Renormalisation of singular SPDEs on Riemannian manifolds

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A class of singular SPDEs

Let M Riemannian manifold, $E \to M$ a vector bundle with a connection and metric. Consider sub-critical equations of the form

$$(\partial_t + \mathfrak{L})u = F(u, \nabla u, ..., \nabla^k u) + \xi \ ,$$

where

- u (generalised) section of a vector bundle $E \to M$
- $\mathfrak{L}$ is an elliptic operator of order $> k$ on E,
- ξ is an irregular bundle valued (stochastic) noise.

Solution theories:

- Para-controlled Calculus [GIP15],(geometric setting [BB16]).
- Regularity Structures [Hai14],(geometric setting [DDD18], [HS23]).
- Renormalisation group flow [Kup14, Duc21].
- Multi-indices [OSSW21].

Some important examples

Φ^4_d-equation

Let $E \to M$ vector bundle with metric and connection.

$$(\partial_t + \triangle^E)u = -|u|^2 \cdot u + \xi \ ,$$

for ξ an E valued space time white noise.

Stochastic Yang–Mills heat flow

Let $P \to M$ principle G-bundle, $|\cdot|_{\mathfrak{g}}$ an Ad-invariant scalar product on $\mathfrak{g}$.
Consider principal connections $\omega = \omega^{ref} + A$

$$\partial_t A = -(d^\omega)^* F_\omega - d^\omega (d^\omega)^*(A) + \xi \ ,$$

for ξ a $\Omega^1(M, \mathrm{Ad}(\mathfrak{g}))$-valued space time white noise.

Content

1. Meta Theorem of subcritical SPDEs [BCCH20]
2. Construction of $\{\mathfrak{Dif}_T\}_{T\in\mathfrak{T}_-}$ on Manifolds

Section 1

Meta Theorem of subcritical SPDEs [BCCH20]

Setup

Meta equation

Consider a subcritical equation

$$\partial_t u + \mathcal{L}u = F(u, \nabla u, ..., \nabla^n u) + \xi \tag{1}$$

where

- $\mathcal{L}$ is an elliptic operator on $\mathbb{T}^d$ of order $> n$
- $\partial_t + \mathcal{L}$ is space-time translation invariant
- $\xi \in \mathcal{D}'(\mathbb{R} \times \mathbb{T}^d)$ space time white noise.

Problem:

- Consider $(\partial_t + \mathcal{L})v = \xi$
- Schauder estimates may not provide enough regularity for the non-linearity $F(v, \nabla v, ..., \nabla^n v)$ to be well defined!

Naive hope

Let ρ_ϵ smooth mollifiers such that and $\xi_\epsilon := \rho_\epsilon \star \xi \to \xi$ as $\epsilon \to 0$.Consider solutions u_ϵ of

$$\partial_t u_\epsilon + \mathcal{L} u_\epsilon = F(u_\epsilon, \nabla u_\epsilon, ..., \nabla^n u_\epsilon) + \xi_\epsilon. \tag{2}$$

Generically, u_ϵ does not converge as $\epsilon \to 0$, one needs *renormalisation*.

The theory of regularity structures provides the following type of result:

Metatheorem

Let G, $\mathcal{L}$, ξ as well as ρ_ϵ and ξ_ϵ be as above. Then, there exists a finite index set $\mathfrak{T}_-$ and non-linearities $\Upsilon^G[\tau]$ depending only on G, $\mathcal{L}$, ξ and as well as constants $c_\epsilon^\tau \in \mathbb{R}$ depending additionally on ρ_ϵ such for u_ϵ satisfying

$$\partial_t + \mathcal{L}u_\epsilon = F(u_\epsilon, \nabla u_\epsilon, ..., \nabla^n u_\epsilon) + \sum_{\tau \in \mathfrak{T}_-} c_\epsilon^\tau \Upsilon^F[\tau](u_\epsilon, \nabla u_\epsilon, ..., \nabla^n u_\epsilon) + \xi_\epsilon \ ,$$

there exists $u \in \mathcal{D}'((0,T) \times \mathbb{R}^d)$ independent of the choice of $\{\rho_\epsilon\}_{\epsilon \in (0,1)}$ such that $u_\epsilon \to u$ in probability as $\epsilon \to 0$.

Note that this metatheorem purposefully kept several aspects vague, see [BCCH20, Theorem 2.22].

Roadmap of the proof

Some vocabulary

- A *structure space* is a vector space/bundle T.
  Elements similar to abstract Taylor polynomials.
- A *model* Z gives analytic "meaning" to elements of T, similarly to the map $P(X) \mapsto p(x)$ abstract polynomial to polynomial function.
- Let $\mathcal{M}$ the *space of models*.
- Given a model $Z \in \mathcal{M}$ and $\gamma > 0$ we denote by $\mathcal{D}^\gamma$ the *space of modelled distributions* (maps $(t, x) \mapsto f(x) \in T$).
- There exists a *reconstruction operator*

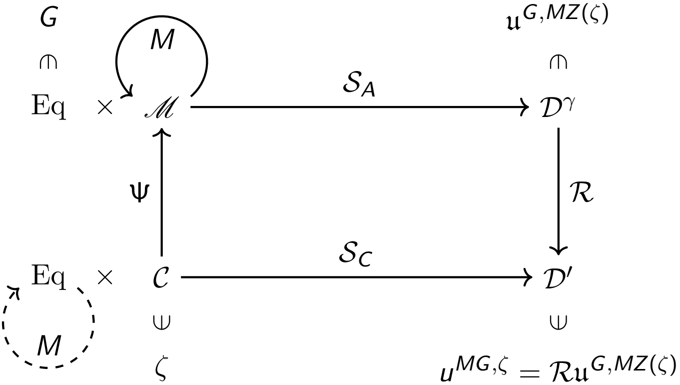
$$\mathcal{R} : \mathcal{D}^\gamma \to \mathcal{D}' \ .$$

Diagram

$$
\begin{array}{ccccc}
G & & Z(\zeta) & & \mathfrak{u}^{G,Z(\zeta)} \\
\Cap & & \Cap & & \Cap \\
\mathrm{Eq} & \times & \mathscr{M} & \xrightarrow{\quad \mathcal{S}_A \quad} & \mathcal{D}^{\gamma} \\
 & & \Big\uparrow \Psi & & \Big\downarrow \mathcal{R} \\
\mathrm{Eq} & \times & \mathcal{C} & \xrightarrow{\quad \mathcal{S}_C \quad} & \mathcal{D}' \\
\Cup & & \Cup & & \Cup \\
G & & \zeta & & u^{G,\zeta} = \mathcal{R}\mathfrak{u}^{G,Z(\zeta)}
\end{array}
$$

This factors the classical solution map $\mathcal{S}_C$. The maps $\mathcal{S}_A$ and $\mathcal{R}$ are continuous,but Ψ is not. In general, as $\xi_\epsilon \to \xi$ the models $\Psi(\xi_\epsilon) = Z(\xi_\epsilon)$ do *not* converge.

There is a renormalisation group $\mathfrak{G}_-$ acting on $\mathcal{M}$ and “space of right hand sides” Eq such that the following diagram commutes



Choosing $M_\epsilon \in \mathfrak{G}_-$ such that $\xi_\epsilon \mapsto M_\epsilon \Psi(\xi_\epsilon)$ is continuous, concludes the sketch.

Section 2

Construction of $\{\mathfrak{Dif}_T\}_{T\in\mathfrak{T}_-}$ on Manifolds

Symmetric sets and Vector bundle assignments

Let $\mathfrak{S}$ a set of types. Let $\mathrm{Iso}(T^1, T^2)$ the set of all type preserving bijections $T^1 \to T^2$.

Definition: Symmetric sets [CCHS22]

A symmetric set $\mathfrak{s}$ consists of an index set $A_\mathfrak{s}$ and a triple

$$\mathfrak{s} = \left(\{T_\mathfrak{s}^a\}_{a\in A_s},\ \{\mathfrak{t}_\mathfrak{s}^a\}_{a\in A_\mathfrak{s}},\ \{\Gamma_\mathfrak{s}^{a,b}\}_{a,b\in A_\mathfrak{s}}\right) ,$$

where $(T_\mathfrak{s}^a, t_\mathfrak{s}^a)$ are finite typed sets and $\Gamma_\mathfrak{s}^{a,b} \subset \mathrm{Iso}(T_\mathfrak{s}^b, T_\mathfrak{s}^a)$ a non-empty set satisfying for $a, b, c \in A_\mathfrak{s}$

$$\begin{aligned} \gamma \in \Gamma_\mathfrak{s}^{a,b} \quad &\Rightarrow \quad \gamma^{-1} \in \Gamma_\mathfrak{s}^{b,a} , \\ \gamma \in \Gamma_\mathfrak{s}^{a,b} , \quad \bar\gamma \in \Gamma_\mathfrak{s}^{b,c} \quad &\Rightarrow \quad \gamma \circ \bar\gamma \in \Gamma_\mathfrak{s}^{a,c} . \end{aligned}$$

(Connected groupoid in the category of typed sets.)

Let $W = (W^{\mathfrak{t}})_{\mathfrak{t}\in\mathfrak{G}}$ be vector bundle assignment.

1. For T a typed set, let

$$W^{\otimes T} := \bigotimes_{x\in T} W^{\mathfrak{t}(x)}\ .$$

2. Any $\psi \in \mathrm{Iso}(T,\bar{T})$ gives a map $W^{\otimes T} \to W^{\otimes \bar{T}}$characterised by

$$W_p^{\otimes T} \ni w_p = \otimes_{x\in T} w_p^x \ \mapsto \psi \cdot w_p = \otimes_{y\in \bar{T}} w_p^{\psi^{-1}(y)}\ .$$

3. Define

$$W^{\otimes \mathfrak{s}} = \Big\{ w \in \prod_{a\in A_{\mathfrak{s}}} W^{\otimes T_{\mathfrak{s}}^a} \,:\, w^{(a)} = \gamma_{a,b}\cdot w^{(b)} \quad \forall a,b \in A_{\mathfrak{s}}\ ,\ \forall \gamma_{a,b} \in \Gamma_{\mathfrak{s}}^{a,b} \Big\}\ .$$

Trees for regularity structures

Consider T with edge types $\mathcal{E} = \mathcal{E}_+ \cup \mathcal{E}_0 \cup \mathcal{E}_-$.

- $\mathcal{E}_+$ will encode kernels
- $\mathcal{E}_0$ place holder for jets
- $\mathcal{E}_-$ will encode noises.

A subcritical rule R defines $\mathfrak{T}$. For a fixed tree T we define

- $E_T^- := \{e \in E_T \; | \mathfrak{e}(e) \in \mathcal{E}_-\}$,
- $\mathfrak{T}_T^- := \{T_E \subset T \mid E \subset E_T^-\}$,

Let

$$\mathfrak{T}_- := \bigcup_{T\in\mathfrak{T}} \mathfrak{T}_T^- / \sim \; .$$

One renormalises equations by associating to each $T \in \mathfrak{T}_-$ a differential operator $\mathfrak{d}_T \in \mathfrak{Dif}_T$.

Multi-linear differential operators associated to negative trees

Let S be a finite set. Let $\{W^s\}_{s\in S}$ and W be vector bundles over M. A map

$$\mathcal{A} : \prod_{s\in S} \mathcal{C}^\infty(W^s) \to \mathcal{C}^\infty(W)$$

is called *multi-linear differential operator* of order k, if it factors through the k-jet bundle via a multi-linear bundle morphism, i.e. there exists $T_\mathcal{A} \in L(\otimes_{s\in S} J^k W^s, W)$, such that the following diagram commutes

$$\begin{array}{ccc}
\prod_{s\in S} \mathcal{C}^\infty(W^s) & \xrightarrow{\ \mathcal{A}\ } & \mathcal{C}^\infty(W) \\
\Big\downarrow {\scriptstyle j^k_\cdot \times \ldots \times j^k_\cdot} & \nearrow {\scriptstyle T_\mathcal{A}} & \\
\mathcal{C}^\infty(\prod_{s\in S} J^k W^s) & & .
\end{array}$$

Let γ be a permutation of the set S such that $W^s = W^{\gamma(s)}$ for each $s \in S$. The operator $\mathcal{A}$ is called γ-*invariant* if for all $f_s \in \mathcal{C}(W^s)$ one has

$$\mathcal{A}(\prod_{s\in S} f_s) = \mathcal{A}(\prod_{s\in S} f_{\gamma(s)})$$

(i.e. $T_{\mathcal{A}}$ is γ-symmetric.)
For $\sigma \in \mathfrak{L}$, $\alpha \in \mathbb{R}$ and a symmetric set $\mathfrak{s} = (S, i, \Gamma)$ where the type set is given by $\mathfrak{L}$ and the index set $A_{\mathfrak{s}}$ consists only of one element. We define

$$\mathfrak{Dif}_\alpha(\mathfrak{s}, \sigma) \tag{3}$$

as the set of all multilinear differential operators which are

- of order $\max\{n \in \mathbb{N} \cup \{-\infty\} \ : n \leq -\alpha\}$
- $\prod_{s\in S} \mathcal{C}^\infty(V^{i(s)})$ to $\mathcal{C}^\infty(V^\sigma)$
- γ invariant for all $\gamma \in \Gamma$.

Definition of $\mathfrak{Dif}_T$

For a tree $T \in \mathfrak{T}_-$,we set

$$\mathfrak{Dif}_T := \mathfrak{Dif}_{|T|}\big(\mathfrak{s}_T, \mathrm{ind}(\rho_T)\big) \tag{4}$$

where $\mathfrak{s}_T = (S_T, i_T, \Gamma_T)$ is given by

- $S_T = N_T \setminus (\{\rho_T\} \cup L_T)$,
- $i_T = \mathrm{dif}_T$,
- Γ_T consists of all tree symmetries restricted to S_T.

Thank you for your attention!

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