g}, d)$ *is called a relative differential Lie algebra.*

Theorem

- *Can define cohomology theory of (relative) differential Lie algebras*
- *Can defined homotopy (relative) differential Lie algebras*
- *Can found the L_∞-algebra on the deformation complex of (relative) Rota-Baxter Lie algebras*

Y. Pei (裴玉峰), Y. Sheng (生云鹤), R. Tang (唐荣) AND K. Zhao (赵开明), *ACTIONS OF MONOIDAL CATEGORIES AND REPRESENTATIONS OF CARTAN TYPE LIE ALGEBRAS*, J. Instit. Math. Jussieu to appear.

J. Jiang (姜军), Y. Sheng (生云鹤), *Deformations, cohomologies and integrations of relative difference Lie algebras*, J. Algerba **614** (2023), 535-563.

Weiguo Lyu(吕为国), Z. Qi (齐子豪), Jian Yang (杨健), G. Zhou (周国栋), *Formal deformations, cohomology theory and L_∞-structures for (absolute and relative) differential Lie algebras of arbitrary weight*, preprint 2022.

Part V.4: (Relative) differential Lie algebras

Theorem

- *Can show the minimal model for (relative) differential Lie algebras*
- *Can find the Koszul dual homotopy cooperad*
- *Can show that the L¡nfty-algebra found before is deduced from the minimal model*

Weiguo Lyu(吕为国), Z. Qi (齐子豪), Jian Yang (杨健), G. Zhou (周国栋), *Formal deformations, cohomology theory and L_∞-structures for (absolute and relative) differential Lie algebras of arbitrary weight*, preprint 2022.

Part V.5: Averaging algebras and embedding tensors

Definition

Let $R = (R, \mu = \cdot)$ be an associative algebra over field k. An averaging operator over R is a k-linear map $A : R \to R$ such that

$$A(x)A(y) = A(A(x)y) = A(xA(y))$$

for all $x, y \in R$ or equivalently

$$\mu \circ (A \otimes A) = A \circ \mu \circ (A \otimes \mathrm{Id}) = A \circ \mu \circ (\mathrm{Id} \otimes A).$$

Definition

Let $\mathfrak{g} = (\mathfrak{g}, \mu = [,])$ be a Lie algebra over a field k and V a representation of $\mathfrak{g}$. A k-linear map $A : V \to \mathfrak{g}$ is an embedding tensor if

$$[A(x), A(y)] = A(A(x)y)$$

for all $x, y \in V$ or equivalently

$$\mu \circ (A \otimes A) = A \circ \mu \circ (A \otimes \mathrm{Id}) = A \circ \mu \circ (\mathrm{Id} \otimes A).$$

Part V.5: Averaging algebras and embedding tensors

Theorem

- *Can define cohomology theory of averaging algebras and embedding tensors.*
- *Can defined homotopy averaging algebras and homotopy embedding tensors.*
- *Can determine the L_∞-structure on deformation complexes for averaging algebras and embedding tensors.*

K. Wang (王凯) and G. Zhou (周国栋), *Cohomology theory of averaging algebras, L_∞-structures and homotopy averaging algebras*, arXiv:2009.11618.

Y. Sheng (生云鹤), R. Tang (唐荣) and C. Zhu (朱晨畅), *The controling L_∞-algebras, cohomology and homotopy of embedding tensors and Lie-Leibniz triples*, Comm. Math. Phys. **386** (2021), no. 1, 269-304

Problem

Prove that the operad of homotopy averaging algebras and of homotopy embedding tensors is the minimal model of the operad of averaging algebras and of embedding tensors

Definition

Let $(A, \mu = \cdot)$ (resp. $\mathfrak{g} = (\mathfrak{g}, \mu = [,])$) be an associative algebra (resp. a Lie algebra) over field $\mathbf{k}$*. A linear operator $P : A \to A$ is said to be a Nijenhuis operator if it satisfies*

$$\mu \circ (P \otimes P) = P \circ \big(\mu \circ (\mathrm{Id} \otimes P) + \mu \circ (P \otimes \mathrm{Id}) - P \circ \mu\big). \tag{3}$$

In this case, (A, μ, P) is called a Nijenhuis algebra.

A. V. Bolsinov, A. Y. Konyaev, V. S. Matveev, *Nijenhuis geometry*. Adv. Math. **394** (2022), Paper No. 108001, 52 pp.

Part V.6: Nijenhuis algebras

Theorem

- *Can define cochain complex of Nijenhuis algebras*
- *Can found the L_∞-structure on the cochain complex*
- *Can define homotopy Nijenhuis algebras*
- *Can prove the minimal model*
- *Can find the Koszul dual homotopy cooperad*
- *Have applications to Nijenhuis geometry*

C. Song (宋朝), K. Wang (王凯), Y. Zhang (张园园), G. Zhou (周国栋), *The homotopy theory of Nijenhuis algebras with applications to Nijenhuis geometry* , in preparasion.

Part V: A hint for relative Koszul duality for operads

Definition

A Rota-Baxter algebra of weight λ is a vector space endowed with and a linear operator $T : R \to R$ subject to

$$\mu \circ (\mu \otimes Id) = \mu \circ (Id \otimes \mu)$$

and

$$\mu \circ (T \otimes T) = T \circ \mu \circ (Id \otimes T) + T \circ \mu \circ (T \otimes Id) + \lambda T \circ \mu .$$

Proposition

The deformation complex of T with μ fixed is a differential graded Lie algebra.
That is, T is "QL-Koszul with respect to μ"!

Thank you very much