Equivariant Index Theorem on $\mathbb{R}^n$ in the Context of Continuous Fields of C^*-algebras

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Algebraic, analytic, geometric structures emerging from quantum field theory
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Noncommutative geometry (NCG)

One aim of NCG is to reformulate invariants in geometry and topology in terms of invariants for algebras, apply them to "more singular spaces."

- compact Hausdorff space $X \leftrightarrow C(X)$algebra of continuous functions;
- Hausdorff space $X \leftrightarrow C_0(X)$ algebra of continuous functions that vanish at infinity;
- Noncommutative generalization of $C(X)$ or $C_0(X)$: C^*-algebra A with(out) a unit, for example,

  $$\mathbb{C}, C_0(\mathbb{R}), M_n(\mathbb{C}), \mathcal{K}.$$

  A C^*-algebra is a closed subalgebra of $\mathcal{B}(H)$ under the operator norm.
- topological K-theory $K^0(X) \leftrightarrow K_0(C(X))$ operator K-theory;
- For a C^*-algebra A, its operator K-theory is an abelian group having the form

  $$K_0(A) := \left\{ [p] - [q] | p, q \in M_\infty(A) := \cup_n M_n(A) / a \sim \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \right\}.$$

Bott periodicity

- $K_0(\mathbb{C}) \cong \mathbb{Z}$;
- $K_0(\mathcal{K}) \cong \mathbb{Z}$ induced by matrix trace Tr;
- Bott periodicity
$$K_0(\mathbb{C}) \cong K_0(C_0(\mathbb{R}^2)).$$

Bott periodicity in other forms:

- homotopy theory: $\pi_k(U) \cong \pi_{k+2}(U)$ for $U = \cup_n U(n)/a \sim \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$;
- topological K-theory: $K^0(X) \cong K^2(X)$ for a topological space X;
- Thom isomorpshim: $K^0(X) \cong K^0(E)$ for a complex vector bundle $E \to X$ (crucial in the proof of Atiyah-Singer index theorem);
- operator K-theory: $K_0(A) \cong K_2(A)$ for C^*-algebra A.

Heisenberg group algebra as a continuous field

- Let H_3 be the Heisenberg group: $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ with multiplication

$$(x, y, t) \cdot (x', y', t') := (x + x', y + y', t + t' + \frac{1}{2}(xy' - x'y)).$$

- The group C^*-algebra $C^*(H_3)$ is the norm closure of $C_c(H_3) \subset \mathcal{B}(L^2(H_3))$ given by convolutions.
- Under the Fourier transform $F = \{F_\lambda\}_{\lambda \in \mathbb{R}}$, where

$$F_\lambda(c) := \begin{cases} \pi_\lambda(c) & \lambda \in \mathbb{R}^* \\ \widehat{\rho(c)} & \lambda = 0 \end{cases} \text{ for } c \in C^*(H_3),$$

$$C^*(H_3) = C_0(\mathbb{R}^2) \times \{0\} \sqcup \mathcal{K}(L^2(\mathbb{R})) \times \mathbb{R}^*$$

a continuous field of C^*-algebras. The continuity at 0 is in line with Connes' tangent groupoid:

$$(x_n, y_n, \lambda_n) \to (x, \xi) \Leftrightarrow x_n - y_n \to 0, \frac{x_n - y_n}{\lambda_n} \to \xi.$$

- The continuous field gives rise to the Bott periodicity

$$K_0(C_0(\mathbb{R}^2)) \cong K_0(\mathcal{K}).$$

Outline

This talk is about an algebraic index theorem, parallel to the Atiyah-Singer index theorem.

- Nest-Tsygan 95' (index theory on compact smooth manifolds)
- Elliott-Natsume-Nest 96' (index theory on $\mathbb{R}^n$)
- Equivariant ENN theorem (equivariant index theory on $\mathbb{R}^n$)

This is the 'equivariant index' perspective of the equivariant Bott periodicity. Main tools are

- continuous fields of C^*-algebras and
- cyclic cohomology from NCG.

Reference:

- Baiying Ren, Hang Wang, Zijing Wang: Equivariant index theorem on $\mathbb{R}^n$ in the context of continuous fields of C^*-algebras, arXiv:2401.07474.

1. ENN Theorem

Shubin's class of pseudodifferential operators

Let $u \in \mathcal{S}(\mathbb{R}^n)$ be a rapidly decreasing function. Then

$$u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle}(Fu)(\xi)d\xi,$$

where F is the Fourier transform.
Let $a \in C^\infty(T^*\mathbb{R}^n)$. A pseudodifferential operator (abbr. ΨDO) P_a is

$$(P_a u)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} a(x,\xi)(Fu)(\xi)d\xi,$$

where $u \in \mathcal{S}(\mathbb{R}^n)$ and a will be called the (total) symbol of P_a.
To let P_a make sense, a needs to satisfy certain conditions.

Definition

$a \in C^\infty(T^*\mathbb{R}^n)$ is a symbol of order $m \in \mathbb{R}$ if $\forall$ multi-index α, $\exists C_\alpha$ s.t.

$$|\partial_z^\alpha(a)| \leq C_\alpha(1+|z|^2)^{\frac{m-|\alpha|}{2}},\ z \in T^*\mathbb{R}^n.$$

Let $\Gamma^m(T^*\mathbb{R}^n)$ be the space of symbols of order m.

Fredholm index

If $a \in \Gamma^m(T^*\mathbb{R}^n)$, then $P_a : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ given by

$$(P_a u)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} a(x,\xi)(Fu)(\xi) d\xi, \ u \in \mathcal{S}(\mathbb{R}^n),$$

is a continuous operator. P_a is also said to be of order m.
If $V = W = \mathbb{C}^k$, $a \in M_k(\Gamma^m(T^*\mathbb{R}^n))$, $P_a : \mathcal{S}(\mathbb{R}^n; V) \to \mathcal{S}(\mathbb{R}^n; W)$.

Definition

$a \in M_k(\Gamma^m(T^*\mathbb{R}^n))$ is said to be elliptic if $\exists C, R > 0$ s.t.

$$a(x,\xi)^* a(x,\xi) \geq C(|x|^2 + |\xi|^2)^m I_k \text{ for } |x|^2 + |\xi|^2 \geq R.$$

P_a is also said to be elliptic.

Remark

If $P_a : \mathcal{S}(\mathbb{R}^n; V) \to \mathcal{S}(\mathbb{R}^n; W)$ is an elliptic ΨDO, then P_a is Fredholm and its formal adjoint P_a^* is also Fredholm. Denote by the Fredholm index

$$\mathrm{ind}(P_a) = \dim(\mathrm{Ker} P_a|_{\mathcal{S}(\mathbb{R}^n;V)}) - \dim(\mathrm{Ker} P_a^*|_{\mathcal{S}(\mathbb{R}^n;W)}).$$

Example

Remark

Furthermore, a ΨDO P_a extends to a possibly unbounded operator

$$P_a : L^2(\mathbb{R}^n; V) \to L^2(\mathbb{R}^n; W)$$

with dense domain $\{f \in L^2(\mathbb{R}^n; V) | P_a f \in L^2(\mathbb{R}^n; W)\}$, which is also Fredholm when P_a is elliptic. In this case,

$$\mathrm{ind}(P_a) = \dim(\mathrm{Ker} P_a|_{L^2(\mathbb{R}^n;V)}) - \dim(\mathrm{Ker} P_a^*|_{L^2(\mathbb{R}^n;W)}).$$

Indeed, if $a \in \Gamma^0(T^*\mathbb{R}^n)$, then P_a extends to a bounded operator from $L^2(\mathbb{R}^n; V)$; while if $a \in \Gamma^m(T^*\mathbb{R}^n)$, $m < 0$, then P_a extends to a compact operator from $L^2(\mathbb{R}^n; V)$.

Example

Define $a \in C^\infty(T^*\mathbb{R}^n)$ by $a(x,\xi) = x + i\xi$. Then $P_a = x + \frac{d}{dx}$ is an elliptic ΨDO on $L^2(\mathbb{R}^n)$ of order 1. Furthermore, P_a is Fredholm and $\mathrm{ind}(P_a) = 1$.

ENN theorem

Let $P_a : \mathcal{S}(\mathbb{R}^n; V) \to \mathcal{S}(\mathbb{R}^n; W)$ be an elliptic ΨDO with symbol a of positive order. Denote by

$$e_a = \begin{pmatrix} (1+a^*a)^{-1} & (1+a^*a)^{-1}a^* \\ a(1+a^*a)^{-1} & a(1+a^*a)^{-1}a^* \end{pmatrix}$$

the graph projection induced by the closed multiplication operator $a : L^2(T^*\mathbb{R}^n; V) \to L^2(T^*\mathbb{R}^n; W)$.

Theorem (Elliott-Natsume-Nest, 96')

$$\mathrm{ind}(P_a) = \frac{1}{(2\pi i)^n n!} \int_{T^*\mathbb{R}^n} \mathrm{tr}(\hat{e}_a (d\hat{e}_a)^{2n}),$$

where $\hat{e}_a = e_a - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

analytic index = topological index

2. Proof of ENN Theorem

The analytic index

View $P_a : L^2(\mathbb{R}^n; V) \to L^2(\mathbb{R}^n; W)$ as an unbounded operator. P_a is closable. Denote by T the closure of P_a.
Denote by e_1 the graph projection of T. Then

$$e_1 - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} (1+T^*T)^{-1} & (1+T^*T)^{-1}T^* \\ T(1+T^*T)^{-1} & (1+TT^*)^{-1} \end{pmatrix} \in \mathcal{K}(L^2(\mathbb{R}^n; V \oplus W)),$$

where $\mathcal{K}(L^2(\mathbb{R}^n))$ stands for compact operators.

Proposition (Elliott-Natsume-Nest, 96')

$$[e_1] - \left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right] = [\mathrm{Ker} P_a] - [\mathrm{Ker} P_a^*] \in K_0\left(\mathcal{K}(L^2(\mathbb{R}^n; V \oplus W))\right).$$

Fact: The canonical trace induces an isomorphism

$$\mathrm{Tr} : K_0\left(\mathcal{K}(L^2(\mathbb{R}^n; V \oplus W))\right) \to \mathbb{Z}.$$

Step 1: Represent $\mathrm{ind}(P_a)$ by the K-theory class:

$$\mathrm{ind}(P_a) = \mathrm{Tr}\left([e_1] - \left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right] \right).$$

Sufficiently large subalgebra $\mathcal{K}^\infty$

Construct the Kohn-Nirenberg quantization as follows.
For $\hbar \in (0,1]$, let $P_\hbar$ be the ΨDO with symbol $a_\hbar(x,\xi) := a(x,\hbar\xi)$.
Denote by $e_\hbar$ the graph projection of $P_\hbar$. Then

$$\mathrm{ind}(P_a) = \mathrm{Tr}\left([e_\hbar] - \left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right]\right),\ \hbar \in (0,1].$$

Fact: $e_\hbar - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ is not a trace-class operator for $\hbar \in (0,1]$.
Idea: Denote by $\mathcal{K}^\infty(L^2(\mathbb{R}^n))$ the subalgebra of integral operators with Schwartz kernels in $\mathcal{S}(T^*\mathbb{R}^n)$. $\mathcal{K}^\infty(L^2(\mathbb{R}^n))$ is dense and stable under the holomorphic functional calculus.

$$\mathrm{Tr} : K_0(\mathcal{K}(L^2(\mathbb{R}^n))) \cong K_0(\mathcal{K}^\infty(L^2(\mathbb{R}^n))) \to \mathbb{Z}.$$

Step 2: Search for $e_\hbar^\infty - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in \mathcal{K}^\infty(L^2(\mathbb{R}^n; V \oplus W))$!

Continuous fields of C^*-algebras

The group C^*-algebra of the $(2n+1)$-dimensional Heisenberg group $C^*(H_{2n+1})$ can be identified as the continuous field of C^*-algebras $(A_\hbar)_{\hbar\in\mathbb{R}}$, where

$$A_0 = C_0(T^*\mathbb{R}^n),$$
$$A_\hbar = \mathcal{K}(L^2(\mathbb{R}^n)),\ \hbar \neq 0.$$

Here we restrict the interval to $[0,1]$.
The continuous field structure is given by the family of sections

$$\{\hbar \in [0,1] \mapsto \rho_\hbar(\hat{f}) \in A_\hbar |\ f \in \mathcal{S}(\mathbb{R}^{2n+1})\}.$$

Here $\rho_\hbar(\hat{f}) \in \mathcal{K}^\infty(L^2(\mathbb{R}^n)) \subset \mathcal{K}(L^2(\mathbb{R}^n))$ for $\hbar \in (0,1]$ is given by

$$\rho_\hbar(\hat{f})\phi(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \hat{f}(x,y,\hbar)\phi(x+\hbar y)dy,\ \phi \in L^2(\mathbb{R}^n),$$

where $\hat{f}$ denotes the Fourier transform with respect to the second variable of $f \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R})$.
Define $\rho_0(\hat{f}) \in \mathcal{S}(T^*\mathbb{R}^n) \subset C_0(T^*\mathbb{R}^n)$ as the restriction $f|_{T^*\mathbb{R}^n\times 0}$.

Continuous section

- Recall that $e_\hbar \in \mathcal{K}(L^2(\mathbb{R}^n))^\sim$ is the graph projection of $P_\hbar$ associated with symbol $a_\hbar(x,\xi) := a(x,\hbar\xi)$, $\hbar \in (0,1]$.
- Let $e_0 := e_a$. Then $e_0 \in C_0(T^*\mathbb{R}^n)^\sim$ because a has positive order.

Theorem (Elliott-Natsume-Nest, 96')

The vector field $e = (e_\hbar)_{\hbar\in[0,1]}$ of $({A_\hbar}^\sim)_{\hbar\in[0,1]}$ is continuous.

Remark

There exists a continuous section $e^\infty = (e^\infty_\hbar)_\hbar$ of projections s.t.

$$\|e - e^\infty\| = \sup_{\hbar\in[0,1]} \|e_\hbar - e^\infty_\hbar\|_\hbar < \epsilon,$$

$$\forall\epsilon > 0, \text{ and } \exists f \in \mathcal{S}(\mathbb{R}^{2n+1}) \text{ s.t. } e^\infty_\hbar - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \rho_\hbar(\hat{f}), \hbar \in [0,1].$$

Fact: $\mathrm{Tr}\left(\rho_\hbar(\hat{f})\right) = \frac{1}{\hbar^n}\int_{\mathbb{R}^n} \hat{f}(x,0,\hbar)dx$ doesn't yield the index formula as $\hbar$ converges to 0.

Step 3: Search for cyclic cocycle ω to replace Tr!

Cyclic cohomology: an overview

Let A be a unital algebra over $\mathbb{C}$.

Definition

A cyclic n-cochain on A is an $(n+1)$-linear functional ϕ s.t.

$$\phi(a_n, a_0, \cdots, a_{n-1}) = (-1)^n \phi(a_0, \cdots, a_n),\ \forall a_0, \cdots, a_n \in A.$$

Denote by $C_\lambda^n(A)$ the space of cyclic n-cochains on A. The differential $b : C_\lambda^n(A) \to C_\lambda^{n+1}(A)$ is defined by:

$$(b\phi)(a_0, \cdots, a_n, a_{n+1}) := \sum_{i=0}^{n} (-1)^i \phi(a_0, \cdots, a_j a_{j+1} \cdots, a_n, a_{n+1}) + (-1)^{n+1} \phi(a_{n+1} a_0, a_1, \cdots, a_n).$$

One checks that $b^2 = 0$. Thus we obtain the cyclic complex of A:

$$C_\lambda^0(A) \xrightarrow{b} C_\lambda^1(A) \xrightarrow{b} C_\lambda^2(A) \xrightarrow{b} \cdots$$

Denote by $H_\lambda^n(A)$, $n = 0, 1, \ldots,$ the cohomology of the cyclic complex, called the cyclic cohomology of A.

Pairing with K-theory

Denote by $Z^n_\lambda(A) := \mathrm{Ker} b|_{C^n_\lambda(A)}$ the space of cyclic n-cocycles on A.
Define the pairing $\langle \ , \ \rangle : K_0(A) \times H^{2n}_\lambda(A) \to \mathbb{C}$ by

$$\langle [e], [\phi] \rangle = (2\pi i)^{-n}(n!)^{-1}\phi\#\mathrm{tr}(e, \cdots, e),$$

where $e \in M(A)$ is an idempotent, $\phi \in Z^{2n}_\lambda(A)$.
There exists the periodic operator $S : H^n_\lambda(A) \to H^{n+2}_\lambda(A)$ s.t.

$$\langle [e], [\phi] \rangle = \langle [e], S[\phi] \rangle.$$

Example: Tr is a cyclic 0-cocycle on $\mathcal{L}_1(L^2(\mathbb{R}^n))$.

Cyclic cohomolgy in ENN theorem

- Define the (2n+1)-linear functional ω on $\mathcal{K}^\infty(L^2(\mathbb{R}^n))$ by

$$\omega(T_0, \cdots, T_{2n}) = \frac{(-1)^n}{n!} \sum_{\sigma \in S_{2n}} \mathrm{sgn}(\sigma) \mathrm{Tr}\left(T_0 \delta_{\sigma(1)}(T_1) \cdots \delta_{\sigma(2n)}(T_{2n})\right),$$

  where $\delta_{2j-1}(T) = [\partial_{x_j}, T]$, $\delta_{2j}(T) = [x_j, T]$, $j \leq n$.
- ω is a cyclic $2n$-cocycle.
  $\forall$ idempotent $T \in \mathcal{K}^\infty(L^2(\mathbb{R}^n))$, $\langle [T], [(2\pi i)^n n! \omega] \rangle = \mathrm{Tr}(T)$.
- Define the (2n+1)-linear functional ϵ on $\mathcal{S}(T^*\mathbb{R}^n)$ as follows,

$$\epsilon(f_0, \cdots, f_{2n}) = \frac{1}{(2\pi i)^n n!} \int_{T^*\mathbb{R}^n} f_0 df_1 \cdots df_{2n}.$$

- ϵ is a cyclic $2n$-cocycle.

Deformation

Finally, we complete the proof.

$$
\begin{aligned}
\mathrm{ind}(P_a) &= \langle [\mathrm{Ker} P_a] - [\mathrm{Ker} P_a^*], [(2\pi i)^n n! \omega] \rangle \\
&= \left\langle [e_1] - \left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right], [(2\pi i)^n n! \omega] \right\rangle \\
&= \left\langle [e_1^\infty] - \left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right], [(2\pi i)^n n! \omega] \right\rangle \\
&= \left\langle [e_\hbar^\infty] - \left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right], [(2\pi i)^n n! \omega] \right\rangle \text{ for all } \hbar > 0 \\
&\overset{\blacktriangle}{=} \left\langle [e_0^\infty] - \left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right], [(2\pi i)^n n! \epsilon] \right\rangle \\
&= \left\langle [e_0] - \left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right], [(2\pi i)^n n! \epsilon] \right\rangle \\
&= \frac{1}{(2\pi i)^n n!} \int_{T^*\mathbb{R}^n} \mathrm{tr}(\hat{e}_a (d\hat{e}_a)^{2n}). \quad \text{topological index}
\end{aligned}
$$

3. Equivariant ENN Theorem

Equivariant index

Let

- compact group $G = SO(n)$ acting on $\mathbb{R}^n$ by isometry
- $P_a : \mathcal{S}(\mathbb{R}^n; V) \to \mathcal{S}(\mathbb{R}^n; W)$: elliptic ΨDO of positive order
- P_a is G-invariant, i.e., V, W are G-spaces and $\forall g \in G$,

$$g|_{L^2(\mathbb{R}^n;W)} P_a = P_a g|_{L^2(\mathbb{R}^n;V)}.$$

Denote by $\mathrm{ind}_G(P_a)$ the equivariant index of P_a given by,

$$\mathrm{ind}_G(P_a) := [\mathrm{Ker} P_a] - [\mathrm{Ker} P_a^*] \in R(G),$$

or equivalently,

$$\begin{aligned} \mathrm{ind}_G(P_a) : \ & G \to \mathbb{C} \\ & g \mapsto \mathrm{ind}_{(g)}(P_a) := \mathrm{Tr}(g|_{\mathrm{Ker} P_a}) - \mathrm{Tr}(g|_{\mathrm{Ker} P_a^*}). \end{aligned}$$

Equivariant index theorem on $\mathbb{R}^n$

Theorem (Ren-W-Wang)

If the fixed-point set $(\mathbb{R}^n)^g$ has positive dimension, then

$$\mathrm{ind}_{(g)}(P_a) = \frac{1}{(2\pi i)^{n_g} n_g! \det(g-1)} \int_{T^*(\mathbb{R}^n)^g} \mathrm{tr}\left[\begin{pmatrix} g^V & 0 \\ 0 & g^W \end{pmatrix} \hat{e}_a (d\hat{e}_a)^{2n_g} \right].$$

- $n_g := \dim(\mathbb{R}^n)^g$
- $\det(g-1)$: the determinant of $g-1$ on $((\mathbb{R}^n)^g)^\perp$
- g^V: the matrix of $g \in Aut(V)$; so is g^W

When g is the group identity, it reduces to ENN theorem.

Theorem (Ren-W-Wang)

If $(\mathbb{R}^n)^g = \{0\}$, then

$$\mathrm{ind}_{(g)}(P_a) = \frac{1}{\det(g-1)} \mathrm{tr}\left[\begin{pmatrix} g^V & 0 \\ 0 & g^W \end{pmatrix} \hat{e}_a(0,0) \right].$$

Application to the Bott-Dirac operator

- $\mathrm{Cliff}_{\mathbb{C}}(\mathbb{R}^{2n})$: the universal unital complex algebra containing $\mathbb{R}^{2n}$ subject to the relations $xx = |x|^2$, $\forall x \in \mathbb{R}^{2n}$.
- Fix an orthonormal basis $\{e_1, \cdots, e_{2n}\}$ of $\mathbb{R}^{2n}$, then $e_{i_1} \cdots e_{i_k}$ for $i_1 < \cdots < i_k$ form a linear basis of $\mathrm{Cliff}_{\mathbb{C}}(\mathbb{R}^{2n})$.
- $\mathrm{Cliff}_{\mathbb{C}}(\mathbb{R}^{2n})$ is a Hermitian inner space by deeming these monomials to be orthonormal. It is $\mathbb{Z}_2$-graded decomposed as

$$\mathrm{Cliff}_{\mathbb{C}}(\mathbb{R}^{2n}) = (\mathrm{Cliff}_{\mathbb{C}}(\mathbb{R}^{2n}))_0 \oplus (\mathrm{Cliff}_{\mathbb{C}}(\mathbb{R}^{2n}))_1.$$

- Clifford multiplication operators $\hat{c}(e_i)$, $c(e_i)$ on $\mathrm{Cliff}_{\mathbb{C}}(\mathbb{R}^{2n})$:

$$\hat{c}(e_i) : w \mapsto (-1)^{\deg(w)} w e_i, \ \ c(e_i) : w \mapsto e_i w.$$

- Bott-Dirac operator on $L^2(\mathbb{R}^{2n}; \mathrm{Cliff}_{\mathbb{C}}(\mathbb{R}^{2n}))$ is defined by

$$B = D + C = \sum_{i=1}^{2n} \hat{c}(e_i) \frac{\partial}{\partial x_i} + c(e_i) x_i.$$

Note that B is an odd, essentially self-adjoint operator.

Equivariant index of the Bott-Dirac operator

Consider $SO(2n)$ acts on $\mathrm{Cliff}_{\mathbb{C}}(\mathbb{R}^{2n})$ by diagonal.

- Firstly, B is an $SO(2n)$-invariant elliptic ΨDO of positive order. Denote by the symbol of B

$$\begin{pmatrix} 0 & a^* \\ a & 0 \end{pmatrix} = \sum_{j=1}^{2n} (\hat{c}(e_j) i\xi_j + c(e_j)x_j).$$

  Then a is elliptic and has order 1 because

$$\begin{pmatrix} a^*a & 0 \\ 0 & aa^* \end{pmatrix} = |z|^2 I.$$

- Next, when $n = 1$, the equivariant index of B is 1 at each $g \in SO(2)$. For $n > 1$, it remains true by induction.
  - $\{1, e_1e_2, e_1, e_2\}$: basis of $(\mathrm{Cliff}_{\mathbb{C}}(\mathbb{R}^2))_0 \oplus (\mathrm{Cliff}_{\mathbb{C}}(\mathbb{R}^2))_1$
  - $a = \begin{pmatrix} x_1 + i\xi_1 & -x_2 + i\xi_2 \\ x_2 + i\xi_2 & x_1 - i\xi_1 \end{pmatrix}$, $\hat{e}_a = \frac{1}{1+|x|^2+|z|^2} \begin{pmatrix} I_2 & a^* \\ a & -I_2 \end{pmatrix}$

$$B:\mathcal{S}(\mathbb{R}^2;(\mathrm{Cliff}_{\mathbb{C}}(\mathbb{R}^2))_0)\to\mathcal{S}(\mathbb{R}^2;(\mathrm{Cliff}_{\mathbb{C}}(\mathbb{R}^2))_1)$$

- $\hat{e}_a=\frac{1}{1+|x|^2+|z|^2}\begin{pmatrix} I_2 & a^* \\ a & -I_2\end{pmatrix}$
- For $g=\begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}\in SO(2)$, $\theta\in[0,2\pi]$:

$$g^{(\mathrm{Cliff}_{\mathbb{C}}(\mathbb{R}^2))_0}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\ g^{(\mathrm{Cliff}_{\mathbb{C}}(\mathbb{R}^2))_1}=\begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}$$

When $g=1\in SO(2)$, the whole $\mathbb{R}^2$ is fixed by g,

$$\mathrm{ind}_{(1)}(B)=\frac{1}{(2\pi i)^2 2!}\int_{T^*\mathbb{R}^2}\mathrm{tr}(\hat{e}_a(d\hat{e}_a)^4)=1.$$

When $g\neq 1\in SO(2)$, $\cos\theta\neq 1$ and $\left(\mathbb{R}^2\right)^g=\{0\}$,

$$\begin{aligned}\mathrm{ind}_{(g)}(B)&=\frac{1}{\det(g-1)}\mathrm{tr}\left[\begin{pmatrix} g^{(\mathrm{Cliff}_{\mathbb{C}}(\mathbb{R}^2))_0} & 0 \\ 0 & g^{(\mathrm{Cliff}_{\mathbb{C}}(\mathbb{R}^2))_1}\end{pmatrix}\hat{e}_a(0,0)\right]\\ &=\frac{2-2\cos\theta}{2-2\cos\theta}=1.\end{aligned}$$

Proof of the equivariant ENN theorem

Step 1: Represent $\operatorname{ind}_{(g)}(P_a)$ by the equivariant K-theory class.

$$[e_1] - \left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right] = [\mathrm{Ker} P_a] - [\mathrm{Ker} P_a^*] \in K_0^G\left(\mathcal{K}(L^2(\mathbb{R}^n; V \oplus W))\right).$$

$$\{\mathrm{Tr}(\hat{g}\cdot)\}_{g\in G} : K_0^G\left(\mathcal{K}(L^2(\mathbb{R}^n; V \oplus W))\right) \to R(G),$$

where $\hat{g} = \begin{pmatrix} g|_{L^2(\mathbb{R}^n;V)} & 0 \\ 0 & g|_{L^2(\mathbb{R}^n;W)} \end{pmatrix}$. Then

$$\operatorname{ind}_{(g)}(P_a) = \mathrm{Tr}\left(\hat{g}\left([\mathrm{Ker} P_a] - [\mathrm{Ker} P_a^*]\right)\right) = \mathrm{Tr}\left(\hat{g}\left([e_1] - \left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right]\right)\right).$$

Step 2: We can choose $e^\infty = (e^\infty_\hbar)_\hbar$ to be G-invariant s.t.

$$[e_\hbar] - \left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right] = [e^\infty_\hbar] - \left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right] \in K_0\left(A^G_\hbar\right).$$

Step 3: Search for $\{\omega_g\}_{g\in G}$ to replace $\{\mathrm{Tr}(\hat{g}\cdot)\}_{g\in G}$ s.t.

- $\operatorname{ind}_{(g)}(P_a) = \langle [\hat{e}_1], [\omega_g] \rangle$ equivariant analytic index,
- $\lim_{\hbar\to 0} \langle [\hat{e}_\hbar], [\omega_g] \rangle = \langle [\hat{e}_a], [\epsilon_g] \rangle$ fixed point formula.

Proof of equivariant ENN theorem

Define the cyclic $2n_g$-cocycle ω_g on $\left(\mathcal{K}^\infty(L^2(\mathbb{R}^n; V \oplus W))\right)^g$:

$$\omega_g(T_0, \cdots, T_{2n_g}) = \frac{(-1)^{n_g}}{n_g!} \sum_{\sigma \in S_{2n_g}} \mathrm{sgn}(\sigma) \mathrm{Tr}\left(\hat{g} T_0 \delta_{\sigma(1)}(T_1) \cdots \delta_{\sigma(2n_g)}(T_{2n_g})\right),$$

$$\left(\mathcal{K}^\infty(L^2(\mathbb{R}^n; V \oplus W))\right)^g = \{T \in \mathcal{K}^\infty(L^2(\mathbb{R}^n; V \oplus W)) | \, \hat{g} T = T \hat{g}\},$$

- We shall consider $(\mathbb{R}^n)^g$ generated by $x_1, \cdots, x_{n_g}$.
- For any idempotent T, $\langle [T], [(2\pi i)^{n_g} (n_g)! \omega_g] \rangle = \mathrm{Tr}(\hat{g} T)$ because $\hat{g}$ commutes with T, ∂_{x_j}, x_j, $\forall j \leq n_g$.

$$\Longrightarrow \mathrm{ind}_{(g)}(P) = \langle [e_1] - \left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right], [(2\pi i)^{n_g} (n_g)! \omega_g] \rangle$$

$$= \langle [e_\hbar^\infty] - \left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right], [(2\pi i)^{n_g} (n_g)! \omega_g] \rangle \text{ for all } \hbar > 0$$

$$\color{blue}{\triangleq} \frac{1}{(2\pi i)^{n_g} n_g! \det(g-1)} \int_{T^*(\mathbb{R}^n)^g} \mathrm{tr}\left[\begin{pmatrix} g^V & 0 \\ 0 & g^W \end{pmatrix} \hat{e}_a (d\hat{e}_a)^{2n_g}\right],$$

$$\text{or } \color{blue}{\triangleq} \frac{1}{\det(g-1)} \mathrm{tr}\left[\begin{pmatrix} g^V & 0 \\ 0 & g^W \end{pmatrix} \hat{e}_a(0,0)\right]. \ \color{blue}{\text{topological index}}$$

Thank You!