Chern-Simons paradigm for Gravity *

Algebraic, analytic and geometric structures emerging from QFT

Chengdu

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* Based on: M. Hassaïe and J.Z., *Chern-Simons (Super) Gravities*, World Scientific (2016)

Gravitation is a manifestation of the curvature of the spacetime geometry

A. Einstein (1915)

The problem with quantum gravity is classical: we don't know what it is we want to "quantize"

$$Z = \int dX \exp\{\tfrac{i}{\hbar} I[X]\}$$

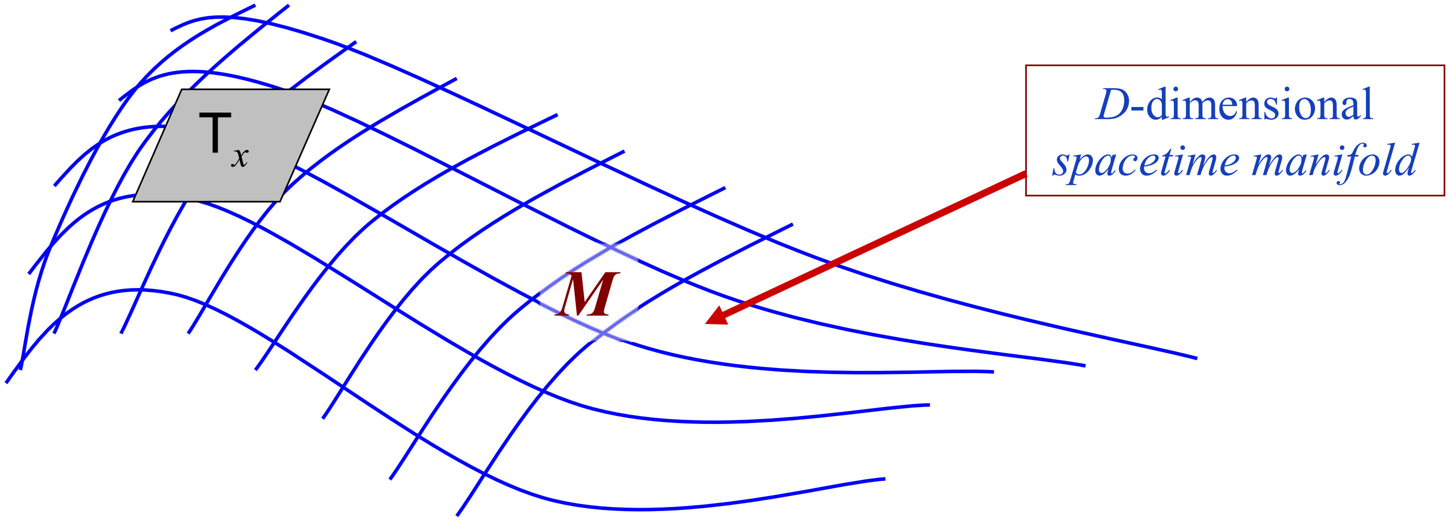
What is X? *What is* I?

1. Spacetime Geometry

General Relativity: Dynamics of the spacetime geometry

- Spacetime *M* is a differentiable manifold (up to isolated singularities: sets of zero-measure).

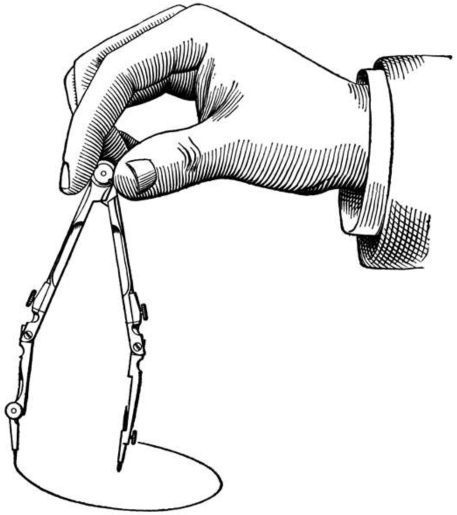
- *M* admits a tangent space T_x at each point.

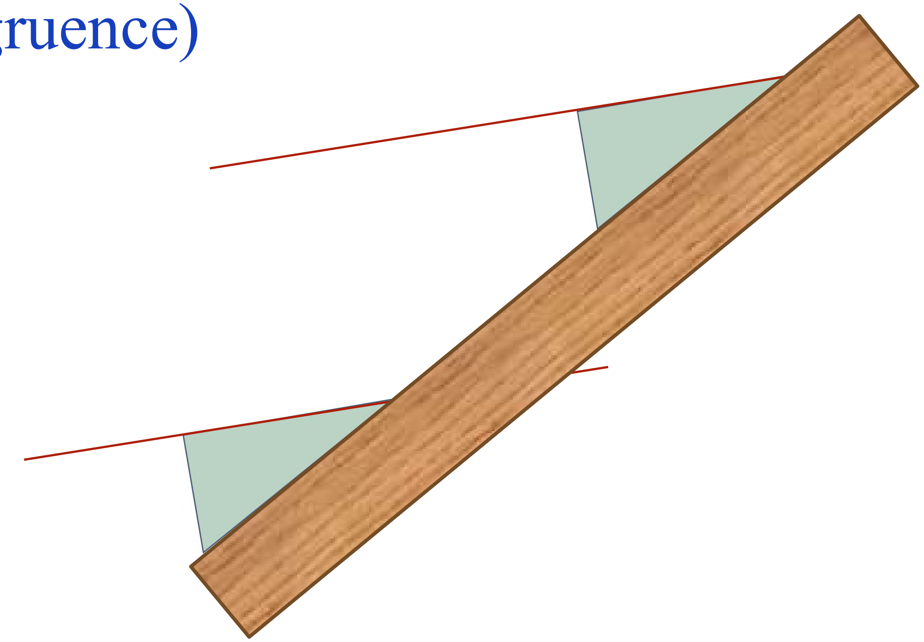


- An open set around any point x is diffeomorphic to an open set in the tangent.

Geometry has two ingredients:

- Metric structure (length/area/volume, scale)
- Affine structure (parallel transport, congruence)





2. The two ingredients of Spacetime Geometry

First ingredient of Geometry: Metric Structure

- Each tangent space T_x is isomorphic to Minkowski space

$$dz^a = e^a_\mu(x) dx^\mu \equiv e^a$$

(e^a: local orthonormal frame, "*vielbein*", "*soldering form*")

- Since Minkowski space is endowed with the Lorentzian metric η_{ab}, the diffeomorphism induces a metric structure on *M*:

$$ds^2 = \eta_{ab}\, dz^a dz^b = \eta_{ab}\, e^a_\mu dx^\mu\, e^b_\nu dx^\nu = g_{\mu\nu}\, dx^\mu dx^\nu$$

$$g_{\mu\nu}(x) \equiv \eta_{ab}\, e^a_\mu(x)\, e^b_\nu(x) \longrightarrow \text{Metric on } M$$

Einstein's interpretation:

Existence of tangent space T_x , isomorphic to Minkowski space at every point of spacetime

$\Updownarrow$

Locally, nature can be described as if the fundamental laws of physics are those of a Lorentz-invariant flat spacetime (~ Special Relativity)

$\Updownarrow$

The laws of physics can be locally cast in a Lorentz-invariant form:
This is what a freely falling observer experiences

The laws of physics can be locally cast in a Lorentz-invariant form:

This is what a freely falling observer experiences

$\Updownarrow$

Going into a freely falling reference frame eliminates gravity.

Conversely, acceleration is locally indistinguishable from gravity.

Principle of Equivalence

"*The happiest thought of my life*"

A.E.

The Equivalence Principle has two important consequences:

1. <u>Covariance:</u> Tensors in M can be related to tensors in T_x

$$T^{a}{}_{b} = e^{a}_{\mu}(x)\, E^{\nu}_{b}(x)\, T^{\mu}{}_{\nu}$$

Tensor in T_x

Tensor in M

Inverse vielbein

Forces, electromagnetic fields, energy, momentum,… and their relations, can be determined by measurements either on M or in T_x . The laws of nature must be invariant under changes of frames: General coordinate invariance

This feature can be used to express every physical quantity as measured in T_x

2. Local Lorentz invariance: Since the tangent space is invariant under the Lorentz group, it must be possible to formulate the Laws of nature in a Lorentz-invariant language. Physical observables must transform locally as representations of the Lorentz group.

Consider a vector field $u^a(x)$ at a point $x \in M$. Under a Lorentz transformation,

$$u'^a(x) = \Lambda^a{}_b(x)\, u^b(x)$$

Element of the Lorentz group acting at the point x

(Fiber bundle structure)

Second ingredient of geometry: Affine Structure (parallelism)

The affine structure is the rule to compare objects at different points in *M*.

Consider a field $u^a(x)$ that transforms as a vector,

$$u'^a(x) = \Lambda^a{}_b(x)\, u^b(x)$$

In order to compare the value of u^a at x and at $x + dx$ one must transport the vector between those two points and this requires a rule (recipe).

Parallelism:

Recipe to compare objects at different points in *M*.

Let $u^a_{||}(x + dx)$ the parallel transported vector from $x + dx$ to x, where

$$u^a_{||}(x + dx \to x) = u^a(x + dx) + \omega^a{}_{b\mu}(x)\, dx^\mu\, u^b\,(x)$$

Recipe for parallel transport (connection)

This notion of parallel transport allows to define a

- *Covariant* derivative:

$$Du^a(x) = u^a_{||}(x + dx \to x) - u^a(x)$$
$$= u^a(x + dx) - u^a(x) + \omega^a{}_{b\mu}(x)\, dx^\mu\, u^b\,(x)$$
$$= dx^\mu[\partial_\mu u^a + \omega^a{}_{b\mu}(x) u^b]$$

$$\boxed{Du^a = du^a + \omega^a{}_b u^b}$$

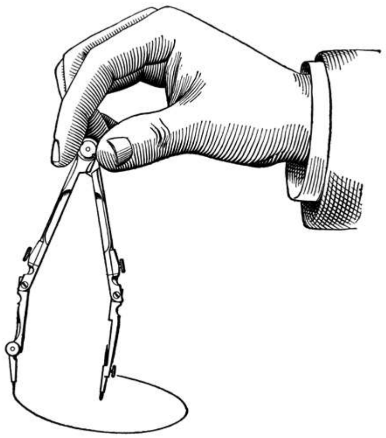
- Under local Lorentz transformations $Du^a = du^a + \omega^a{}_b u^b$ must transform like u^a:

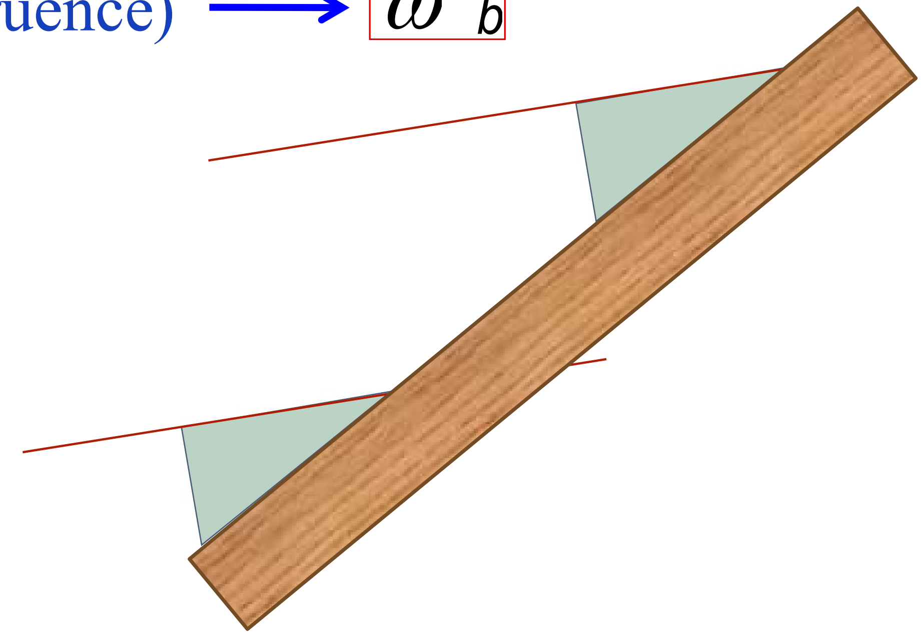
$$u^a(x) \to u'^a(x) = \Lambda^a{}_b(x) u^b(x)\,, \quad \Lambda^a{}_b \in SO(1, D-1)$$

$$Du^a \to (Du^a)' = \Lambda^a{}_b(x) Du^b\,,$$

This requires $\omega \to \omega' = \Lambda[\omega + d\,]\Lambda^{-1}$ Lorentz connection

- Metric structure (length/area/volume, scale) → e^a
- Affine structure (parallel transport, congruence) → $\omega^a{}_b$





3. Spacetime Recipe

Hilbert's idea: The equations that govern the spacetime geometry must be obtained from an action principle. The equations should be the stationarity condition for the action functional, $\delta I = 0$

Action:

$$I[e,\omega] = \int_M L(e,\omega)$$

$$\delta I = \int \left[\frac{\delta L}{\delta e}\delta e + \frac{\delta L}{\delta \omega}\delta\omega\right] = 0 \quad \Longrightarrow \quad \begin{cases} \dfrac{\delta L}{\delta e} = 0 \\ \dfrac{\delta L}{\delta \omega} = 0 \end{cases}$$

Einstein's equations

What is $L(e,\omega)$?

The covariant derivative defines the curvature and torsion 2-forms:

- *Lorentz curvature:* $DDu^a = R^a{}_b\, u^b$

$$R^a{}_b = d\omega^a{}_b + \omega^a{}_c\, \omega^c{}_b$$

- *Torsion:* $T^a = De^a = de^a + \omega^a{}_b\, e^b$

$$D\eta^{ab} = d\eta^{ab} + \omega^a{}_c\eta^{cb} + \omega^b{}_c\eta^{ac} = 0 = \omega^{ab} + \omega^{ba}, \rightarrow \boxed{\omega^{ab} = -\omega^{ba}}$$

Everything that has been said about Lorentz symmetry is true for $SO(n, D-n)$

Taking additional covariant derivative does not produce new geometric objects

$$\left.\begin{array}{c} D(R^a{}_b\, u^b) = R^a{}_b\, Du^b \,, \Longrightarrow DR^a{}_b \equiv 0 \\ DT^a = R^a{}_b\, e^b \end{array}\right\} \text{Bianchi/Jacobi identities}$$

From e^a and $\omega^a{}_b$ and their derivatives, a very limited number of objects can be produced: e^a, $\omega^a{}_b$, $R^a{}_b$, T^a.

- The most general gravity action must involve these elements only.

4. Mixing and cooking procedure

Pure gravity action in *D*-dimensions $I[e,\omega] = \int_M L(e,\omega)$

L must be a *D*-form constructed only out of exterior products and derivatives of

1-forms: $\begin{cases} e^a = e^a_\mu \, dx^\mu & \text{Vector} \\ \omega^a{}_b = \omega^a{}_{b\,\mu} \, dx^\mu & \text{Connection} \end{cases}$

2-forms: $\begin{cases} R^a{}_b = \frac{1}{2} R^a{}_{b\,\mu\,\nu} dx^\mu \wedge dx^\nu & \text{Tensor} \\ T^a = \frac{1}{2} T^a_{\mu\nu} \, dx^\mu \wedge dx^\nu & \text{Vector} \end{cases}$

0-forms: η_{ab} , $\epsilon_{a_1 a_2 \cdots a_D}$ Invariant tensors

One way to make sure that the resulting equations express the same relations under Lorentz transformations is to demand that $L(e,\omega)$ be Lorentz invariant

With these restrictions there is a very limited number of possibilities for L in each dimension.

$D=2$:

$$L_2 = \alpha\, \epsilon_{ab}\, e^a \wedge e^b + \beta \epsilon_{ab} R^{ab} \quad \Rightarrow \quad I_2(e,\omega) = \alpha\, V(M) + \beta \chi(M)$$

Volume form — Euler density

$$\sim \left(\alpha' \sqrt{|g|} + \beta' \sqrt{|g|} R\right) d^2x$$

Euler characteristic: does not vary under continuous deformations of the geometry

$\underline{D = 3}$:

$$L = \alpha_0\, \epsilon_{abc}\, e^a\, e^b\, e^c + \alpha_1 \epsilon_{abc}\, R^{ab}\, e^c + \beta\, e^a\, T_a$$

If torsion is discarded (as Einstein did), $L \sim \sqrt{|g|}[\Lambda + \kappa\, R] d^3x$

Einstein-Hilbert action

$\underline{D = 4}$:

$$L = \alpha_0\, \epsilon_{abcd}\, e^a\, e^b\, e^c\, e^d + \alpha_1 \epsilon_{abcd}\, e^a\, e^b\, R^{cd} + \alpha_2 \epsilon_{abcd}\, R^{ab}\, R^{cd} +$$

$$+\beta R^{ab} R_{ab} + \gamma\, T^a\, T_a + \lambda\, R_{ab}\, e^a\, e^b$$

Pontryagin form

Euler form

$T^a T_a - R_{ab} e^a e^b =$ Nieh-Yan topological invariant

If torsion is discarded (Einstein), $L \sim \sqrt{|g|}[\Lambda + \kappa\, R] d^4 x$ Einstein-Hilbert

$D = 2n$: $L_{2n} = \sum_{k=0}^{n-1} \alpha_k \, \epsilon_{a_1 \cdots a_{2n}} [R]^k \, [e]^{2(n-k)} +$

$+(Euler) + (Pontryagin) + (Nieh - Yan). +(Torsional\ terms)$

$E_{2n} = \epsilon_{a_1 \cdots a_{2n}} R^{a_1 a_2} \cdots R^{a_{2n-1} a_{2n}}$ Euler form

$P_{4k} = R^{a_1}{}_{a_2} R^{a_2}{}_{a_3} \cdots R^{a_{2k}}{}_{a_1}$ Pontryagin forms

$N_{4k} = [R^{2k-2}]^{ab} T_a T_b - [R^{2k-1}]^{ab} e_a e_b$ Nieh-Yan forms

5. More exotic recipes

In odd dimensions there is a surprise. For each topological invariant density in $D=2n$, there exists in addition to the Lorentz invariant terms, quasi-invariant ones that also give rise to Lorentz-invariant equations.

This happens because those densities can be locally written as the exterior derivative of a Chern-Simons term.

Consider the Euler and Pontryagin densities in D=4:

$$E_4 = \epsilon_{abcd} R^{ab} R^{cd} = dC_3^E \qquad P_4 = R^a{}_b \; R^b{}_a = dC_3^P$$

where C_3^E & C_3^P are 3-forms that define geometric actions for 3-dimensional geometries. (And similarly in dimensions 5, 7, 9,...)

What are these Lorentz quasi-invariant actions?

D=3: Combine the $SO(1,2)$ connection ω^{ab} and 3-dimensional vielbein e^a into a connection for a larger group, w^{AB}:

$$\left\{\begin{bmatrix} 0 & \omega^{01} & \omega^{02} \\ -\omega^{01} & 0 & \omega^{12} \\ -\omega^{02} & -\omega^{12} & 0 \end{bmatrix}, \begin{bmatrix} e^0 \\ e^1 \\ e^2 \end{bmatrix}\right\} \Longrightarrow w^{AB} = \begin{bmatrix} 0 & \omega^{01} & \omega^{02} & e^0 \\ -\omega^{01} & 0 & \omega^{12} & e^1 \\ -\omega^{02} & -\omega^{12} & 0 & e^2 \\ -e^0 & -e^1 & -e^2 & 0 \end{bmatrix}$$

$$SO(1,2) \Longrightarrow SO(1,3) \text{ [or } SO(2,2)\text{]}$$

The $SO(1,3)$ [or $SO(2,2)$] connection w^{AB} defines an $SO(1,3)$ [or $SO(2,2)$] curvature out of $SO(1,2)$ geometric ingredients:

$$F^{AB} = dw^{AB} + w^{A}{}_{C}\, w^{C}{}_{B} = \begin{bmatrix} R^{ab} - \sigma e^{a} e^{b} & T^{a} \\ -T^{b} & 0 \end{bmatrix}, \sigma = \pm 1 \quad \begin{cases} +: SO(1,3) \\ -: SO(2,2) \end{cases}$$

The signs $\pm$ correspond to the 2 possible choices:

$$\eta_{AB} = \begin{bmatrix} \eta_{ab} & 0 \\ 0 & \sigma \end{bmatrix} = diag(-1, 1, 1, \sigma)$$

Euler and Pontryagin invariants can be defined using F^{AB}

Euler form: $E_4 = \epsilon_{ABCD} F^{AB} F^{CD} = 4\epsilon_{abc}\left(R^{ab} \pm e^a e^b\right) T^c$

$$= 4\, d\left[\epsilon_{abc}\left(R^{ab} e^c \pm \tfrac{1}{3} e^a e^b e^c\right)\right]$$

Euler Chern-Simons 3-form C_3^E

This CS form can be included as a piece of the Lagrangian in 3 dimensions

- C_3^E is $SO(1,2)$ invariant (Lorentz scalar)
- It changes by a locally exact form (boundary term) under $SO(1,3)$ [$SO(2,2)$]

Pontryagin form: $P_4 = F^{AB}F_{BA} = \left(R^{ab} \pm e^a e^b\right)(R_{ba} \pm e_b e_a) \mp 2T^a T_a$

$$= R^{ab}R_{ba} \mp 2(R^{ab}\, e_b e_a + T^a T_a)$$

$$= d\left[d\omega^{ab}\omega_{ba} + \tfrac{2}{3}\omega^a{}_b\omega^b{}_c\,\omega^c{}_a\right] \mp 2d[e^a T_a]$$

Pontryagin C-S 3-form for $SO(1,2)$ C_3^P

Nieh-Yan C-S 3-form C_3^{NY}

These CS forms can also be included in the in the 3-D Lagrangian.

- C_3^{NY} is local Lorentz invariant
- C_3^P is quasi invariant (it changes by a boundary term)

What's going on?

Let U be a characteristic class: $\int U = \tau(M) \in \mathbb{Z}$ is a topological invariant of M.

Since U is closed, locally can be written as $U = dC$. ("Boundary term")

U is also invariant under local $SO(2,2n\text{-}2)$ transformations, $0 = \delta U = \delta(dC) = d(\delta C)$,

$\Longrightarrow \delta C$ is also locally exact, $\delta C = d(something)$.

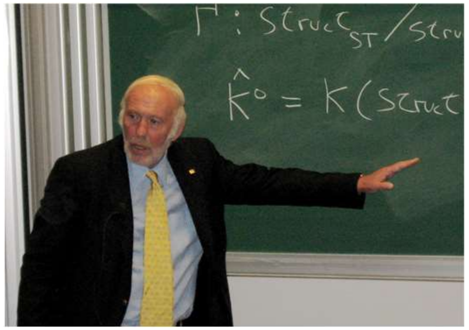
Hence, C changes under $SO(2,2n\text{-}2)$ by a boundary term: it is quasi-invariant at most.

S.-S. Chern 1911-2004

Characteristic Forms and Geometric Invariants

Shiing-Shen Chern; James Simons

The Annals of Mathematics, 2nd Ser., Vol. 99, No. 1 (Jan., 1974), 48-69.



J. H. Simons 1938

(Received June 13, 1972)

1. Introduction

This work, originally announced in [4], grew out of an attempt to derive a purely combinatorial formula for the first Pontrjagin number of a 4-manifold. The hope was that by integrating the characteristic curvature form (with respect to some Riemannian metric) simplex by simplex, and replacing the integral over each interior by another on the boundary, one could evaluate these boundary integrals, add up over the triangulation, and have the geometry wash out, leaving the sought after combinatorial formula. This process got stuck by the emergence of a boundary term which did not yield to a simple combinatorial analysis. The boundary term seemed interesting in its own right and it and its generalization are the subject of this paper.

General recipe: For every characteristic class in a given dimension $2n$ there is an associated CS Lagrangian in $D = 2n - 1$.

The CS form contributing to the Lagrangian in $D = 2n - 1$ are quasi-invariant under local Lorentz transformations, which is sufficient to make the resulting theory Lorentz invariant.

The most general Lagrangian for three-dimensional gravity

$$L_3 = \alpha_0 \, \epsilon_{abc} \, e^a \, e^b \, e^c + \alpha_1 \epsilon_{abc} \, R^{ab} \, e^c + \\ + \beta \, e^a \, T_a + \gamma \left[d\omega^{ab} \omega_{ba} + \tfrac{2}{3} \omega^a{}_b \omega^b{}_c \, \omega^c{}_a \right]$$

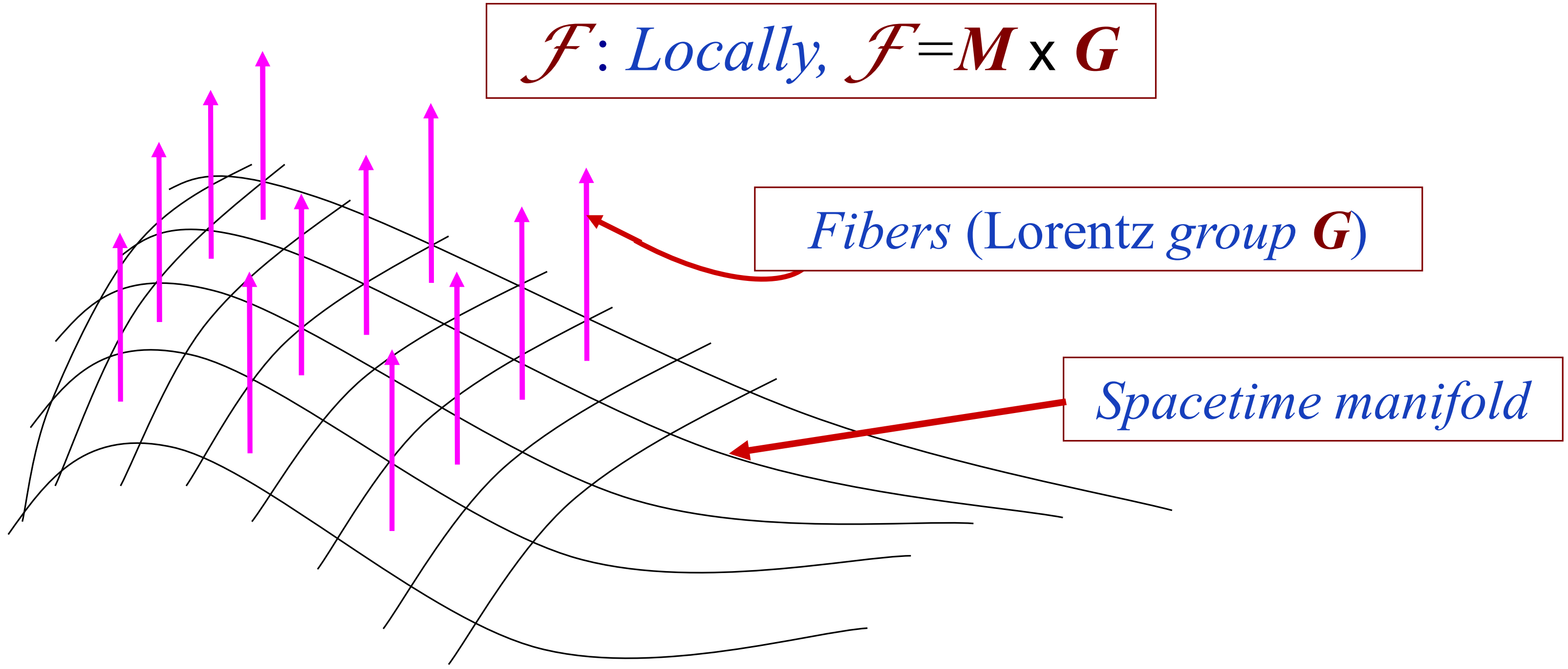
- Quasi-invariant under local $SO(1,2)$ transformations
- For $\alpha_1 = 3\alpha_0$ and $\beta = -2\gamma$ the action is quasi-invariant under local $SO(2,2)$

⟹ Symmetry enhancement at the CS point

⟹

6. Summary and scope

- Spacetime can be conceived as a fiber bundle



- The most general pure gravity action is a functional of the metric and affine structures e, ω

$$I[e,\omega] = \int_M L(e,\omega)$$

where L is a D-form invariant or quasi invariant under local Lorentz transformations. In general, for $D = 2n-1$,

$$C^E_{2n-1} = \alpha_0\, \epsilon_{a_1 \cdots a_{2n-1}} [e^{2n-1} + \alpha_1 e^{2n-3} R + \alpha_2 e^{2n-5} R^2 + \cdots + \alpha_{n-1} e R^{n-1}]$$

fixed, dimensionless combinatorial coefficients

- In odd dimensions I can, by a judicious choice of coefficients, be made quasi invariant under local $SO(1,D)$ [or $SO(2,D-1)$] transformations.
  - Symmetry enhancement: $SO(1,D-1) \to SO(1,D)$
  - Fewer arbitrary parameters, protected by the gauge symmetry
  - Dimensionless Lagrangian parameters, scale-invariant action principle
  - Supersymmetric extensions
  - Built-in conformal symmetry
  - Dualities?

Thanks!
Sylvie, Bin & Li for the warm hospitality,

... and to all of you for your patience!