Rota-Baxter Lie bialgebras and Rota-Baxter Poisson Lie groups

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Motivation and goal

Motivation:
Rota-Baxter operators are closely related with r-matrices. $(\mathfrak{g}^*, \mathfrak{g}^*_B) \to (G, G_{\mathfrak{B}})$

M. A. Semenov-Tian-Shansky, Integrable Systems: the r -matrix Approach, Research Institute For Mathematical Sciences, Kyoto University, Kyoto, Japan, 2008.

- Finite-dimensional semisimple Lie algebras $\Longrightarrow$ open Toda lattices;
- loop algebras$\Longrightarrow$ periodic Toda lattices;
- double loop algebras $\Longrightarrow$ Schroedinger equation, sine-Gordon equation;
- Lie algebra of formal pseudo-differential operators $\Longrightarrow$ KdV equation;
- $\cdots$.

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Goal:

- Introduce Rota-Baxter operators on Poisson Lie groups;
- Find applications in integrable system.

Plan

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- Lie bialgebras and Poisson Lie groups
- Rota-Baxter Lie bialgebras
- Rota-Baxter Poisson Lie groups

L. Guo, H. Lang and Y. Sheng, Integration and geometrization of Rota-Baxter Lie algebras, *Adv. Math.* 387 (2021), 107834.

H. Lang and Y. Sheng, Factorizable Lie bialgebras, quadratic Rota-Baxter Lie algebras and Rota-Baxter Lie bialgebras, *Commun. Math. Phys.* 397 (2023), 763-791.

H. Lang and Y. Sheng, Rota-Baxter Poisson Lie groups, in preparation.

1: Lie bialgebras and Poisson Lie groups

Definition (Drinfeld 83, 86)

A Lie bialgebra is a pair of Lie algebras $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ and $(\mathfrak{g}^*, [\cdot,\cdot]_{\mathfrak{g}^*})$ s.t.

$$d_*[x,y]_{\mathfrak{g}} = [d_*(x), y]_{\mathfrak{g}} + [x, d_*(y)]_{\mathfrak{g}}, \quad \forall x, y \in \mathfrak{g},$$

where $d_* : \mathfrak{g} \to \wedge^2\mathfrak{g}$ is defined by $\langle d_*(x), \xi \wedge \eta \rangle := \langle x, [\xi,\eta]_{\mathfrak{g}^*} \rangle$.

$d[\xi,\eta]_{g^*} = [d(\xi),\eta]_{\mathfrak{g}^*} + [\xi, d(\eta)]_{\mathfrak{g}^*}$ for $\xi, \eta \in \mathfrak{g}^*$.

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$d[\xi,\eta]_{g^*} = [d(\xi), \eta]_{\mathfrak{g}^*} + [\xi, d(\eta)]_{\mathfrak{g}^*}$ for $\xi, \eta \in \mathfrak{g}^*$.

Definition (Drinfeld 83, 86)

A Poisson Lie group is a Lie group G with a Poisson structure $(\pi \in \mathfrak{X}^2(G), [\pi,\pi] = 0)$ s.t. the group multiplication $G \times G \to G$ is a Poisson map.

$C^\infty(G)$ is a Poisson algebra. TFAE

- Multiplication is a Poisson map;
- $\pi_{gh} = l_{g*}\pi_h + r_{h*}\pi_g$ $(\Rightarrow \pi_e = 0)$;
- $\{f_1, f_2\}(g_0 h_0) = \{f_1 \circ l_{g_0}, f_2 \circ l_{g_0}\}(h_0) + \{f_1 \circ r_{h_0}, f_2 \circ r_{h_0}\}(g_0)$.

Drinfeld theorem

Theorem (Drinfeld)

There is a one-one correspondence between connected and simply connected Poisson Lie groups and Lie bialgebras:

$$(G, \pi_G) \xrightarrow{1-1} (\mathfrak{g} = T_e G, d_* = d_e \pi), \qquad \big((d_e \pi)_x = (L_{\tilde{x}} \pi)(e)\big).$$

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Poisson Lie groups come in pairs.

$$(G, \pi_G) \to (\mathfrak{g}, \mathfrak{g}^*) \to (\mathfrak{g}^*, \mathfrak{g}) \to (G^*, \pi_{G^*}).$$

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Example

$$(G, \pi_G = 0) \to (\mathfrak{g}, \mathfrak{g}^*) \to (\mathfrak{g}^*, \mathfrak{g}) \to (\mathfrak{g}^*, \pi_{KKS}).$$

Drinfeld doubles

- The double $\mathfrak{d} := \mathfrak{g} \oplus \mathfrak{g}^*$ of a Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$ is a Lie algebra

  $$[x+\xi, y+\eta]_{\bowtie} = \big([x,y]_{\mathfrak{g}} + \mathfrak{ad}^*_{\xi} y - \mathfrak{ad}^*_{\eta} x, [\xi,\eta]_{\mathfrak{g}^*} + \mathrm{ad}^*_x \eta - \mathrm{ad}^*_y \xi\big), \quad \forall x, y \in \mathfrak{g}, \xi, \eta \in \mathfrak{g}^*.$$

  The triple $(\mathfrak{d}, \mathfrak{g}, \mathfrak{g}^*)$ is called a Manin triple.

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  The triple $(\mathfrak{d}, \mathfrak{g}, \mathfrak{g}^*)$ is called a Manin triple.
- Let G, G^* and D be the connected and simply connected Lie groups of $\mathfrak{g}, \mathfrak{g}^*$ and $\mathfrak{g} \bowtie \mathfrak{g}^*$. If G is complete, we have $D \cong G^* \bowtie G$. The multiplication is

  $$(u,g)(v,h) = (u\lambda_g(v), \rho_{v^{-1}}(g)h), \qquad g, h \in G, u, v \in G^*.$$

Example 1

- Let $sl(n, \mathbb{C}) = \{X \in \mathfrak{gl}(n, \mathbb{C}); tr(X) = 0\}$. Consider

$$sl(n, \mathbb{C}) = su(n) \oplus sb(n, \mathbb{C}),$$

where $su(n) = \{X \in sl(n, \mathbb{C}); X + \bar{X}^T = 0\}$ and $sb(n, \mathbb{C})$ is the Lie algebra of all $n \times n$ traceless upper triangular complex matrices with real diagonal entries. With the scalar product $S(X, Y) = \mathrm{Im}(tr(XY))$, we have a Manin triple $((sl(n, \mathbb{C}), S), su(n), sb(n, \mathbb{C}))$.

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- Let $SL(n,\mathbb{C}) = \{A \in GL(n,\mathbb{C}); |A| = 1\}$. By Schmidt orthogonalization, we have the global decomposition

$$SL(n,\mathbb{C}) = SU(n)SB(n),$$

  where $SU(n) = \{A \in SL(n,\mathbb{C}); A^*A = I\}$ and $SB(n)$ is the subgroup of all $n \times n$ determinant 1 upper triangular complex matrices with real diagonal elements. Then we obtain a pair of Poisson Lie groups: $G = SU(n)$ and $G^* = SB(n)$.

Example 2

- Let $\mathfrak{g} = sl(n,\mathbb{C})$ and consider the direct sum Lie algebra $\mathfrak{g} \oplus \mathfrak{g}$. Define

$$\begin{aligned} \mathfrak{g}_{diag} &= \{(X,X); X \in sl(n,\mathbb{C})\}; \\ \mathfrak{g}^*_{st} &= \{(Y + X_+, -Y + X_-); Y \in \mathfrak{h}, X_+ \in n_+, X_- \in n_-\}, \end{aligned}$$

where $\mathfrak{h}, n_+, n_-$ are the Lie algebras of diagonal, strictly upper and lower triangular matrices in $sl(n,\mathbb{C})$. W.r.t the scalar product

$$\langle (X_1, Y_1), (X_2, Y_2) \rangle_{\mathfrak{g}\oplus\mathfrak{g}} = \mathrm{Im}(tr(X_1 X_2)) - \mathrm{Im}(tr(Y_1 Y_2)),$$

the triple $(\mathfrak{g} \oplus \mathfrak{g}, \mathfrak{g}_{diag}, \mathfrak{g}^*_{st})$ consists a Manin triple of Lie algebras.

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the triple $(\mathfrak{g} \oplus \mathfrak{g}, \mathfrak{g}_{diag}, \mathfrak{g}^*_{st})$ consists a Manin triple of Lie algebras.
- We thus have a Poisson Lie group $G = SL(n, \mathbb{C})$ with dual Poisson Lie group

$$SL(n, \mathbb{C})^* = \{(n_+ h, h^{-1} n_-); n_\pm \in N_\pm, h \in H\} = B_+ \times_H B_-,$$

where $H, N_\pm$ denote the diagonal, strictly upper and lower triangular matrices in G and $B_\pm$ are upper and lower triangular matrices in G.

The dual Poisson Lie group (G^*, π_{st}) plays important roles in Ginzburg-Weinstein map $(u(n)^* \cong U(n)^*)$, Riemann-Hilbert map $(\mathfrak{g}^* \to G^*)$, etc. It can be identified with a certain moduli space of meromorphic connections over the unit disk having an irregular singularity at the origin.

P. Boalch, Stokes matrices, Poisson Lie groups and Frobenius manifolds, *Invent. Math.* 146 (2001), 479-506.

Factorizable Lie bialgebras

Let $\mathfrak{g}$ be a Lie algebra and $r \in \mathfrak{g} \otimes \mathfrak{g}$. Introduce a bracket on $\mathfrak{g}^*$:

$$r_+ : \mathfrak{g}^* \to \mathfrak{g}, \quad r_+(\xi) := r(\xi, \cdot), \qquad r_- = -r_+^* : \mathfrak{g}^* \to \mathfrak{g}, \qquad I = r_+ - r_- : \mathfrak{g}^* \to \mathfrak{g}.$$

$$[\xi, \eta]_r := \mathrm{ad}^*_{r_+\xi}\eta - \mathrm{ad}^*_{r_-\eta}\xi. \qquad (d_*(x) = [x, r]).$$

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Question: When is $(\mathfrak{g}, \mathfrak{g}^*_r)$ a Lie bialgebra?

- skew-symmetry $\Leftrightarrow [x, r + \sigma(r)] = 0 \Leftrightarrow I \circ \mathrm{ad}^*_x = \mathrm{ad}_x \circ I$;
- Jacobi identity $\Leftarrow [r_{12}, r_{13}] + [r_{13}, r_{23}] + [r_{12}, r_{23}] = 0$ (CYBE).

Definition

Such a Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*_r)$ is called a quasitriangular Lie bialgebra.

- Factorizable: $I : \mathfrak{g}^* \to \mathfrak{g}$ is an isomorphism of v.s.;
- Triangular: $I = 0$ $(r \in \wedge^2\mathfrak{g})$.

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Example

Given a Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$, then $(\mathfrak{d}, \mathfrak{d}^*_r)$ is a factorizable Lie bialgebra with $r = \sum_i \xi^i \otimes x_i \in \mathfrak{g}^* \otimes \mathfrak{g} \subset \mathfrak{d} \otimes \mathfrak{d}$. In fact, $\mathfrak{d}^*_r = \mathfrak{g} \times \overline{\mathfrak{g}^*}$ and

$$I = \mathrm{id}, \qquad r_- : \mathfrak{d}^*_r \to \mathfrak{d}, (x, \xi) \mapsto -(0, \xi).$$

Factorizable Poisson Lie groups

Let G be a Poisson Lie group corresponding to a factorizable Lie bialgebra $(\mathfrak{g}, \mathfrak{g}_r^*)$. Then $\pi_G = \overleftarrow{r} - \overrightarrow{r}$. Let G^* be its simply connected dual. Integrating the Lie algebra homo. $r_\pm : \mathfrak{g}^* \to \mathfrak{g}$ to Lie group homo. $R_\pm : G^* \to G$. Define

$$J : G^* \to G, \qquad J(u) = R_+(u)R_-(u)^{-1}.$$

If J is a global diffeomorphism, we call G is a factorizable Poisson Lie group.

N. Reshetikhin and M. A. Semenov-Tian-Shansky, Quantum R-matrices and factorization problems, *J. Geom. Phys.* 5 (1988), 533-550.

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Example

Given simply connected Poisson Lie group (G, π_G) and its dual (G^*, π_{G^*}). Suppose it is complete. The lifts $R_+, R_-, J : D^* \to D$ are

$$R_+(g,u) = g, \qquad R_-(g,u) = u^{-1}, \qquad J(g,u) = gu, \qquad \forall g \in G, u \in G^*.$$

So J is a global diffeomorphism and (D, π_D) is a factorizable Poisson Lie group.

Example

$$(\mathfrak{g}, \mathfrak{g}^*), \qquad (\mathfrak{g}^*, \mathfrak{g}), \qquad (\mathfrak{d}, \mathfrak{d}^*), \qquad (\mathfrak{d}^*, \mathfrak{d}).$$

$$G, \qquad G^*, \qquad D = G^* \bowtie G, \qquad D^* = G \times \overline{G^*}.$$

$$SU(n), \qquad SB(n, \mathbb{C}), \qquad SL(n, \mathbb{C}), \qquad SU(n) \times \overline{SB(n, \mathbb{C})}.$$

Example

$$(\mathfrak{g},\mathfrak{g}^*),\qquad (\mathfrak{g}^*,\mathfrak{g}),\qquad (\mathfrak{d},\mathfrak{d}^*),\qquad (\mathfrak{d}^*,\mathfrak{d}).$$

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$$SU(n),\qquad SB(n,\mathbb{C}),\qquad SL(n,\mathbb{C}),\qquad SU(n)\times\overline{SB(n,\mathbb{C})}.$$

Here

$$\pi_D=\overleftarrow{r}-\overrightarrow{r},\qquad r=\sum_i \xi^i\otimes x_i\in\mathfrak{d}\otimes\mathfrak{d}.$$

$$\pi_{D^*}=(-\pi_G,\pi_{G^*})-\sum_{i=1}^{n}\Big((0,\overrightarrow{\xi^i})\wedge(\overrightarrow{x_i},0)-(0,\overleftarrow{\xi^i})\wedge(\overleftarrow{x_i},0)\Big).$$

2. RB algebras and RB Lie algebras

Definition

A Rota-Baxter algebra of weight λ is an algebra $\mathcal{A}$ with a linear map $B : \mathcal{A} \to \mathcal{A}$ such that

$$Bx \cdot By = B(Bx \cdot y + x \cdot By + \lambda x \cdot y).$$

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Example

Let $\mathcal{A} = C^\infty(\mathbb{R})$ and $B(f)(x) = \int_0^x f(t)dt =: F(x)$. So $F'(x) = f(x)$. Then $(C^\infty(\mathbb{R}), B)$ is a RB algebra of weight 0.

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$$[x, y]_B := [Bx, y]_{\mathfrak{g}} + [x, By]_{\mathfrak{g}} + \lambda[x, y]_{\mathfrak{g}}.$$

$(\mathfrak{g}, [\cdot, \cdot]_B)$ is called the descendent Lie algebra and denoted by $\mathfrak{g}_B$.

RB Lie bialgebras

Definition (Lang-Sheng)

A RB operator of weight λ on a Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$ is a linear map $B : \mathfrak{g} \to \mathfrak{g}$ s.t.

(i) B is a RB operator of weight λ on $\mathfrak{g}$;

(ii) $\widetilde{B}^* := -\lambda \mathsf{id} - B^*$ is a RB operator of weight λ on $\mathfrak{g}^*$. ($\Leftrightarrow B^*$ is a RB on $\mathfrak{g}^*$.)

A Lie bialgebra with a RB operator is called a RB Lie bialgebra.

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Theorem (Lang-Sheng)

$(\mathfrak{g}, \mathfrak{g}^*, B)$ is a RB Lie bialgebra,

(1) *iff $((\mathfrak{g}, B), (\mathfrak{g}^*, \widetilde{B}^*); \mathrm{ad}^*, \mathfrak{ad}^*)$ is a matched pair of RB Lie algebras.*

(2) *then $(\mathfrak{d}, \mathfrak{d}^*, B \oplus \tilde{B}^*)$ is also a RB Lie bialgebra, where $B \oplus \tilde{B}^* : \mathfrak{g} \oplus \mathfrak{g}^* \to \mathfrak{g} \oplus \mathfrak{g}^*$ is*

$$(B \oplus \tilde{B}^*)(x + \xi) = Bx - \lambda\xi - B^*\xi, \quad x \in \mathfrak{g}, \ \xi \in \mathfrak{g}^*.$$

(3) *$(\mathfrak{g}_B, \mathfrak{g}^*_{\tilde{B}^*})$ is a descendent matched pair of RB Lie algebras with*

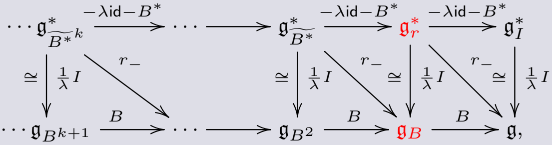
$$\mathfrak{g}_B \bowtie \mathfrak{g}^*_{\tilde{B}^*} = \mathfrak{d}_{B \oplus \tilde{B}^*}.$$

RB Lie bialgebras given by factorizable Lie bialgebras

Theorem

Let $(\mathfrak{g}, \mathfrak{g}_r^*)$ *be a factorizable Lie bialgebra with* $I = r_+ - r_- : \mathfrak{g}^* \xrightarrow{\cong} \mathfrak{g}$*. Then*

- $(\mathfrak{g}, \mathfrak{g}_r^*, B)$ *is a RB Lie bialgebra of weight* λ*, where* $B := \lambda r_- \circ I^{-1}$*;*
- *we have the following commutative diagram of Lie algebra homo.:*



where $B = \lambda r_- \circ I^{-1}$ *and* $\widetilde{B^*} = \lambda I^{-1} \circ r_-$ *and* $\mathfrak{g}_{B^k}$ *is the descendent Lie algebra of the RB operator* B *on* $\mathfrak{g}_{B^{k-1}}$*.*

RB Lie bialgebras given by factorizable Lie bialgebras

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Let $(\mathfrak{g}, \mathfrak{g}^*_r)$ *be a factorizable Lie bialgebra with* $I = r_+ - r_- : \mathfrak{g}^* \xrightarrow{\cong} \mathfrak{g}$. *Then*

- $(\mathfrak{g}, \mathfrak{g}^*_r, B)$ *is a RB Lie bialgebra of weight* λ, *where* $B := \lambda r_- \circ I^{-1}$;
- *we have the following commutative diagram of Lie algebra homo.:*

$$\begin{array}{ccccccccc}
\cdots \mathfrak{g}^*_{\widetilde{B^*}^k} & \xrightarrow{-\lambda \mathrm{id} - B^*} & \cdots & \longrightarrow & \mathfrak{g}^*_{\widetilde{B^*}} & \xrightarrow{-\lambda \mathrm{id} - B^*} & \mathfrak{g}^*_r & \xrightarrow{-\lambda \mathrm{id} - B^*} & \mathfrak{g}^*_I \\
\cong \downarrow \frac{1}{\lambda} I & \searrow^{r_-} & & & \cong \downarrow \frac{1}{\lambda} I & \searrow^{r_-} & \cong \downarrow \frac{1}{\lambda} I & \searrow^{r_-} & \cong \downarrow \frac{1}{\lambda} I \\
\cdots \mathfrak{g}_{B^{k+1}} & \xrightarrow{B} & \cdots & \longrightarrow & \mathfrak{g}_{B^2} & \xrightarrow{B} & \mathfrak{g}_B & \xrightarrow{B} & \mathfrak{g},
\end{array}$$

where $B = \lambda r_- \circ I^{-1}$ *and* $\widetilde{B^*} = \lambda I^{-1} \circ r_-$ *and* $\mathfrak{g}_{B^k}$ *is the descendent Lie algebra of the RB operator* B *on* $\mathfrak{g}_{B^{k-1}}$.

Example

Let $(\mathfrak{g}, \mathfrak{g}^*)$ be any Lie bialgebra. The Lie bialgebra $(\mathfrak{d}, \mathfrak{d}^*)$ of the Drinfeld double $\mathfrak{d} = \mathfrak{g} \bowtie \mathfrak{g}^*$ with

$$B : \mathfrak{d} \to \mathfrak{d}, \qquad x + \xi \mapsto -\lambda \xi$$

is a RB Lie bialgebra.

3: RB Poisson Lie groups

A RB operator of weight 1 on a Lie group G is a smooth map $\mathfrak{B} : G \to G$ such that

$$\mathfrak{B}(g)\mathfrak{B}(h) = \mathfrak{B}(g\mathfrak{B}(g)h\mathfrak{B}(g)^{-1}), \qquad \forall g, h \in G.$$

Descendent Lie group: $G_{\mathfrak{B}}$ with the multiplication $g *_{\mathfrak{B}} h = g\mathfrak{B}(g)h\mathfrak{B}(g)^{-1}$.

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Definition (Lang-Sheng)

Let (G, π) be a Poisson Lie group with Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$. A smooth map $\mathfrak{B} : G \to G$ is called a RB operator on (G, π) if

(1) $\mathfrak{B}$ is a RB operator on the Lie group G;

(2) $\widetilde{B^*} := -\mathrm{id} - B^* : \mathfrak{g}^* \to \mathfrak{g}^*$ is a RB operator of weight 1 on $\mathfrak{g}^*$, where $B = \mathfrak{B}_{*e}$.

The triple $(G, \pi, \mathfrak{B})$ is called a RB Poisson Lie group.

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Differentiation: RB Poisson Lie group $\longrightarrow$ RB Lie bialgebra.
Integration: Have problem! In fact, a RB operator of a Lie algebra can not be integrated to a RB operator on the corresponding Lie group; See

J. Jiang, Y. Sheng and C. Zhu, Lie theory and cohomology of relative Rota-Baxter operators, *J. London Math. Society* 109 (2024).

Examples

(1) If $\mathfrak{B}$ is a RB operator on the Poisson Lie group G, then $\hat{\mathfrak{B}} : G \to G$ defined by $\hat{\mathfrak{B}}(g) = g^{-1}\mathfrak{B}(g^{-1})$ is also a RB operator on the Poisson Lie group G.

(2) The maps

$$\mathfrak{B} : G \to G, \qquad g \mapsto g^{-1}; \qquad g \to e$$

are RB operators on an arbitrary Poisson Lie group (G, π).

(3) $(G, \pi = 0, \mathfrak{B})$ is a RB Poisson Lie group iff $(G, \mathfrak{B})$ is a RB Lie group.

(4) Let $\mathfrak{g}$ be a Lie algebra and $(\mathfrak{g}^*, \pi_{KKS})$ the corresponding linear Poisson Lie group. Then $(\mathfrak{g}^*, \pi_{KKS}, \mathfrak{B})$ is a RB Poisson Lie group iff $(\mathfrak{g}, \mathfrak{B}^*)$ is a RB Lie algebra.

Questions

Given a RB Poisson Lie group $(G, \pi_G, \mathfrak{B})$,

- what is the explicit relation of $\mathfrak{B}$ with π_G or $\{\cdot, \cdot\}$?
- what's the relation of it with RB Poisson algebras (RB ass. RB Lie)?
- do we have a descendent RB Poisson Lie group $(G_{\mathfrak{B}}, \pi_{\mathfrak{B}}, \mathfrak{B})$?
- dual RB Poisson Lie group $(G^*, \pi_{G^*}, \mathcal{C})$?
- Given a RB Poisson Lie group $(G, \pi, \mathfrak{B})$ and a dual RB Poisson Lie group $(G^*, \pi_{G^*}, \mathcal{C})$ s.t. $\mathfrak{B}^*_{*e} + \mathcal{C}_{*e} = -\mathrm{id}$, is there a double RB Poisson Lie group $(G \bowtie G^*, \pi_D, \mathfrak{B}_D)$?

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Done!

(1) Lie bialgebras;

(2) $\pi_G = 0$ case, not trivial;

(3) Factorizable Poisson Lie groups.

Proposition

Let $(G, \pi_G = 0)$ *be a Poisson Lie group and* $(\mathfrak{g}^*, \pi_{KKS})$ *its dual Poisson Lie group. Let* $\mathfrak{B}$ *be a RB operator on the Lie group* G*. Then*

(1) $(G, 0, \mathfrak{B})$ *and* $(\mathfrak{g}^*, \Pi_{KKS}, -\mathrm{id} - B^*)$ *are both RB Poisson Lie groups.*

(2) *The double Poisson Lie group* $D = \mathfrak{g}^* \ltimes G$ *with the RB operator* $\tilde{\mathfrak{B}}$

$$\tilde{\mathfrak{B}}(u, g) = \Big(\mathrm{Ad}^*_{\mathfrak{B}(g)}\big((-\mathrm{id} - B^*)\mathrm{Ad}^*_{\mathfrak{B}(\mathfrak{g})^{-1}g^{-1}}u\big), \mathfrak{B}(g)\Big), \qquad \forall g \in G, u \in \mathfrak{g}^*.$$

is a RB Poisson Lie group;

(3) *The dual* $D^* = G \times \overline{\mathfrak{g}^*}$ *with the RB operator*

$$\mathfrak{B} \times (-\mathrm{id} - B^*) : D^* \to D^*, \qquad (g, u) \mapsto \big(\mathfrak{B}(g), (-\mathrm{id} - B^*)(u)\big)$$

is a RB Poisson Lie group.

RB operators on factorizable Poisson Lie groups

$$R_+, R_- : G^* \to G, \qquad J : G^* \xrightarrow{\cong} G, \quad J(u) = R_+(u)R_-(u)^{-1}.$$

Proposition

If G be a factorizable Poisson Lie group and G^* the dual Poisson Lie group, then

(i) *the maps*

$$\mathfrak{B} : G \to G, \quad \mathfrak{B}(g) := R_-(J^{-1}g); \quad \mathcal{C} : G^* \to G^*, \quad \mathcal{C}(u) := J^{-1}(R_- u)$$

are RB operators on G and G^*. Note that $\mathfrak{B}^*_{*e} + \mathcal{C}_{*e} = -\mathrm{id}$.

(ii) $(G_{\mathfrak{B}}, J_*\pi_{G^*}, \mathfrak{B})$ *is a RB Poisson Lie group.*

(iii) $J : (G^*, \pi_{G^*}, \mathcal{C}) \to (G_{\mathfrak{B}}, J_*\pi_{G^*}, \mathfrak{B})$ *is an isomorphism of RB Poisson Lie groups.*

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- Two RB Poisson Lie groups: $(G, \pi_G, \mathfrak{B} = R_- \circ J^{-1})$, $(G_{\mathfrak{B}}, J_*\pi_{G^*}, \mathfrak{B})$.
- In general, it is not clear how to make $G_{\mathfrak{B}}$ a Poisson Lie group. Let $G = SL(n, \mathbb{C})$ be the Poisson Lie group with dual

$$SL(n, \mathbb{C})^* = \{(n_+h, h^{-1}n_-); n_\pm \in N_\pm, h \in H\} = B_+ \times_H B_-,$$

Due to the fact the Gauss decomposition has restrictions,

$$SL(n, \mathbb{C})^* \to SL(n, \mathbb{C}), \qquad (b_+, b_-) \mapsto b_+b_-^{-1},$$

is a local Poisson diffeomorphism.

RB operators on the double of factorizable Poisson Lie groups

$$R_+, R_- : G^* \to G, \qquad J : G^* \xrightarrow{\cong} G, \quad J(u) = R_+(u)R_-(u)^{-1}.$$

Lemma

Let (G, π, J) be a factorizable Poisson Lie group. Then we have a Lie group isomorphism from the double Lie group $D = G^* \bowtie G$ to the direct product Lie group $G \times G$:

$$G^* \bowtie G \xrightarrow{\Psi} G \times G, \qquad \Psi(u, g) = (R_+(u)g, R_-(u)g), \qquad \forall u \in G^*, g \in G.$$

Later, we will make Ψ an isomorphism of RB Poisson Lie groups.

Theorem (Lang-Sheng)

Let (G, π, J) be a factorizable Poisson Lie group and G^* be the dual Poisson Lie group. Then there is a unique RB operator $\tilde{\mathfrak{B}}$ on the double $D = G^* \bowtie G$ s.t.

$$\tilde{\mathfrak{B}}(u, 1) = \mathcal{C}(u), \qquad \tilde{\mathfrak{B}}(1, g) = \mathfrak{B}(g), \qquad \forall u \in G^*, g \in G,$$

and Ψ is an isomorphism of RB Lie groups

$$\begin{array}{ccc} G^* \bowtie G & \xrightarrow{\Psi} & G \times G \\ \tilde{\mathfrak{B}} \downarrow & & \downarrow \mathfrak{B}\times\mathfrak{B} \\ G^* \bowtie G & \xrightarrow[\Psi]{} & G \times G \end{array} .$$

Moreover, $(G^* \bowtie G, \pi_D, \tilde{\mathfrak{B}})$ is a RB Poisson Lie group.

Proposition

Let (G, π, J) be a factorizable Poisson Lie group, there is a unique Poisson structure $\pi_{G\times G}$ on $G \times G$ making $(G \times G, \pi_{G\times G}, \mathfrak{B} \times \mathfrak{B})$ with $\mathfrak{B} = R_- \circ J^{-1}$ a RB Poisson Lie group s.t. Ψ is an isomorphism of RB Poisson Lie groups.

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Proposition

The RB Lie bialgebra $(\mathfrak{g} \oplus \mathfrak{g}, \mathfrak{g}^* \bowtie \mathfrak{g}^*, B \times B)$ of the above RB Poisson Lie group is as follows: $B = r_- \circ I^{-1}$ and the Lie bracket on $\mathfrak{g}^* \bowtie \mathfrak{g}^*$ is

$$\begin{aligned}
[\xi_1, \xi_1']_{\mathfrak{g}^*\bowtie\mathfrak{g}^*} &= -[\xi_1, \xi_1'], \\
[\xi_2, \xi_2']_{\mathfrak{g}^*\bowtie\mathfrak{g}^*} &= -[\xi_2, \xi_2'], \\
[\xi_1, \xi_2]_{\mathfrak{g}^*\bowtie\mathfrak{g}^*} &= (\mathrm{ad}^*_{r_+\xi_2}\xi_1, -\mathrm{ad}^*_{r_-\xi_1}\xi_2),
\end{aligned}$$

for $(\xi_1, \xi_2), (\xi_1', \xi_2') \in \mathfrak{g}^* \bowtie \mathfrak{g}^*$.

The Poisson structure $\pi_{G\times G}$ is quite interesting and it is a mixed product Poisson structure.

Poisson structure: $\pi_{G\times G}$

Let $\{x_i\}_{i=1}^n$ be a basis of $\mathfrak{g}$ s.t. $\{x_i\}_{i=1}^k$ $(k \leq n)$ is a basis of $\mathrm{Im} r_+$ and let $\{\xi_i\}_{i=1}^n$ be the dual basis of $\mathfrak{g}^*$. We have

$$\pi_{G\times G} = (-\pi_G, -\pi_G) + \sum_{i=1}^{k} \left((\overleftarrow{r_+\xi_i}, 0) \wedge (0, \overleftarrow{x_i}) + (\overrightarrow{r_+\xi_i}, 0) \wedge (0, \overrightarrow{x_i}) \right),$$

where $\pi_G = \overleftarrow{r} - \overrightarrow{r}$. This Poisson structure has properties as follows:

- $(G \times G, \pi_{G\times G})$ is also factorizable with

$$\pi_{G\times G} = \overrightarrow{r^{(2)}} - \overleftarrow{r^{(2)}}, \qquad r^{(2)} \in (\mathfrak{g} \oplus \mathfrak{g})^{\otimes 2} = (\mathfrak{g} \oplus \mathfrak{g}) \otimes (\mathfrak{g} \oplus \mathfrak{g}).$$

$$r^{(2)} = (r)_1 - (r^{21})_2 - \sum_{i=1}^{n} (y_i)_1 \wedge (x_i)_2,$$

  where if $r = \sum_i x_i \otimes y_i \in \mathfrak{g}^{\otimes 2}$, define $r^{21} = \sum_i y_i \wedge \otimes x_i$ and for $X \in \mathfrak{g}^{\otimes m} (m = 1, 2)$, the notation $(X)_j \in (\mathfrak{g} \oplus \mathfrak{g})^{\otimes m}$ means the image of X under the embedding of $\mathfrak{g}$ in $\mathfrak{g} \oplus \mathfrak{g}$ as the j-th summand;
- $\pi_{G\times G}$ is a mixed product Poisson structure on $G \times G$ in the sense that the projections $p_i : G \times G \to G$ are well-defined Poisson structures on G.

J-H. Lu and V. Mouquin, Mixed product Poisson structures associated to Poisson Lie groups and Lie bialgebras, *Int. Math. Res. Notices* 2017 (2017), 5919-5976.

Example: Factorizable Poisson Lie group (D, π_D)

Proposition

Let $(D = GG^*, \pi_D)$ be the double of a Poisson Lie group (G, π_G). Then

(i) *the two maps*

$$\mathfrak{B} : D = GG^* \to D, \quad (g, u) \mapsto u^{-1}, \qquad \mathcal{C} : D^* \to D^*, \quad (g, u) \mapsto u^{-1}$$

are RB operators on the Poisson Lie groups D and D^*.

(ii) *the triple $(D, *_{\mathfrak{B}}, J_* \pi_{D^*})$ is also a Poisson Lie group with $\mathfrak{B}$ a RB operator on it, where $(gu) *_{\mathfrak{B}} (hv) = ghvu$. Moreover, the map*

$$J : (D^*, \pi_{D^*}, \mathcal{C}) \to (D, *_{\mathfrak{B}}, J_* \pi_{D^*}, \mathfrak{B}), \qquad (g, u) \mapsto gu$$

defines an isomomorphism of RB Poisson Lie groups.

Thanks for your attention!