Segal's Axioms & Probability.

I. Quantum Mechanics of a Single (free) particle moving in $\mathbb{R}$

$*$ (possible) position of the particle described by $x \in \mathbb{R}$.

Message from QM: DO NOT know precise position/momentum of the particle, but $\exists$ "wave function" $f \in L^2(\underbrace{\mathbb{R}}_{\text{possible positions}}, \underbrace{d_{\mathbb{R}}}_{\text{Lebesgue measure}})$

s.t. $|f(x)|^2$ = prob. density of finding particle at x. ($\Rightarrow \|f\|_{L^2} = 1$).

Schrödinger's picture of time evolution

particle starts at time $t=0$ with w.f. f_0

$\Rightarrow$ w.f. at time $t>0$ is $f_t = e^{itH} f_0$, $H = -\partial_x^2$ (free particle)

i.e. $f_t(x) = \int_{\mathbb{R}} e^{itH}(x,y) f_0(y)\,dy$, $\quad e^{itH}(x,y)$ = integral kernel

$$\approx \langle \delta_x, e^{itH} \delta_y \rangle_{L^2},$$

= "amplitude" for particle to travel from y to x.

$\boxed{\text{Want more details of what happens during the process "}e^{itH}\text{".}}$

Recall: heat kernel $e^{-tH}(x,y) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{1}{4}\cdot\frac{|x-y|^2}{t}}$

$t \mapsto -it$ (Wick rotation) $\Rightarrow e^{itH}(x,y) = \sqrt{\frac{i}{4\pi t}}\, e^{\frac{i}{4}\cdot\frac{|x-y|^2}{t}}$

$\boxed{\underline{\underline{\text{magic}}} \quad \left(\frac{i}{4\pi t}\right)^{\frac{1}{2}} e^{\frac{i}{4}\cdot t\frac{|x-y|^2}{t^2}}}$

$\boxed{q(t_j) = y_j = \text{position of particle at } t_j = j(\delta t). \quad t_0 = 0,\ t_N = t.\quad y_0 = y,\ y_N = x.}$

Now pick $N \gg 1$. $\delta t \stackrel{\text{def}}{=} t/N$.

$$e^{itH}(x,y) = \left(e^{i\frac{t}{N}H}\right)^N(x,y)$$

$$= \underbrace{\int\cdots\int}_{N-1} e^{i\frac{t}{N}H}(x,y_{N-1})\, e^{i\frac{t}{N}H}(y_{N-1},y_{N-2})\cdots e^{i\frac{t}{N}H}(y_1,y)\, dy_1\cdots dy_{N-1}$$

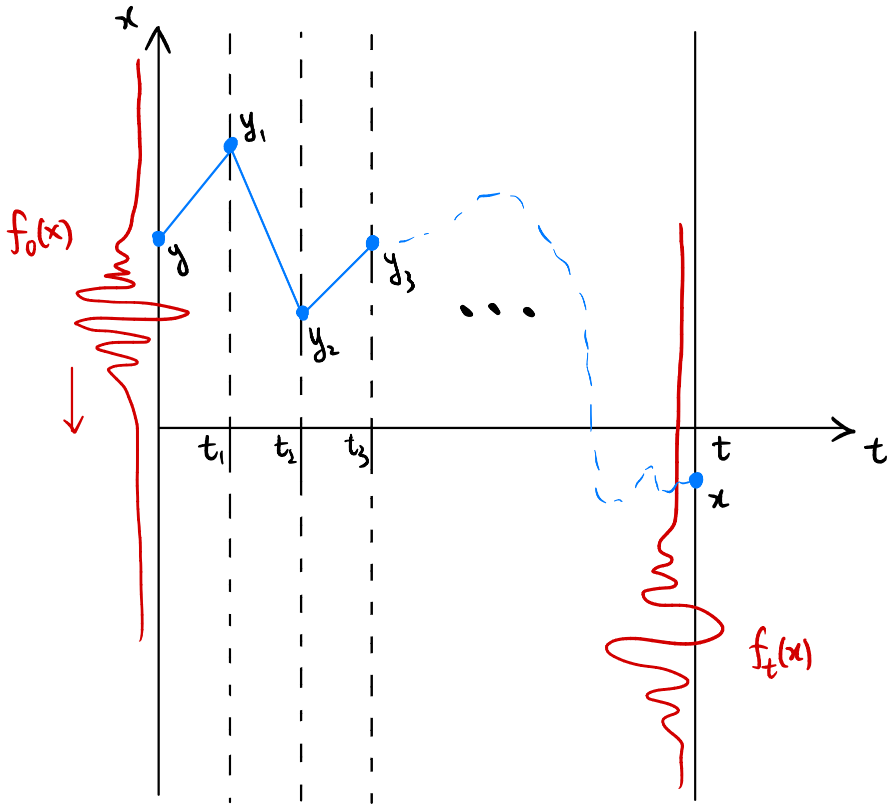
$$= \int\cdots\int \prod_{j=0}^{N-1} e^{\frac{i}{4}(\delta t)\frac{|q(t_{j+1})-q(t_j)|^2}{(\delta t)^2}} \left(\frac{i}{4\pi\delta t}\right)^{\frac{N}{2}} \prod_{j=1}^{N-1} dq(t_j).$$

$$\underset{\delta t\to 0}{\overset{N\to\infty}{=\!=}} \int_{\left\{\substack{\text{paths } q:[0,t]\to\mathbb{R} \\ \text{with } q(0)=x,\ q(t)=y}\right\}} \underbrace{e^{\frac{i}{4}\int_0^t |\dot q(s)|^2 ds}\, [\mathcal{D}q]}_{A}$$

"Lebesgue measure" on path space.

$$[\mathcal{D}q] \approx \lim_{N\to\infty} \left(\frac{i}{4\pi\delta t}\right)^{\frac{N}{2}} \prod_{j=1}^{N-1} dq(t_j).$$

$\boxed{\text{Rmk. Without } i,\ A_{\text{(normalized)}} = \text{Wiener measure for Brownian motion (Bridge), rigorous.}}$



x
$f_0(x)$
y
y_1
y_2
y_3
t_1
t_2
t_3
t
x
t
$f_t(x)$

II. QM/QFT of a (rough) closed string moving in $\mathbb{R}$

* (possible) configuration of string described by a real distribution $\varphi \in \mathcal{D}'(\mathbb{S}^1)$

Alternatively, $\varphi \in \mathcal{D}'(\mathbb{S}^1)$ is config. of real scalar field over $\mathbb{S}^1$.

Message from QM: $\exists$ "wave function" $F \in L^2(\mathcal{D}'(\mathbb{S}^1), \mu_{???})$

s.t. $|F(\varphi)|^2 =$ prob. density of finding string at config. $\varphi \in \mathcal{D}'(\mathbb{S}^1)$.

depends on $\mu_{???}$

Moral from previous story (version without i) for time evolution:

$\Rightarrow$ define directly integral kernel using integration on path space.

a priori

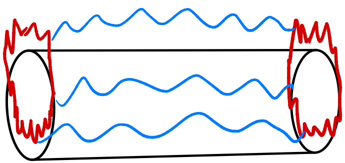
i.e. $(U_t F)(\varphi_{out}) = \int A_\Omega(\varphi_{in}, \varphi_{out}) F(\varphi_{in}) \, d\mu_{???}(\varphi_{in})$. $\Omega = [0,t] \times \mathbb{S}^1$, with a metric.

with

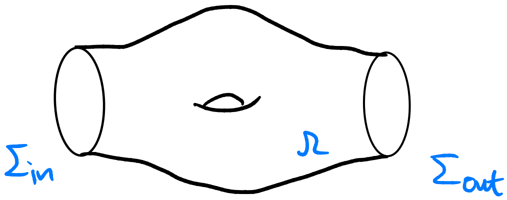
$$A_\Omega(\varphi_{\text{in}}, \varphi_{\text{out}}) = \int_{\underbrace{\{\phi \mid \partial\Omega = (\varphi_{\text{in}}, \varphi_{\text{out}})\}}_{\subseteq \mathcal{D}'(\Omega)}} \mathrm{e}^{-\int_\Omega \frac{1}{2}(|\nabla\phi|_g^2 + m^2\phi^2) + P(\phi(x)) \mathrm{d}V_\Omega(x)} [\mathcal{L}\phi],$$

mass > 0

interaction potential
"
polynomial bounded below



Example: $P(\phi) = \phi^4$.

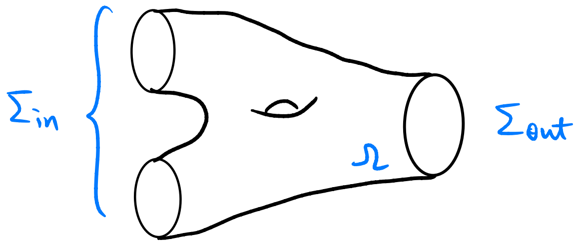


$\boxed{\text{Atiyah-Segal Axioms}}$ for an abstract QFT.

It is a Rule of association

① Circle $\Sigma \rightsquigarrow$ (real) Hilbert Space $\mathcal{H}_\Sigma$.
$\Sigma_1 \sqcup \Sigma_2 \rightsquigarrow \mathcal{H}_{\Sigma_1} \otimes \mathcal{H}_{\Sigma_2}$.

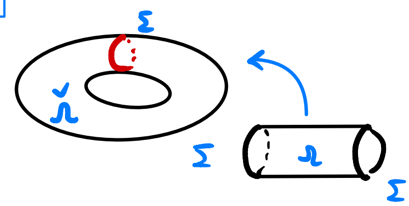
② Cobordism $\Omega \rightsquigarrow U_\Omega : \mathcal{H}_{\Sigma_1} \to \mathcal{H}_{\Sigma_2}$



such that

④ $\boxed{U_{\Omega_2 \cup_{\Sigma'} \Omega_1} = U_{\Omega_2} \circ U_{\Omega_1}}$

③ $Z_X = \mathrm{tr}(U_\Omega)$.



⑤ $U_{\Omega^*} = U_\Omega^\dagger$.
↳ co-orient. reversed

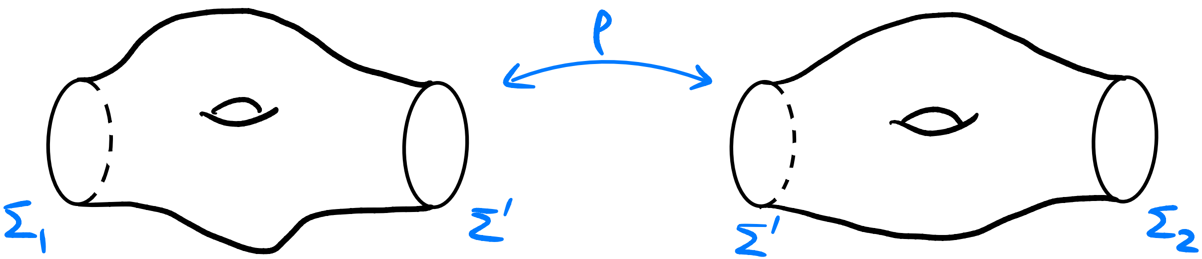
Rmk. Ω has a metric.

Rmk. Metric cobordisms does not form a category (no identities).

For M = closed Riemannian Surface.

partition function
$\parallel$

$$Z_M = \int_{\mathcal{D}'(M)} \mathrm{e}^{-\int_M P(\phi(x))\mathrm{d}V_M(x)} \mathrm{e}^{-\frac{1}{2}\int_M (|\nabla\phi|_g^2 + m^2\phi^2)\mathrm{d}V_M} [\mathcal{L}\phi]$$



$$\mathcal{A}_{\Omega_1 \cup_\rho \Omega_2}(\varphi_2, \varphi_1) \stackrel{?}{=} \int \mathcal{A}_{\Omega_2}(\varphi_2, \varphi')\,\mathcal{A}_{\Omega_1}(\varphi', \varphi_1)\,[d\varphi']$$

[1] Pickrell, Doug, P(φ) 2 Quantum Field Theories and Segal's Axioms, Commun. Math. Phys. 280, 403–425, 2008.

[2] Guillarmou, C., Kupiainen, A., Rhodes, R., and Vargas, V. (2021). Segal's axioms and bootstrap for Liouville Theory. arXiv:2112.14859.

[3] S. Kandel, P. Mnev and K. Wernli, Two-dimensional perturbative scalar QFT and Atiyah-Segal gluing, Adv. Theor. Math. Phys. 25 (2021) no.7, 1847-1952.

$\boxed{\text{Case } P=0.}$ First let M = closed surface.

$\rightsquigarrow$ $$e^{-\frac{1}{2}\int_M (|\nabla\phi|_g^2+m^2\phi^2)\mathrm{d}V_M}[\mathcal{L}\phi] \overset{\text{heu}}{=} \text{“}\det\text{”}(\Delta+m^2)^{-\frac{1}{2}}\boxed{\mathrm{d}\mu_{\text{GFF}}^M(\phi)}$$

Idea: $$\int_{\mathbb{R}^N} e^{-\frac{1}{2}\langle \mathbf{x}, C^{-1}\mathbf{x}\rangle}\mathrm{d}\mathcal{L}^N(\mathbf{x}) = \frac{(2\pi)^{N/2}}{(\det C^{-1})^{1/2}}$$

a Gaussian Proba measure on $\mathcal{D}'(M)$.

∞-dim'l generalization of determinant. (ζ-reg. det.).

$\boxed{\text{(massive) Gaussian Free Field (GFF).}}$ is a Prob. measure on $\mathcal{D}'(M)$. μ_{GFF}^M

$\Rightarrow$ $\phi \mapsto \langle\phi, f\rangle = \phi(f)$, $f \in C^\infty(M)$, defines a R.V. on Sample Space $\mathcal{D}'(M)$.

s.t. (i) each $\phi(f)$ is Gaussian $\mathbb{R}$-valued R.V.

covariance operator.

(ii) $$\mathbb{E}[\phi(f)\phi(h)] = \langle f, (\Delta+m^2)^{-1}h\rangle_{L^2(M)}, \quad \mathbb{E}[\phi(f)] \equiv 0,$$

$\forall$ Test functions $f, h \in C^\infty(M)$.

Existence: Bochner-Minlos. Uniqueness: Kolmogorov / Fourier transform.

(Many other equivalent constructions).

☆ $\|\Phi(f)\|_{L^2(\mu_{GFF})} = \|f\|_{H^{-1}(M)}$ → Sobolev space.

$\Rightarrow H^{-1}(M) =$ Gaussian Hilbert Space. $\quad H^{-1}(M) \hookrightarrow L^2(\mu_{GFF})$.

(for Gaussian R.V. $F \perp_{L^2} G \Leftrightarrow F, G$ indep).

$\Rightarrow \Phi(f)$ makes sense as R.V. $\forall f \in H^{-1}$ & not as number

Formally, take $f = \delta_x$, $h = \delta_y$,

$\Rightarrow \mathbb{E}[\phi(x)\phi(y)] \approx G(x, y) =$ Green's function = 2-pt function

= (free) "propagator" in physics.

$$\boxed{\text{How to define } \mathcal{A}^0_\Omega(\varphi_{in}, \varphi_{out}) \text{ for } P=0\,?}$$

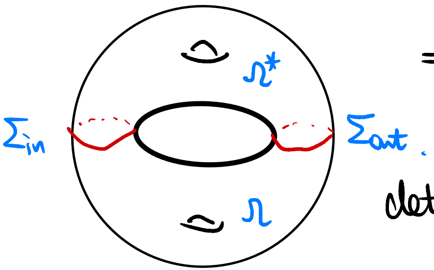
☆ $\mathcal{A}^0_\Omega$ is meant to be integrated against some measure on $\mathcal{D}'(\Sigma_{in})$ or $\mathcal{D}'(\Sigma_{out})$.

$\Rightarrow$ its value will depend on choice of these measures.

(different choice being mutually abs. cont.

$\Rightarrow$ value related by R.-N densities)

Now, we apply Segal's rules ③ – ⑤.



$$\Rightarrow Z_{(\Omega^* \cup_{\Sigma_{out}} \Omega)^\vee} \overset{|\Omega|^2}{=} \mathrm{tr}\left(U_{\Omega^*} \circ U_\Omega\right) = \int |A_\Omega(\varphi,\psi)|^2 d\mu^?(\varphi) \otimes d\mu^?(\psi).$$

$$\| $$

$$\det \times \int d\mu_{GFF}^{|\Omega|^2}$$

$\Rightarrow$ simple minded choice: $A_\Omega(\varphi,\psi) \equiv 1 \times \det$

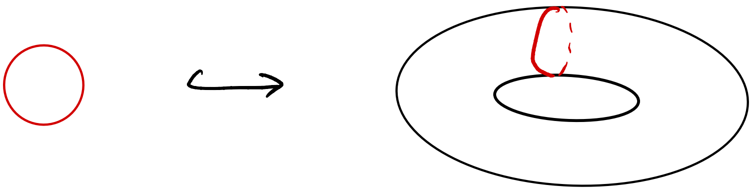
↳ a const.

if we let $d\mu^?(\varphi,\psi) \overset{def}{=} \underbrace{\tau_{\Sigma_{in} \cup \Sigma_{out}}\left(\mu_{GFF}^{|\Omega|^2}\right)}_{\text{measure image under "trace" (restriction) } \phi \longmapsto (\phi|_{\Sigma_{in}}, \phi|_{\Sigma_{out}}).}$.

! this will NOT be the actual choice.

A property of the GFF.

For ANY isometric smooth embedding $\Sigma \hookrightarrow M$.



the image measures $\tau_\Sigma(\mu^M_{GFF})$ on $\mathcal{D}'(\Sigma)$ are always mutually absolutely continuous with each other.

What is the law of a random field under restriction onto a hypersurface $\Sigma \subseteq M$?

$\rightsquigarrow$ this will be a random dist. in $\mathcal{D}'(\Sigma)$.

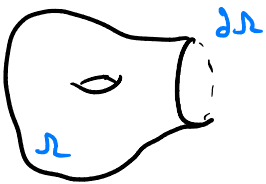
(?) Is it Gaussian? What is the covariance?

Yes

$$\mathbb{E}[\phi|_\Sigma(x)\,\phi|_\Sigma(y)] = G|_{\Sigma\times\Sigma}(x,y) \;!$$

$\Rightarrow$ i.e. the cov. op. must have int. kernel $= G|_{\Sigma\times\Sigma}$.

$\Rightarrow$ this cov. op. is a "2-sided" version of what is called "Dirichlet-to-Neumann Operator".



$$DN : C^\infty(\partial\Omega) \longrightarrow C^\infty(\partial\Omega)$$
$$\varphi \longmapsto \partial_\nu \underbrace{(PI\,\varphi)}_{\text{Harmonic extension}}\big|_{\partial\Omega}.$$

DN " 1-sided version.

ν: outward normal

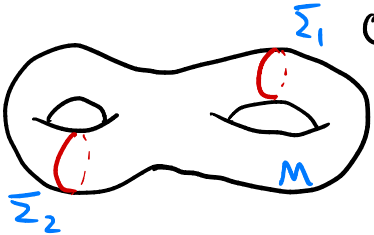
Dirichlet data $\longmapsto$ Neumann data.

$\boxed{\text{Moral.}}$ $\mathcal{A}_\Omega$ is related to induced Prob. measures under restriction to hypersurface.

$\Rightarrow$ Canonical choice in this class of mutually absolutely continuous Prob. measures: Gaussian measure with Covariance

$(\Delta_\Sigma + m^2)^{\frac{1}{2}}$.

What is behind Segal's composition axiom?



① $$d\tau_{\Sigma_1 \sqcup \Sigma_2}(\mu^{M}_{GFF})(\varphi_1, \varphi_2)$$
$$= d\tau_{\Sigma_1}(\mu^{M}_{GFF})(\varphi_1) \text{ ``}\otimes\text{'' } d\mathbb{P}_{\tau_{\Sigma_2}\phi \,|\, \tau_{\Sigma_1}\phi = \varphi_1}(\varphi_2)$$
$$= d\tau_{\Sigma_2}(\mu^{M}_{GFF})(\varphi_2) \text{ ``}\otimes\text{'' } d\mathbb{P}_{\tau_{\Sigma_1}\phi \,|\, \tau_{\Sigma_2}\phi = \varphi_2}(\varphi_1)$$

$\rightsquigarrow$ analogue of "Bayes formula".

② "Markov decomposition". for $\Sigma \subseteq M$.

$$\mu^{M}_{GFF} = PI \circ \tau_{\Sigma}(\mu^{M}_{GFF}) \otimes \mu^{M \setminus \Sigma, D}_{GFF} \quad \Leftarrow \quad W^{-1}(M) = \underbrace{W^{-1}_{\Sigma}(M)}_{\{f \in W^{-1} \,|\, \mathrm{supp}\, f \subseteq \Sigma\}} \overset{\perp}{\oplus} (\perp).$$

$$\phi = PI \circ \tau_{\Sigma}\phi + \underbrace{\phi^{D}_{M \setminus \Sigma}}$$

Gaussian random field w/ covariance $G_D(x,y) =$ Green's func. w/ Dirichlet condition on Σ.

$P \neq 0$. $\dim M = 2$. $\rightarrow$ define as R.V. in $L^1(\mu_{GFF})$.

$$\boxed{\int \boxed{e^{-\int_M \phi(x)^4 dx}} \, d\mu_{GFF}^M(\phi) < \infty}$$

Nelson's argument, '60s.

<u>Rmk</u>. does NOT work in 3D. (target measure is mutually singular WRT μ_{GFF}^M) $\rightsquigarrow$ SPDE method (regularity str. / paracontrol)

$\boxed{\text{Problem}}$ ϕ = dist. low regularity, ϕ^2, ϕ^3, etc. not defined.

$\rightsquigarrow$ need renormalization.

<u>Example</u>. How to define $\frac{1}{x} \cdot 1_{(0,+\infty)} \in \mathcal{D}'(\mathbb{R})$?

Idea: $\exists$ distribution in $\mathcal{D}'(\mathbb{R}\setminus\{0\})$ agrees w/ $\frac{1}{x} \cdot 1_{(0,+\infty)}$.

Pick $\varphi \in C_c^\infty(\mathbb{R})$, $\varepsilon > 0$, I.B.P.

$$\Rightarrow \int_\varepsilon^\infty \varphi(x) \frac{\mathrm{d}x}{x} = \underbrace{-\int_\varepsilon^\infty \varphi'(x) \log(x) \mathrm{d}x}_{\text{agree with } \frac{1}{x}\cdot 1_{(0,+\infty)} \text{ for } \varphi \in C_c^\infty(\mathbb{R}\setminus\{0\})} - \varphi(\varepsilon) \log(\varepsilon).$$

$\| $

$\frac{1}{x} \cdot 1_{(0,+\infty)} - \infty \cdot \delta_0$ = renormalized version.

In our case, ① first regularize $\phi_\varepsilon := K_\varepsilon \phi$, K_ε = Smoothing Op.

② replace $\phi_\varepsilon(x)^4 \rightsquigarrow :\phi_\varepsilon(x)^4: = \phi_\varepsilon(x)^4 - 6\mathbb{E}[\phi_\varepsilon(x)^2]\phi_\varepsilon(x)^2 + 3\mathbb{E}[\phi_\varepsilon(x)^2]^2$

$\rightsquigarrow$ Corresponds to $\overline{\text{proj. of R.V. } \phi_\varepsilon(x)^4}$ onto $\mathrm{Sym}^4(H^{-1}(M))$ in $L^2(\mu_{GFF})$.

③ discover that $\int_M :\phi_\varepsilon(x)^4: dx \xrightarrow{\varepsilon \to 0}$ well-defined R.V. $\in L^2(\mu_{GFF})$

$= S_M(\phi)$

☆ $\varepsilon, \varepsilon' > 0$,

$$G_{\varepsilon,\varepsilon'}(x,y) = \mathrm{Kernel}\left(K_{\varepsilon}^{*}(\Delta + m^2)^{-1} K_{\varepsilon}\right) = \mathbb{E}\left[\phi_\varepsilon(x)\,\phi_{\varepsilon'}(y)\right]$$

$$|G_{\varepsilon,\varepsilon'}(x,y)| \leq \text{integrable func. uniformly in } \varepsilon, \varepsilon'.$$

$$|G_{\varepsilon,\varepsilon}(x,y) - G_{\varepsilon,\varepsilon'}(x,y)| \leq \text{integrable func.} \cdot o(|\varepsilon - \varepsilon'|)$$

<u>Example.</u> $K_\varepsilon = e^{-\varepsilon(\Delta + m^2)}$. Remember $\dim M = 2$

$$\Rightarrow \Big\{ \underbrace{\int_M :\phi_\varepsilon(x)^4: \, dx}_{= S_{M,\varepsilon}} \Big\} \text{ Cauchy in } L^2(\mu_{\mathrm{GFF}}).$$

④ Hypercontractivity.

$X =$ deg $\leq n$ poly. of Gaussian R.V.s

$$\Rightarrow \quad \mathbb{E}[|X|^p]^{\frac{1}{p}} \leqslant (p-1)^{\frac{n}{2}} \mathbb{E}[X^2]^{\frac{1}{2}}$$

$\Rightarrow$ Cauchy in L^p, $\forall\, 1 \leq p < \infty$

⑤
$$\begin{aligned}
\mathbb{P}\big(\mathrm{e}^{-S_{M,\chi}} \geqslant \mathrm{e}^{b_2|\log(2\varepsilon)|^n+1}\big) &= \mathbb{P}\left(S_{M,\chi} \leqslant -b_2|\log(2\varepsilon)|^n - 1\right) \\
&\leqslant \mathbb{P}\left(|S_{M,\chi} - S_{M,\chi,\varepsilon}| \geqslant 1\right) \\
&\leqslant \|S_{M,\chi} - S_{M,\chi,\varepsilon}\|^p_{L^p(\mu_{\mathrm{GFF}})} && \leftarrow \text{Chebychev} \\
&\leqslant (p-1)^{\frac{np}{2}} C_1^p \varepsilon^{\frac{p}{2}} \|\chi\|^p_{L^4} && \leftarrow \text{③ \& ④, } \forall p! \\
&\lesssim \|\chi\|^p_{L^4} p^{\frac{n}{2}p} (C_1 \varepsilon^{\frac{1}{2}})^p, \\
&\lesssim \exp\left(-C_2(\varepsilon^{\frac{1}{2}}\|\chi\|_{L^4})^{-\frac{1}{n}}\right). && \leftarrow \text{minimize over } 1 \leq p < \infty
\end{aligned}$$

(Goal: $e^{-S_M} \in L^1$)

⑥
$$\mathbb{E}\left[\mathrm{e}^{-S_{M,\chi}}\right] = \underbrace{\int_0^\infty \mathbb{P}\left(\mathrm{e}^{-S_{M,\chi}} \geqslant t\right) \mathrm{d}t}_{< \infty} \Rightarrow 🏆$$

More technical aspects of the Segal problem for $P \neq 0$.

① locality of the interaction S_M

Roughly speaking, $M = A \cup B$, then

$$\int_M :P(\phi(x)): dx = \int_A :P(\phi(x)): dx + \int_B :P(\phi(x)): dx.$$

this necessitates: ① K_ε needs to be local.

② $:X:$ needs to be local.

② "ultraviolet stability": Operator K_ε can be chosen from a class of smoothing operators and they define the same R.V. $S_M(\phi)$ in the limit.

Thank you!