Quantum N-toroidal algebras and extended quantized GIM algebras of N-fold affinization

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Contents

Contents

Contents

Background

For a simple Lie algebra $\mathfrak{g}$, there exist two important natural kinds of generalizations:

- Affinizations:

$$\mathfrak{g} \quad \rightarrow \hat{\mathfrak{g}} \quad \rightarrow \mathfrak{g}_{tor} \quad \rightarrow \mathfrak{g}_{N,tor}$$

- Quantizations:

$$\mathfrak{g} \quad \rightarrow U_q(\mathfrak{g}) \rightarrow \quad U_q(\hat{\mathfrak{g}}) \rightarrow \quad U_q(\mathfrak{g}_{tor}) \rightarrow \quad U_q(\mathfrak{g}_{N,tor})$$

Background

Affinizations of simple Lie algebra $\mathfrak{g}$

$$\mathfrak{g} \qquad \rightarrow \hat{\mathfrak{g}} \qquad \rightarrow \mathfrak{g}_{tor} \qquad \rightarrow \mathfrak{g}_{N,tor}$$

- Affine Lie algebra $\hat{\mathfrak{g}}$: the central extension of $\mathfrak{g} \otimes \mathbb{C}[t_0^{\pm 1}]$ by the one-dimensional center $\mathbb{C}c_0$.
- Toroidal Lie algebra $\mathfrak{g}_{tor}$: also called the double affine Lie algebra.
- N-Toroidal Lie algebra $\mathfrak{g}_{N,tor}$: the infinite dimensional universal central extensional of $\mathfrak{g} \otimes \mathbb{C}[t_0^{\pm 1}, \cdots, t_{N-1}^{\pm 1}]$.

S. Rao and R. Moody, *Vertex representations for N-toroidal Lie algebras and a generalization of the Virasoro algebra*, Comm. Math. Phys., **159** (1994), 239–264.

Background

Quantizations of simple Lie algebra $\mathfrak{g}$

$$\mathfrak{g} \quad \to U_q(\mathfrak{g}) \to \quad U_q(\hat{\mathfrak{g}}) \to \quad U_q(\mathfrak{g}_{tor}) \to \quad U_q(\mathfrak{g}_{N,tor})$$

- Quantum group $U_q(\mathfrak{g})$: $q-$deformation of the universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$(Drinfeld, Jimbo, Lusztig ...)

M. Jimbo, *A q-difference analogue of U(g) and the Yang-Baxter equation*, Lett. Math. Phys. **10** (1985), 63C69.

V. G. Drinfeld, *Quantum groups*, Proc. of the ICM, Berkeley, 1986, pp. 798-820. American Mathematical Society, Providence, 1987.

- Quantum affine algebra $U_q(\hat{\mathfrak{g}})$: admits two realizations Drinfeld-Jimbo realization and Drinfeld realization, associated to affine Lie algebra $\hat{\mathfrak{g}}$.

V. G. Drinfeld, *A new realization of Yangians and quantized affine algebras*, Soviet Math. Dokl. **36** (1988), 212–216.

Background

Quantizations of simple Lie algebras

$$\mathfrak{g} \quad \rightarrow U_q(\mathfrak{g}) \rightarrow \quad U_q(\hat{\mathfrak{g}}) \rightarrow \quad U_q(\mathfrak{g}_{tor}) \rightarrow \quad U_q(\mathfrak{g}_{N,tor})$$

- Quantum toridal algebra $U_q(\mathfrak{g}_{tor})$ introduced by Ginzburg-Kapranov -Vasserot, were found to have many applications in geometry, algebra, and mathematical physics.

V. Ginzburg, M. Kapranov, E. Vasserot, *Langlands reciprocity for algebraic surfaces*, Math. Res. Lett. **2** (1995), 147-160.

Background

Quantum toridal algebra $U_q(\mathfrak{g}_{tor})$:

- Schur-Weyl duality for type A ([VV1])

M. Varagnolo and E. Vasserot, *Schur duality in the toroidal setting*, Comm. Math. Phys. **182** (1996), 469-484.

- q-Fock space representation for type A ([VV2], [STU])

M. Varagnolo and E. Vasserot, *Double-loop algebras and the Fock space*, Invent. Math. **133** (1998), 133-159.

Y. Saito, K. Takemura and D. Uglov, *Toroidal actions on level 1 modules of* $U_q(\hat{sl}_n)$, Transform. Groups *3* (1998), 75-102.

- Simple integrable highest weight modules for type A ([M])

K. Miki, *Representations of quantum toroidal algebra* $U_q(sl_{n+1}, tor)(n \leqslant 2)$, J. Math. Phys., **41** (2000), 7079-7098.

- Quantum vertex representations via McKay correspondence for type ADE.

I. B. Frenkel, N. Jing, W. Wang, *Quantum vertex representations via finite groups and the McKay correspondence*, Comm. Math. Phys. **211** (2000), 365-393.

Background

Quantizations of simple Lie algebras

$$\mathfrak{g} \quad \to U_q(\mathfrak{g}) \to \quad U_q(\hat{\mathfrak{g}}) \to \quad U_q(\mathfrak{g}_{tor}) \to \quad U_q(\mathfrak{g}_{N,tor})$$

- How about the structures and representations of quantum N-toridal algebras $U_q(\mathfrak{g}_{N,tor})$?

Y. Gao, N. Jing, L. Xia, H. Zhang, Quantum N-toroidal algebras and extended quantized GIM algebrasof N-fold affinization, Comm. Math Stat., (2023) doi.org/10.1007/s40304-022-00316-4.

C. Ying, L. Xia, H. Zhang, Vertex representation of quantum N -toroidal algebra for type F_4, Comm. Algebra, 48(9)(2020), 3780-3799.

N. Jing, Z. Xu, H. Zhang, Vertex representation of quantum N -toroidal algebra for type C, J. Alg. Appl. 20(10)(2021), 0-2150185.

N. Jing, Q. Wang, H. Zhang, Level -1/2 realization of quantum N-toroidal algebra, Algebra Colloq., 29 (2022) 79-98.

Goal

- Give the definition of quantum N-toroidal algebras.
- Find the relation between quantum N-toroidal algebras and quantized GIM algebras.
- Construct the vertex representations of quantum N-toroidal algebras.

Contents

Quantum toroidal algebras $U_q(\mathfrak{g}_{tor})$

- $U_q(\mathfrak{g}_{tor})$ is an associative algebra over $\mathbb{C}$ generated by $x_i^{\pm}(k)$, $a_i(\ell)$, $K_i^{\pm 1}$, $\gamma^{\pm\frac{1}{2}}$, $q^{\pm d}$, $(i \in I,\ k \in \mathbb{Z},\ \ell \in \mathbb{Z}\backslash\{0\})$ satisfying the defining relations.

  (D1) $\gamma^{\pm\frac{1}{2}}$ is central, $K_i^{\pm 1} K_i^{\mp 1} = 1$, $[\,K_i^{\pm 1}, K_j^{\pm 1}\,] = [\,K_i^{\pm 1}, a_j(r)\,] =0$, and $q^{\pm d_i}, K_i^{\pm}$ commute with each other

  (D2) $K_i x_j^{\pm}(k) K_i^{-1} = q^{\pm a_{ij}} x_j^{\pm}(k)$,

  (D3) $[\,a_i(r), a_j(s)\,] = \delta_{r+s,0} \frac{[\,r\,a_{ij}\,]}{r} \frac{\gamma^r - \gamma^{-r}}{q-q^{-1}} p^{rb_{ij}}$,

  (D4) $q^{d_1} a_i(r) q^{-d_1} = q^r a_i(r)$, $\quad q^{d_1} x_i^{\pm}(k) q^{-d_1} = q^k x_i^{\pm}(k)$,

  (D5) $q^{d_2} a_i(r) q^{-d_2} = a_i(r)$, $\quad q^{d_2} x_i^{\pm}(k) q^{-d_2} = q^{\delta_{i0}} x_i^{\pm}(k)$,

  (D6) $[\,a_i(r), x_j^{\pm}(k)\,] = \pm \frac{[\,r\,a_{ij}\,]}{r} \gamma^{\mp\frac{|r|}{2}} x_j^{\pm}(r+k) p^{rb_{ij}}$,

  (D7) $p^{b_{ij}} [\,x_i^{\pm}(k+1),\, x_j^{\pm}(l)\,]_{q^{\pm(\alpha_i,\alpha_j)}} + [\,x_j^{\pm}(l+1),\, x_i^{\pm}(k)\,]_{q^{\pm(\alpha_i,\alpha_j)}} = 0$,

  (D8) $[\,x_i^{+}(k),\, x_j^{-}(l)\,] = \delta_{ij} \frac{\gamma^{-l}\phi_i(k+l) - \gamma^{-k}\varphi_i(k+l)}{q-q^{-1}}$,

  (D9) Quantum Serre relations.

Quantum toroidal algebras $U_q(\mathfrak{g}_{tor})$

- $U_q(\mathfrak{g}_{tor})$ contains two remarkable subalgebras: vertical subalgebra $U_{(1)}$ and horizontal subalgebra $U_{(2)}$.

$$U_{(1)} = \langle x_i^{\pm}(k),\, a_i(r),\, K_i^{\pm 1},\, \gamma^{\pm\frac{1}{2}} | i \in I/\{0\},\, k \in \mathbb{Z},\, r \in \mathbb{Z}/\{0\}\rangle.$$

$$U_{(2)} = \langle x_i^{\pm}(0),\, K_i^{\pm 1},\, \gamma^{\pm\frac{1}{2}} | i \in I\rangle.$$

- $U_{(1)} \cong$ Drinfeld realization, $U_{(2)} \cong$ Drinfeld-Jimbo realization.

Quantum N-toroidal algebras $U_q(\mathfrak{g}_{N,tor})$

Let $I=\{0,1,\cdots n\}$, $J=\{1,\cdots N-1\}$, $\underline{k}=(k_1,k_2,\cdots,k_{N-1})\in\mathbb{Z}^{N-1}$,

$e_s=(0,\cdots,0,1,0,\cdots,0)$ be the unit vector of $(N-1)$-dimension.

Definition (Gao-Jing-Xia-Z)

The quantum N-toroidal algebras $U_q(\mathfrak{g}_{N,tor})$ is an associative algebra over $\mathbb{C}$ generated by $x_i^{\pm}(\underline{k})$, $a_i^{(s)}(\ell)$, $K_i^{\pm 1}$, $\gamma_s^{\pm\frac{1}{2}}$, $q^{\pm d}$, $(i\in I, s\in J$, $\underline{k}\in\mathbb{Z}^{N-1}$, $\ell\in\mathbb{Z}\backslash\{0\})$ satisfying the following relations,

$$\gamma_j^{\pm\frac{1}{2}} \quad \text{are central and} \quad K_i^{\pm 1}K_i^{\mp 1}=1,\ q^d, K_i^{\pm} \text{ commute with each other,} \tag{1}$$

$$[\,a_i^{(s)}(\ell),\ K_j^{\pm 1}\,]=0, \qquad [K_j^{\pm 1}, q^{\pm d}]=0=[a_i^{(s)}(\ell), q^{\pm d}], \tag{2}$$

$$[\,a_i^{(s)}(\ell), a_j^{(s')}(\ell')\,]=\delta_{s,s'}\delta_{\ell+\ell',0}\frac{[\,\ell a_{ij}\,]_i}{\ell}\cdot\frac{\gamma_s^{\ell}-\gamma_s^{-\ell}}{q-q^{-1}}, \tag{3}$$

$$q^d\, x_i^{\pm}(\underline{k})\, q^{-d}=q^{\pm\delta_{i,0}}\, x_i^{\pm}(\underline{k}), \tag{4}$$

$$K_i\, x_j^{\pm}(\underline{k})\, K_i^{-1}=q_i^{\pm a_{ij}} x_j^{\pm}(\underline{k}), \tag{5}$$

Quantum N-toroidal algebras $U_q(\mathfrak{g}_{N,tor})$

$$[\, a_i^{(s)}(\ell), x_j^{\pm}(\underline{k})\,] = \pm\frac{[\ell a_{ij}]_i}{\ell}\gamma_s^{\mp\frac{\ell}{2}} x_j^{\pm}(\ell e_s + \underline{k}), \tag{6}$$

$$[x_i^{\pm}(ke_s), x_i^{\pm}(le_{s'})\,] = 0, \qquad \text{for} \quad s \neq s' \quad \text{and} \quad kl \neq 0, \tag{7}$$

$$[x_i^{\pm}((t+1)e_s), x_j^{\pm}(t'e_s)\,]_{q_i^{\pm a_{ij}}} + [x_j^{\pm}((t'+1)e_s), x_i^{\pm}(te_s)\,]_{q_i^{\pm a_{ij}}} = 0, \tag{8}$$

$$[\, x_i^{+}(te_s),\ x_j^{-}(t'e_s)\,] = \frac{\delta_{ij}}{q_i - q_i^{-1}}\Big(\gamma_s^{\frac{t-t'}{2}}\, \phi_i^{(s)}(t+t') - \gamma_s^{\frac{t'-t}{2}}\, \varphi_i^{(s)}(t+t')\Big), \tag{9}$$

Quantum Serre relations (10)

Remarks

- When $N = 2$, the definition of quantum N-toroidal algebra is just that of quantum toroidal algebra. When $N \geqslant 3$, quantum N-toroidal algebra is a natural generalization of quantum toroidal algebra.
- For fixed $s \in J$, there exists a subalgebra $U_q^{(s)}$ of $U_q(\mathfrak{g}_{N,tor})$ generated by the elements $x_i^{\pm}(ke_s)$, $a_i^{(s)}(\ell)$, $K_i^{\pm 1}$, $\gamma_s^{\pm\frac{1}{2}}$, $q^{\pm d}$ for $i \in I$. It is easy to see that every $U_q^{(s)}$ is exactly isomorphic to quantum toroidal algebra.
- In fact, there exists another central element $\gamma_0 = K_0 K_\theta$, where θ is the highest root of simple Lie algebra $\mathfrak{g}$.

Algebra $\mathcal{U}_0(\mathfrak{g}_{N,tor})$

Furthermore, we define an algebra $\mathcal{U}_0(\mathfrak{g}_{N,tor})$ generated by finitely many Drinfeld generators with finitely many Drinfeld relations and claim that the quantum N-toroidal algebra $U_q(\mathfrak{g}_{N,tor})$ is isomorphic to a quotient of $\mathcal{U}_0(\mathfrak{g}_{N,tor})$ or $\mathcal{U}_0(\mathfrak{g}_{N,tor})$ itself.

Definition

The algebra $\mathcal{U}_0(\mathfrak{g}_{N,tor})$ is an associative algebra generated by $x_i^{\pm}(\underline{0})$, $x_0^{-\epsilon}(\epsilon e_s)$, $K_i^{\pm 1}$, $q^{\pm d}$ and $\gamma_s^{\pm\frac{1}{2}}$ ($\epsilon = \pm 1$ or $\pm$, $i \in I$, $s \in J$) satisfying the following relations:

$$\gamma_s^{\pm\frac{1}{2}} \text{are central, and } K_i^{\pm} \text{ commute with each other,} \tag{11}$$

$$q^{\pm d}, K_i^{\pm} \text{ commute with each other,} \tag{12}$$

$$K_i x_j^{\pm}(\underline{0}) K_i^{-1} = q_i^{\pm a_{ij}} x_j^{\pm}(\underline{0}), \quad K_i x_0^{-\epsilon}(\epsilon e_s) K_i^{-1} = q_i^{\pm a_{ij}} x_0^{-\epsilon}(\epsilon e_s), \tag{13}$$

$$q^d x_i^{\pm}(\underline{0}) q^{-d} = q^{\pm\delta_{0,i}} x_i^{\pm}(\underline{0}), \quad q^d x_0^{-\epsilon}(\epsilon e_s) q^{-d} = q^{\epsilon} x_0^{-\epsilon}(\epsilon e_s), \tag{14}$$

$$[\, x_i^{+}(\underline{0}),\, x_j^{-}(\underline{0})\,] = \delta_{ij}\frac{K_i - K_i^{-1}}{q - q^{-1}}, \qquad [\, x_0^{+}(-e_s),\, x_0^{-}(e_s)\,] = \frac{\gamma_s^{-1} K_0 - \gamma_s K_0^{-1}}{q - q^{-1}}, \tag{15}$$

Subalgebra of quantum N-toroidal algebra $U_q(\mathfrak{g}_{N,tor})$

$$[\,x_0^{-\epsilon}(\epsilon e_s),\, x_0^{-\epsilon}(\underline{0})\,]_{q_0^{-2}} = 0, \qquad [\,x_0^{-\epsilon}(\epsilon e_s),\, x_0^{-\epsilon}(\epsilon e_{s'})\,] = 0, \text{for } s \neq s' \in J, \tag{16}$$

$$[\,x_i^{\epsilon}(\underline{0}),\, x_0^{-\epsilon}(\epsilon e_s)\,] = 0, \quad \text{for} \quad i \neq 0, \tag{17}$$

$$[x_i^{\pm}(\underline{0}), x_j^{\pm}(\underline{0})] = 0, \qquad [x_0^{-\epsilon}(\epsilon e_s), x_k^{-\epsilon}(\underline{0})] = 0, \quad \text{for} \quad a_{ij} = a_{0k} = 0, \tag{18}$$

$$\sum_{s=0}^{\ell=1-a_{ij}} (-1)^s \begin{bmatrix} \ell \\ s \end{bmatrix}_i (x_i^{\pm}(\underline{0}))^{n-s} x_j^{\pm}(\underline{0}) (x_i^{\pm}(\underline{0}))^s = 0, \qquad a_{ij} < 0, \tag{19}$$

$$\sum_{t=0}^{\ell=1-a_{0j}} (-1)^t \begin{bmatrix} \ell \\ t \end{bmatrix}_j (x_j^{\epsilon}(\underline{0}))^{n-t} x_0^{\epsilon}(-\epsilon e_s) (x_j^{\epsilon}(\underline{0}))^t = 0, \qquad a_{0j} < 0, \tag{20}$$

Quantum Serre relations involving $x_i^{\pm}(0),\ x_0^{+}(-1), x_0^{-}(1)$. (21)

N-symmetry

That is,
$$\mathcal{U}_0(\mathfrak{g}_{N,tor}) := \left\langle x_i^{\pm}(\underline{0}),\, x_0^{-\epsilon}(\epsilon e_s), K_i^{\pm 1}, q^{\pm d}, \gamma_s^{\pm\frac{1}{2}} \;\middle|\; i \in I, s \in J \right\rangle.$$

Proposition

For $s \in J$, there exists a $\mathbb{Q}$-algebra automorphism τ_s of $\mathcal{U}_0(\mathfrak{g}_{N,tor})$ such that,

$$\tau_s(x_0^{\epsilon}(-\epsilon e_{s'})) = \begin{cases} x_0^{\epsilon}(0), & \text{if } s = s'; \\ x_0^{\epsilon}(-\epsilon e_{s'}), & \text{if } s \neq s' \end{cases}$$

$$\tau_s(q) = q, \quad \tau_s(x_0^{\epsilon}(0)) = x_0^{\epsilon}(-\epsilon e_s), \quad \tau_(x_i^{\pm}(0)) = x_i^{\pm}(0),$$

where $i = 1, 2, \cdots, n$, $s' = 1, \cdots, N-1$ and $\epsilon = \pm$ or ± 1.

Theorem

As associate algebras, quantum N-toroidal algebra $U_q(\mathfrak{g}_{N,tor})$ is isomorphic to the quotient algebra of $\mathcal{U}_0(\mathfrak{g}_{N,tor})$ for type A and subalgebra $\mathcal{U}_0(\mathfrak{g}_{N,tor})$ for other types, respectively.

$$U_q(\mathfrak{g},\, N\text{-}tor) \cong \begin{cases} \mathcal{U}_0(\mathfrak{g}, N\text{-}tor)/J_0, & \text{for type A;} \\ \mathcal{U}_0(\mathfrak{g}, N\text{-}tor), & \text{otherwise.} \end{cases}$$

The proof of Theorem4

- For the case $N = 2$,
  - We have checked it for type A in Jing-Z.
  - For other types, we need to check the Serre relations.
- For the case of $N > 2$, it can be verified similarly.

Remark

It is clear that $\mathcal{U}_0(\mathfrak{g}_{N,tor})$ is finitely generated with finitely many relations, and has much fewer generators and relations than Drinfeld's original form. Actually, $\mathcal{U}_0(\mathfrak{g}_{N,tor})$ provides an alternative realization of quantum N-toroidal algebra $U_q(\mathfrak{g}_{N,tor})$.

Contents

Generalized intersection matrix (GIM)

Let J be a finite index set, a square matrix $M = (m_{ij})_{i,j\in J}$ over $\mathbb{Z}$ is called a generalized intersection matrix if it satisfies:
(C1) $m_{ii} = 2$ for $i \in J$;
(C2) $m_{ij} \cdot m_{ji}$ are nonnegative integers for $i \neq j$;
(C3) $m_{ij} = 0$ implies $m_{ji} = 0$.

- As m_{ij} can be positive, a GIM generalizes the notion of the generalized Cartan matrix.

GIM algebras

The GIM algebra $\mathcal{L}(M)$ associated to a GIM $M = (m_{ij})_{i,\, j \in J}$ are an associative algebra over $\mathbb{C}$ generated by e_i, f_i, h_i for $i \in J$ satisfying the following relations,
$(R1)$ For $i, j \in J$,

$$[h_i, e_j] = m_{ij} e_j, \qquad [h_i, f_j] = -m_{ij} f_j,$$
$$[e_i, f_i] = h_i.$$

$(R2)$ For $m_{ij} \leq 0$,

$$[e_i, f_j] = 0 = [f_i, e_j], \quad (ade_i)^{m_{ij}+1} e_j = 0 = (adf_i)^{m_{ij}+1} f_j.$$

$(R3)$ For $m_{ij} > 0$ and $i \neq j$,

$$[e_i, e_j] = 0 = [f_i, f_j], \quad (ade_i)^{m_{ij}+1} f_j = 0 = (adf_i)^{m_{ij}+1} e_j.$$

Theorem (Berman-Moody 92' Inv. Math.)

There exists a surjective homomorphism $\varphi : \mathcal{L}(M) \rightarrow \mathfrak{g}_{N,tor}$.

Quantized GIM algebras $U_q(\mathcal{L}(M))$

- In [T, LT], the authors studied the structures of quantized GIM algebras for simply-laced cases of 2-affinization.
- In [GHX], the authors proved that a quantized GIM algebra $U_q(\mathcal{L}(M))$ for simple-laced cases is isomorphic to a subalgebra of a quantum universal enveloping algebra $U_q(A)$

[GHX] Y. Gao, N. Hu and L. Xia, *Quantized GIM algebras and their images in quantized Kac-Moody algebras*, Alg. Rep.Theory, **24**(3) (2021).

[LT] R. Lv and Y. Tan, *On quantized generalized intersection matrix algebras associated to 2-fold affinization of Cartan matrices*, J. Algebra Appl. **12** (2013), 125–141.

[T] Y. Tan, *Drinfeld-Jimbo coproduct of quantized GIM Lie algebras* , J. Algebra, **313** (2007), 617–641.

Quantized GIM algebras

The extended quantized GIM algebra $U_q(\mathcal{L}(M))$ of N-fold affinization is a unital associative algebra over $\mathbb{K}$ generated by the elements E_i, F_i, $K_i^{\pm 1}, q^{\pm d}(i \in \tilde{J})$, satisfying the following relations:

$(M1)$ For $i, j \in \tilde{J}$, $K_i^{\pm 1} K_i^{\mp 1} = 1$, $q^{\pm d}$ and $K_j^{\pm 1}$ commute with each other.

$(M2)$ For $i \in J_1$ and $j \in J_2$,

$$\begin{aligned} q^d\, E_i\, q^{-d} &= q\, E_i, & \qquad q^d\, F_i\, q^{-d} &= q^{-1}\, F_i, \\ q^d\, E_j\, q^{-d} &= E_j, & \qquad q^d\, F_j\, q^{-d} &= F_j. \end{aligned}$$

$(M3)$ For $i \in \tilde{J}$ and $j \in \tilde{J}$,

$$K_j\, E_i\, K_j^{-1} = q_i^{m_{ij}}\, E_i, \qquad K_j\, F_i\, K_j^{-1} = q_i^{-m_{ij}}\, F_i.$$

$(M4)$ For $i \in \tilde{J}$, we have that

$$[\, E_i, F_i\,] = \frac{K_i - K_i^{-1}}{q - q^{-1}}.$$

Quantized GIM algebras

$(M5)$ For $m_{ij} < 0$, we have that

$$[\,E_i, F_j\,] = 0,$$

$$\sum_{s=0}^{1-m_{ij}} (-1)^s \begin{bmatrix} 1-m_{ij} \\ s \end{bmatrix}_i E_i^{1-m_{ij}-s} E_j E_i^s = 0,$$

$$\sum_{s=0}^{1-m_{ij}} (-1)^s \begin{bmatrix} 1-m_{ij} \\ s \end{bmatrix}_i F_i^{1-m_{ij}-s} F_j F_i^s = 0.$$

$(M6)$ For $m_{ij} > 0$ and $i \neq j$, we have that

$$[\,E_i, E_j\,] = 0 = [F_i, F_j],$$

$$\sum_{s=0}^{1+m_{ij}} (-1)^s \begin{bmatrix} 1+m_{ij} \\ s \end{bmatrix}_i E_i^{1+m_{ij}-s} F_j E_i^s = 0,$$

$$\sum_{s=0}^{1+m_{ij}} (-1)^s \begin{bmatrix} 1+m_{ij} \\ s \end{bmatrix}_i F_i^{1+m_{ij}-s} E_j F_i^s = 0.$$

$(M7)$ For $m_{ij} = 0$ and $i \neq j$, we have that

$$[\,E_i, E_j\,] = 0 = [E_i, F_j] = [F_i, F_j].$$

Definition

Let $A = (a_{ij})_{i,j\in I_0}$ be the Cartan matrix of finite type. Define

$$M = (m_{ij})_{i,j\in \tilde{J}} = \begin{pmatrix} T & P \\ Q & A \end{pmatrix},$$

where T is the $N \times N$ matrix $\sum_{i,j} 2E_{ij}$, and $P = (p_{ij})$ (resp. $Q = (q_{ij})$) is the $N \times n$ (resp. $n \times N$) matrix given by $p_{ij} = a_{0j}$ (resp. $q_{ij} = a_{i0}$).

Remark

Note that M is a symmetrizable GIM of N-fold affinization of A: $D_M M$ is symmetric for the diagonal matrix $D_M = \sum_{i\in\tilde{J}} d_i E_{ii} = \begin{pmatrix} d_0 I_N & 0 \\ 0 & D_0 \end{pmatrix}$ *where* $D_0 = diag(d_i | i \in I_0)$.

Quantized GIM algebra of N-fold affinizations

Now we focus on showing that the algebra $\mathcal{U}_0(\mathfrak{g}_{N,tor})$ can be realized as a quotient of the extended quantized GIM algebra of N-fold affinization. First let us deonte that for $i \in I$ and $s \in J$,

$$F_{-s} = x_0^-(e_s)q_0^{-2d},\ E_{-s} = q_0^{2d}x_0^+(-e_s),\ K_{-s} = \gamma_s K_0^{-1},$$
$$E_i = x_i^+(0),\ F_i = x_i^-(0),\ K_i = K_i.$$

Let $\tilde{J} = \{-N+1, \cdots, -1, 0, 1, \cdots, n\}$.

Proposition

Using the above notations, the algebra $\mathcal{U}_0(\mathfrak{g}_{N,tor})$ is an associative algebra generated by $E_i, F_i, K_i, (i \in \tilde{J})$, satisfying the following relations.

$$\begin{aligned}
&K_iK_i^{-1} = K_i^{-1}K_i = 1, \quad q^{\pm d} \textit{ and } K_i^{\pm} \textit{ commute with each other},\\
&K_iE_jK_i^{-1} = q_i^{m_{ij}}E_j, \qquad K_iF_jK_i^{-1} = q_i^{-m_{ij}}F_j,\\
&[\,E_i,\, F_j\,] = \delta_{i,j}\frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},\\
&q^dE_iq^{-d} = qE_i, \qquad q^dF_iq^{-d} = qF_i,\\
&q^dE_jq^{-d} = E_j, \qquad q^dF_jq^{-d} = F_j,
\end{aligned}$$

Quantized GIM algebra of N-fold affinizations

$$
\begin{aligned}
&[\,E_i, F_j\,] = 0,\\
&\sum_{s=0}^{1-m_{ij}} (-1)^s \begin{bmatrix} 1-m_{ij} \\ s \end{bmatrix}_i E_i^{1-m_{ij}-s} E_j E_i^s = 0,\\
&\sum_{s=0}^{1-m_{ij}} (-1)^s \begin{bmatrix} 1-m_{ij} \\ s \end{bmatrix}_i F_i^{1-m_{ij}-s} F_j F_i^s = 0,
\end{aligned}
$$

where $i \neq j \in \tilde{J}$ such that $m_{ij} < 0$,

$$
\begin{aligned}
&[\,E_i, E_j\,] = 0 = [F_i, F_j],\\
&\sum_{s=0}^{1+m_{ij}} (-1)^s \begin{bmatrix} 1+m_{ij} \\ s \end{bmatrix}_i E_i^{1+m_{ij}-s} F_j E_i^s = 0,\\
&\sum_{s=0}^{1+m_{ij}} (-1)^s \begin{bmatrix} 1+m_{ij} \\ s \end{bmatrix}_i F_i^{1+m_{ij}-s} E_j F_i^s = 0,
\end{aligned}
$$

where $i \neq j \in \tilde{J}$ such that $m_{ij} > 0$,

$$[\,E_i, E_j\,] = 0 = [E_i, F_j] = [F_i, F_j], \quad \text{for } \ i \neq j \in \tilde{J} \ \text{ such that } \ m_{ij} = 0,$$

Quantized GIM algebra of N-fold affinizations

$$[E_{-j},[E_0,E_i]_{q_0}]_{q_0}+[E_0,[E_{-j},E_i]_{q_0}]_{q_0^{-3}}=0,\\ [F_{-j},[F_0,F_i]_{q_0^{-1}}]_{q_0^3}+[F_0,[F_{-j},F_i]_{q_0^{-1}}]_{q_0^{-1}}=0, \tag{22}$$

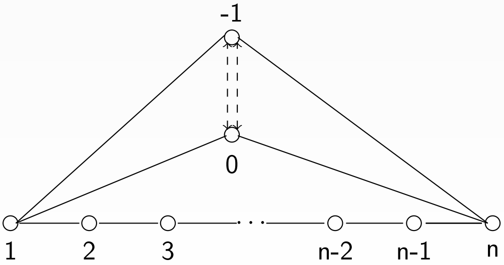
where $m_{i0}=1$ and $i\in I_1=\{1,2,\cdots,n\}, j\in J$,

$$\begin{aligned}
&[E_{-j},[E_0,[E_0,E_i]_{q_0^2}]_{q_0^{-2}}]_{q_0^4}+[[E_0,[E_{-j},[E_0,E_i]_{q_0^2}]]_1]_{q_0^{-2}}\\
&\qquad+[[E_0,[E_0,[E_{-j},E_i]_{q_0^2}]]_{q_0^{-4}}]_{q_0^{-2}}=0,\\
&[F_{-j},[F_0,[F_0,F_i]_{q_0^{-2}}]_{q_0^4}]_{q_0^2}+[[E_0,[E_{-j},[E_0,E_i]_{q_0^{-2}}]]_1]_{q_0^2}\\
&\qquad+[[E_0,[E_0,[E_{-j},E_i]_{q_0^{-2}}]]_{q_0^2}]_{q_0^{-4}}=0,\\
&[E_{-j},[E_{-j},[E_0,E_i]_{q_0^2}]_1]_{q_0^2}+[[E_{-j},[E_0,[E_{-j},E_i]_{q_0^2}]]_{q_0^{-4}}]_{q_0^2}\\
&\qquad+[[E_0,[E_{-j},[E_{-j},E_i]_{q_0^2}]]_1]_{q_0^2}=0,\\
&[F_{-j},[F_{-j},[F_0,F_i]_{q_0^{-2}}]_1]_{q_0^{-2}}+[[F_{-j},[F_0,[F_{-j},F_i]_{q_0^{-2}}]]_{q_0^4}]_{q_0^{-2}}\\
&\qquad+[[F_0,[F_{-j},[F_{-j},F_i]_{q_0^{-2}}]]_1]_{q_0^{-2}}=0,
\end{aligned} \tag{23}$$

Take type A for example

For the case of $N = 2$,

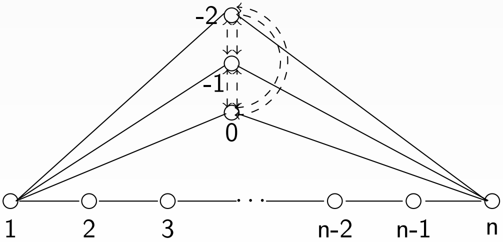
$$M = \begin{pmatrix} 2 & 2 & -1 & 0 & \cdots & 0 & -1 \\ 2 & 2 & -1 & 0 & \cdots & 0 & -1 \\ -1 & -1 & 2 & -1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 2 & -1 \\ -1 & -1 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix},$$



Take type A for example

For the case of $N > 2$,

$$M = \begin{pmatrix} 2 & \cdots & 2 & -1 & 0 & \cdots & 0 & -1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 2 & \cdots & 2 & -1 & 0 & \cdots & 0 & -1 \\ -1 & \cdots & -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & -1 & 2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 2 & -1 \\ -1 & \cdots & -1 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix},$$



For N=3

Remark

From Proposition of N-symmetry, there exists an automorphism τ_σ of the algebra $\mathcal{U}_0(\mathfrak{g}_{N,tor})$ for $\sigma \in S_X$ where $X = \{0, -1, \cdots, -N+1\}$, such that $\tau_\sigma(q^d) = q^d$, $\tau_\sigma(\gamma_s) = \gamma_{-\sigma(-i)}$ for $s \in J$ and $i \in K$,

$$\tau_\sigma(E_j) = \begin{cases} E_{\sigma(j)}, & \text{if} \quad j \in X; \\ E_j, & \text{if} \quad j \notin X, \end{cases} \qquad \tau_\sigma(F_j) = \begin{cases} F_{\sigma(j)}, & \text{if} \quad j \in X; \\ F_j, & \text{if} \quad j \notin X, \end{cases}$$

$$\tau_\sigma(K_j) = \begin{cases} K_{\sigma(j)}, & \text{if} \quad j \in X; \\ K_j, & \text{if} \quad j \notin X, \end{cases}$$

Therefore we have the following Corollary immediately.

Corollary

The algebra $\mathcal{U}_0(\mathfrak{g}_{N,tor})$ is isomorphic to the quotient algebra $U_q(\mathcal{L}(M))/K$ of the extended quantized GIM algebra of the N-fold affinization $U_q(\mathcal{L}(M))$. That is,

$$\mathcal{U}_0(\mathfrak{g}_{N,tor}) \cong U_q(\mathcal{L}(M))/K,$$

where K is the ideal of $U_q(\mathcal{L}(M))$ generated by Serre relations.

Moreover, we have the following two theorems.

Theorem

As an associative algebra, the quantum 2-toroidal algebra $U_q(\mathfrak{g}_{2,tor})$ is isomorphic to a quotient algebra of $\mathcal{U}_0(\mathfrak{g}_{2,tor})$ for type A and $\mathcal{U}_0(\mathfrak{g}_{2,tor})$ itself for other types.

$$U_q(\mathfrak{g}_{2,tor}) \cong \begin{cases} \mathcal{U}_0(\mathfrak{g}_{2,tor})/H_1, & \text{for type } A; \\ \mathcal{U}_0(\mathfrak{g}_{2,tor}), & \text{otherwise,} \end{cases}$$

$U_q(\mathfrak{g}_{N,tor})$ and extended quantized GIM algebra of N-affinization

Theorem

As an associative algebra, the algebra $U_q(\mathfrak{g}_{N,tor})$ ($N > 2$) is isomorphic to a quotient algebra of $\mathcal{U}_0(\mathfrak{g}_{N,tor})$:

$$U_q(\mathfrak{g}_{N,tor}) \cong \mathcal{U}_0(\mathfrak{g}_{N,tor})/H_2.$$

Combining the above Theorems with Corollary, we obtain the following main theorem, which generalizes a well-known result of Berman and Moody for Lie algebras.

Theorem

The algebra $U_q(\mathfrak{g}_{N,tor})$ is isomorphic to quotient algebra of the extended quantized GIM algebras of N-fold affinization $U_q(\mathcal{L}(M))$.

Contents

Vertex representation

- Let $I = \{0, 1, \cdots, n\}$ and $I_0 = \{1, \cdots, n\}$. Let $\mathfrak{g}$ be the finite dimensional simple Lie algebra of simply-laced type over $\mathbb{K}$ with the Cartan matrix $(a_{ij})_{i,j \in I_0}$.
- Denote by $\hat{\mathfrak{g}}$ the affine Kac-Moody Lie algebra associated to $\mathfrak{g}$ and its Cartan matrix by $(a_{ij})_{i,j \in I}$.
- Let $\mathfrak{h}$ and $\hat{\mathfrak{h}}$ be their Cartan subalgebras, Δ and $\hat{\Delta}$ their root systems, respectively.
- Also let $\Pi = \{\bar{\alpha}_1, \cdots, \bar{\alpha}_n\}$ be a basis of Δ, where $\alpha_0, \alpha_1, \cdots, \alpha_n$ are the simple roots of $\hat{\mathfrak{g}}$.
- Let $\bar{Q} = \bigoplus_{i=1}^{n} \mathbb{Z}\bar{\alpha}_i$ and $Q = \bigoplus_{i=0}^{n} \mathbb{Z}\alpha_i$ be the root lattice of $\mathfrak{g}$ and $\hat{\mathfrak{g}}$ respectively.
- The affine weight lattice P is $P = \bigoplus_{i=0}^{n} \mathbb{Z}\Lambda_i \bigoplus \mathbb{Z}\delta$, where $\Lambda_0, \cdots, \Lambda_n$ are the fundamental weights of $\hat{\mathfrak{g}}$ and δ the null root.

Quantum Heisenberg algebra

Definition

The Heisenberg algebra $U_q(\widehat{\mathfrak{h}}, N\text{-tor})$ is an associative algebra generated by $\{\, a_i(l) \mid l \in \mathbb{Z}\backslash\{0\},\, i \in I \,\}$, satisfying the following relations for $m, l \in \mathbb{Z}\backslash\{0\}$,

$$[\, a_i(m), a_j(l)\,] = \delta_{m+l,0}\frac{[ma_{ij}]}{m}[m]. \tag{24}$$

Fock space

- We denote by $U_q(\widehat{\mathfrak{h}}^+, N\text{-tor})$ (resp.$U_q(\widehat{\mathfrak{h}}^-, N\text{-tor})$) the commutative subalgebra of $U_q(\widehat{\mathfrak{h}}, N\text{-tor})$ generated by $a_i(l)$ (resp. $a_i(-l)$) with $l \in \mathbb{Z}_{>0}$, $i \in I, j \in J$.
- Let $\mathrm{S}(\widehat{\mathfrak{h}}^-, N\text{-tor})$ be the symmetric algebra generated by $a_i(-l)$ with $l \in \mathbb{Z}_{>0}$.
- Then $\mathrm{S}(\widehat{\mathfrak{h}}^-, N\text{-tor})$ is a $U_q(\widehat{\mathfrak{h}}, N\text{-tor})$-module with the action defined by

$$\gamma_s^{\pm\frac{1}{2}} \cdot v = q^{\pm\frac{1}{2}} v,$$
$$a_i(-l) \cdot v = a_i(-l)\, v,$$
$$a_i(l) \cdot v = \sum_j \frac{[la_{ij}]}{l} \frac{q^l - q^{-l}}{q - q^{-1}} \frac{d\, v}{d\, a_j(-l)}.$$

for any $v \in S(\widehat{\mathfrak{h}}^-, N\text{-tor})$, $l \in \mathbb{Z}_{>0}$ and $i \in I$.

Fock space

- First we define a 2-cocycle $\varepsilon(\ ,\ ):\ Q\times Q\to\pm 1$ such that
$$\varepsilon(\alpha,\,\beta) \;=\; (-1)^{(\alpha,\,\beta)}\varepsilon(\beta,\,\alpha).$$
- Let $\mathbb{K}[Q]=\sum\limits_{\alpha\in Q}\mathbb{K}e^{\alpha}$ be a twisted group algebra with base elements of the form e^{α} $(\alpha\in Q)$, and the product is $e^{\alpha}e^{\beta}=\varepsilon(\alpha,\beta)e^{\alpha+\beta}$.
- Then we have that for $i,j\in I$,
$$e^{\alpha_i}e^{\alpha_j}=(-1)^{(\alpha_i,\,\alpha_j)}e^{\alpha_j}e^{\alpha_i}.$$
- Define the Fock space $\mathcal{F}$=S($\widehat{\mathfrak{h}}^-$, N-tor)$\otimes\mathbb{K}[Q]$, and the operators $a_i(l)$, e^{α}, K_i, q^d and $z^{a_i(0)}$ act on $\mathcal{F}$ as follows ($v\otimes e^{\beta}\in\mathcal{F}$):
$$a_i(-l)(v\otimes e^{\beta})=(a_i(-l)v)\otimes e^{\beta},$$
$$e^{\alpha}(v\otimes e^{\beta})=v\otimes e^{\alpha}e^{\beta},$$
$$a_i(0)(v\otimes e^{\beta})=(\alpha_i,\,\beta)v\otimes e^{\beta},$$
$$z^{a_i(0)}(v\otimes e^{\beta})=z^{(\alpha_i,\,\beta)}v\otimes e^{\beta},$$
$$q^{d}(v\otimes e^{\beta})=q^{m_0}(v\otimes e^{\beta}),$$

Normal order

Let : : be the usual normal order defined as follows:

$$: a_i(m)a_j(l) := \begin{cases} a_i(m)a_j(l), & m \leq l; \\ a_i(l)a_j(m), & m > l, \end{cases}$$

$$: e^{\alpha_i} a_i(0) :=: a_i(0) e^{\alpha_i} := e^{\alpha_i} a_i(0).$$

Vertex operators

We introduce the main vertex operators.

$$Y_i^{\pm}(z) = \exp\left(\pm\sum_{k=1}^{\infty}\frac{a_i(-k)}{[k]}q^{\mp k/2}z^k\right)\exp\left(\mp\sum_{k=1}^{\infty}\frac{a_i(k)}{[k]}q^{\mp k/2}z^{-k}\right) \times e^{\pm\alpha_i}z^{\pm a_i(0)},$$

$$\Phi_i(z) = q^{a_i(0)}\exp\left((q-q^{-1})\sum_{\ell=1}^{\infty}a_i(\ell)z^{-\ell}\right),$$

$$\Psi_i(z) = q^{-a_i(0)}\exp\left(-(q-q^{-1})\sum_{\ell=1}^{\infty}a_i(-\ell)z^{\ell}\right).$$

Denote that $Y_i^{\pm}(z) = \sum_{n\in\mathbb{Z}} Y_i^{\pm}(n)z^{-n}$.

Vertex representation

Theorem

For $i \in I$ and $s \in J$, the Fock space $\mathcal{F}$ is a $U_q(\mathfrak{g}_{N,tor})$-module for simply-laced types of level 1 under the action ρ defined by :

$$
\begin{aligned}
\gamma_s^{\pm\frac{1}{2}} &\mapsto q^{\pm\frac{1}{2}}, \\
q^{\pm d} &\mapsto q^{\pm d}, \\
K_i &\mapsto q^{a_i(0)}, \\
x_i^{\pm}(\underline{k}) &\mapsto Y_i^{\pm}(ht(\underline{k})), \\
\phi_i^{(s)}(z) &\mapsto \Phi_i(z), \\
\varphi_i^{(s)}(z) &\mapsto \Psi_i(z),
\end{aligned}
$$

where $ht(\underline{k}) \doteq k_1 + \cdots + k_{N-1}$ for $\underline{k} = (k_1, \cdots, k_{N-1})$.

Thank for your attention!