Renormalisation in Fermionic SPDEs

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Algebraic, Analytic and Geometric Structures Emerging from QFT

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Problem of Constructive Quantisation

- (Bosonic) Constructive QFT $\implies$ find measure on $\mathscr{D}'(\mathbb{R}^d; E)$

$$\mathrm{d}\mu(\varphi) = Z^{-1} e^{-S(\varphi)} \mathcal{D}\varphi$$

- Problems: $\nexists \mathcal{D}\varphi$, S is nonlinear
- Solution(?): Get to $\mathrm{d}\mu$ dynamically $\implies$ Stochastic Quantisation

Fokker-Planck Equation

- Let $V\colon \mathbb{R}^n \to \mathbb{R}$ potential, probability measure: $\mathrm{d}\mu(x) = Z^{-1} e^{-V(x)} \mathrm{d}^n x$
- Fokker-Planck Equation for μ: Let $\mathrm{d}\mu_t(x) = p_t(x)\mathrm{d}^n x$, $F = -\nabla V$

$$\partial_t p_t = \Delta p_t - \nabla \cdot (F p_t)$$

- If $\mu_0 = \mu$, then $\partial_t p_t = 0$; if $\mu_0 \neq \mu$ (under suitable assumptions)

$$\mu_t \xrightarrow{t \to \infty} \mu$$

- FP-Equation "makes sense" only in finite dimension. Dual formulation makes sense "always".

Stochastic Differential Equation

- Let ν be the Gaussian measure on $\mathscr{D}'(\mathbb{R})$, s.t.

$$\int_{\mathscr{D}'(\mathbb{R})} \xi(t)\xi(s)\mathrm{d}\nu(\xi) = \delta(t-s)$$

- Solve Stochastic Differential Equation (SDE)

$$\partial_t X_t = F(X_t) + \sqrt{2}\xi_t$$

- If X_t solves SDE, then the measure

$$\mu_t(A) := (X_t)_* \nu(A) := \nu\big(X_t^{-1}(A)\big)$$

solves FP equation.

SDE Crash Course

- What is a solution of SDE?
- Fix $\xi \in \mathscr{D}'(\mathbb{R})$, solve
$$\partial_t X_t = F(X_t) + \sqrt{2}\xi_t$$
for ν-almost every ξ independently.
- Solution to SDE is the measurable map
$$\xi \longmapsto X(\xi)$$
- This is a Pathwise/Pointwise Solution

SDE Crash Course

- What is a solution of an SDE?
- Alternative point of view: $\xi \in \mathscr{D}'\Big(\mathbb{R}; \mathcal{M}\big(\mathscr{D}'(\mathbb{R})\big)\Big)$

$$\xi \colon f \longmapsto \big(\varphi \mapsto \varphi(f)\big) \ .$$

- SDE is a regular ODE with values in the commutative algebra of measurable functions.

Langevin Stochastic Quantisation

- Classical Field Theory:

$$S(\varphi) = \int_{\mathbb{R}^2} \frac{1}{2}|\nabla\varphi|^2 + \frac{m^2}{2}\varphi^2 + \frac{1}{4}\varphi^4$$

- Stochastic PDE:

$$\partial_t\varphi = -\frac{\delta S}{\delta\varphi} + \xi = (\Delta - m^2)\varphi - \varphi^3 + \xi$$

- Space-Time White Noise: $\xi \in \mathscr{D}'\big(\mathbb{R}^3; \mathcal{M}(\mathscr{D}'(\mathbb{R}^3))\big)$

$$\int \xi(t,x)\xi(s,y)\mathrm{d}\nu(\xi) = \delta(t-s)\delta(x-y)$$

- Solutions of SPDE constructed pointwise for a.e. ξ

Renormalisation

- Singular nonlinear PDEs solved by running Picard iteration (like ODE)

$$T[\varphi] = \left(\partial_t - \Delta - m^2\right)^{-1}\left(-\varphi^3 + \xi\right)$$

and we hope it has a fixed-point!

- Singular nonlinear PDEs solved by running Picard iteration, using fixed-point argument.
- Picard iteration: Start $\RSI := (\partial_t - \Delta - m^2)^{-1}\xi \in \mathcal{C}^{0-}(\mathbb{R}^3)$
- At 2nd step singular products appear: $\big(\RSI(x)\big)^2$ and $\big(\RSI(x)\big)^3$
- Define using L^2-limit in the probability space of

$$\RSV(x) := \big(\RSI(x)\big)^2 - \mathbb{E}\left[\big(\RSI(x)\big)^2\right], \quad \RSW(x) := \big(\RSI(x)\big)^3 - 3\mathbb{E}\left[\big(\RSI(x)\big)^2\right]\RSI(x)$$

  when removing some regularisation.
- Remainder Equation: $\varphi = u + \RSI$

$$-(\partial_t - \Delta + m^2)u = u^3 + 3u^2\RSI + 3u\RSV + \RSW$$

  well-defined!

Fermions

- What are Fermions? Matter particles! Why? Fermions anticommute!

$$\psi(x)\psi(y) = -\psi(y)\psi(x)$$

- $\implies$ Pauli exclusion principle, $V \propto N$
- $$\psi(x) \in \mathscr{D}'(\mathbb{R}^d;\mathbb{C}^n) \subset \bigwedge \mathscr{D}'(\mathbb{R}^d;\mathbb{C}^n) =: \mathcal{G}$$

  Grassmann/Exterior Algebra of vector space $\mathscr{D}'(\mathbb{R}^d;\mathbb{C}^n)$.

Fermions

- $$\psi(x)\psi(y) = -\psi(y)\psi(x)$$
  $$\psi(x) \in \mathscr{D}'(\mathbb{R}^d; \mathbb{C}^n) \subset \bigwedge \mathscr{D}'(\mathbb{R}^d; \mathbb{C}^n) =: \mathcal{G}$$
- Bosonic observables described by commutative algebra of measurable functions/random variables and their expectations w.r.t. measure μ
- Fermionic observables described by anticommutative algebra. What is the measure?

  A linear functional $\omega \colon \mathcal{G} \longrightarrow \mathbb{C}$.
- Topology?

Canonical Anticommutation Relation (CAR) Algebra

- Let $\mathfrak{H} := L^2(\mathbb{R}^3; \mathbb{C}^4)$. CAR algebra $\mathcal{A}(\mathfrak{H})$ of $\mathfrak{H}$: C^*-algebra given by (anti)linear generators $a(f), a(g)^\dagger$, $f, g \in \mathfrak{H}$, satisfying

$$[a(f), a(g)^\dagger]_+ := a(f)a(g)^\dagger + a(g)^\dagger a(f) = \langle f, g \rangle_{L^2}$$

- $a(g)^\dagger$ Creation Operator/Skorohod Integral
  $a(f)$ Annihilation Operator/Malliavin Derivative
- $\mathcal{A}(\mathfrak{H})$ subalgebra of bounded operators on

$$\mathcal{F}_a(\mathfrak{H}) := \bigoplus_{n=0}^{\infty} \mathfrak{H}^{\wedge n}$$

Vacuum State – CAR Algebra

- Let $\mathfrak{H} := L^2(\mathbb{R}^3; \mathbb{C}^4)$. CAR algebra $\mathcal{A}(\mathfrak{H})$ of $\mathfrak{H}$: C^*-algebra given by (anti)linear generators $a(f), a(g)^\dagger$, $f, g \in \mathfrak{H}$, satisfying

$$[a(f), a(g)^\dagger]_+ := a(f)a(g)^\dagger + a(g)^\dagger a(f) = \langle f, g\rangle_{L^2}$$

- For $\Omega = 1 \in \mathbb{C} = \mathfrak{H}^{\wedge 0} \subset \mathcal{F}_a(\mathfrak{H})$, define state on $\mathcal{A}(\mathfrak{H})$

$$\omega : \begin{array}{rcl} \mathcal{A}(\mathfrak{H}) & \longrightarrow & \mathbb{C} \\ A & \longmapsto & \langle \Omega, A\Omega\rangle_{\mathcal{F}_a(\mathfrak{H})} \end{array}$$

Fermionic White Noise

- Let $U := \begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix}$. Fermionic space-time white noise on $\mathbb{R}^{1+2}$ is given by the operator-valued distribution

$$L^2(\mathbb{R}^3; \mathbb{C}^4) \ni f \longmapsto \Psi(f) := a(f)^\dagger + a(Uf)$$

$$[\Psi(f), \Psi(g)]_+ = 0$$

- In components $\Psi = (\psi, \overline{\psi})$, this satisfies $i, j \in \{1, 2\}$

$$\omega(\psi^i(t,x)\overline{\psi}^j(s,y)) = -\omega(\overline{\psi}^j(s,y)\psi^i(t,x)) = \delta_{ij}\delta(t-s)\delta(x-y)$$

Fermionic White Noise

- Ψ is Gaussian w.r.t. ω, satisfies for $f_i, g_i \in L^2(\mathbb{R}^3; \mathbb{C}^2)$

$$\begin{aligned} &\omega\Big(\psi(f_1)\bar{\psi}(g_1)\cdots\psi(f_n)\bar{\psi}(g_n)\Big) \\ &= \sum_{\sigma\in\mathfrak{S}_n} \operatorname{sgn}(\sigma)\omega\Big(\psi(f_1)\bar{\psi}(g_{\sigma(1)})\Big)\cdots\omega\Big(\psi(f_n)\bar{\psi}(g_{\sigma(n)})\Big) \end{aligned}$$

- By changing U, you can define any free Fermionic theory.

Yukawa$_2$-Model

- Model describing a pair of particle and antiparticle Fermions $\upsilon, \bar{\upsilon}$ interacting via a Boson φ

$$\int_{\mathbb{R}^2} \left(\frac{1}{2} |\nabla \varphi|^2 + \frac{m^2}{2} \varphi^2 + \left\langle \bar{\upsilon}, (-\not{\nabla} + M) \upsilon \right\rangle_{\mathbb{R}^2} + g \varphi \left\langle \bar{\upsilon}, \upsilon \right\rangle_{\mathbb{R}^2} \right) \mathrm{d}x$$

- Here $\not{\nabla}$ is the Dirac operator

$$\not{\nabla} := \begin{pmatrix} 0 & -\partial_1 + i\partial_2 \\ -\partial_1 - i\partial_2 & 0 \end{pmatrix}$$

Langevin Yukawa Equation

- For fixed Bosonic noise $\xi \in \mathscr{D}'(\mathbb{R}^3)$ solve the set of equations as elements of $\mathscr{D}'(\mathbb{R}^3; \mathcal{A}(\mathfrak{H}))$.

$$
\begin{aligned}
\partial_t \varphi &= (\Delta - m^2)\varphi - g \langle \bar{\upsilon}, \upsilon \rangle_{\mathbb{R}^2} + \xi \\
\partial_t \upsilon &= (\nabla\!\!\!/ - M)u - g\varphi\upsilon + \psi \\
\partial_t \bar{\upsilon} &= (-\overline{\nabla\!\!\!/} - M)\bar{\upsilon} - g\varphi\bar{\upsilon} + \bar{\psi}\ .
\end{aligned}
$$

- Correlation functions at $t \to \infty$, give correlation functions of interacting theory.

Singular Product

- For the Picard iteration in the Yukawa$_2$ model, define

$$\overset{\blacksquare}{|} := \left(\partial_t - \not{\nabla} + M\right)^{-1}\psi\ , \qquad \overset{\square}{|} := \left(\partial_t + \overline{\not{\nabla}} + M\right)^{-1}\bar{\psi}$$

- In $(2+1)$D, the product $\left\langle \overset{\square}{|}(x), \overset{\blacksquare}{|}(x) \right\rangle_{\mathbb{R}^2}$ is ill-defined
- As before,

$$\overset{\square\,\blacksquare}{\vee}(x) := :\left\langle \overset{\square}{|}(x), \overset{\blacksquare}{|}(x) \right\rangle_{\mathbb{R}^2}: := \left\langle \overset{\square}{|}(x), \overset{\blacksquare}{|}(x) \right\rangle_{\mathbb{R}^2} - \omega\left(\left\langle \overset{\square}{|}(x), \overset{\blacksquare}{|}(x) \right\rangle_{\mathbb{R}^2}\right)$$

is a well-defined unbounded(!) operator-valued distributions.

The Problem

- Space of needed unbounded operators on $\mathcal{F}_a(\mathfrak{H})$ is not Banach algebra. There is no set of submultiplicative seminorms $\mathfrak{p}$, s.t.

$$\mathfrak{p}(ab) \leqslant \mathfrak{p}(a)\mathfrak{p}(b)$$

- Need submultiplicativity to solve non-linear equation using fixed-point argument!
- How to solve equation without submultiplicativity?

Points?

- Same problem in Bosonic case. Solution: Work Pointwise!
- Instead of topologising $\mathcal{M}(\Sigma; \mathscr{D}'(\mathbb{R}^d)) \sim \mathscr{D}'(\mathbb{R}^d; \mathcal{M}(\Sigma))$ work at each point $p \in \Sigma$ and solve problem in $\mathscr{D}'(\mathbb{R}^d)$
- Clear what points are when target C^*-algebra is commutative (Gel'fand Isomorphism)
- Algebraic Geometry: Points are (finite-dimensional) irreducible representations of your algebra

CAR Points?

- Does it work for Grassmann/CAR algebra?
- No!
- Infinite dimensional CAR algebra does not admit finite dimensional reps!
- If $\pi\colon \mathcal{A}(\mathfrak{H}) \to \mathcal{B}(\mathbb{C}^n)$ rep, $a(f) \in \ker(\pi)$, $f \neq 0$

$$\|f\|^2 = \pi([a(f)^\dagger, a(f)]_+) = \left[\pi(a(f))^\dagger, \pi(a(f))\right]_+ = 0$$

- Have to extend the CAR algebra!

Extended CAR Algebra

- Construction based on ideas from [DV75].
- Define a free (algebraic) $*$-algebra $\widehat{\mathfrak{A}}(\mathfrak{H})$ over Hilbert space $\mathfrak{H}$, i.e. freely generated by

$$\left\{\alpha(f), \alpha(f)^{\dagger} \,\middle|\, f \in \mathfrak{H}\right\}$$

  subject to (anti)-linearity and $*$-relations.
- Universal Property: $\forall$ $*$-algebra M $\forall \widehat{\pi} \colon \mathfrak{H} \to M$ linear $\exists! \pi \colon \widehat{\mathfrak{A}}(\mathfrak{H}) \to M$ $*$-algebra morphism extension

Extended CAR Algebra

- Define

$$\mathrm{Gr}(\mathfrak{H}) := \left\{ b \,\middle|\, b \subset \mathfrak{H} \text{ subspace}, \dim(b) < \infty \right\} .$$

- Let $P_b \colon \mathfrak{H} \to b$ projection. Define $\pi_b \colon \widehat{\mathfrak{A}}(\mathfrak{H}) \to \mathcal{A}(b)$ via

$$\pi_b(\alpha(f)^\dagger) = a(P_b f)^\dagger \; , \qquad \pi_b(\alpha(f)) = a(P_b f)$$

- $\mathcal{A}(b)$ is finite dimensional, no unbounded operators!
- Define

$$\mathfrak{A}(\mathfrak{H}) := \widehat{\mathfrak{A}}(\mathfrak{H}) \Big/ \bigcap_{b \in \mathrm{Gr}(\mathfrak{H})} \ker \pi_b$$

with seminorms

$$\|A\|_n := \sup_{\substack{b \in \mathrm{Gr}(\mathfrak{H}) \\ \dim(b) \leqslant n}} \|\pi_b(A)\|$$

Extended CAR Algebra

- The final object is a locally C^*-algebra, the Extended CAR Algebra,

$$\mathscr{A}(\mathfrak{H}) := \overline{\mathfrak{A}(\mathfrak{H})}^{(\|\cdot\|_n)_n}$$

- It contains a C^*-algebra

$$\mathfrak{A}_\infty(\mathfrak{H}) := \left\{ A \in \mathscr{A}(\mathfrak{H}) \,\middle|\, \sup_{n\in\mathbb{N}} \|A\|_n < \infty \right\}$$

with surjective morphism $\mathsf{F}\colon \mathfrak{A}_\infty(\mathfrak{H}) \to \mathcal{A}(\mathfrak{H})$.

- Under certain conditions one can extend F to certain unbounded elements of $\mathscr{A}(\mathfrak{H})$ to be unbounded operators associated with a von Neumann completion of $\mathcal{A}(\mathfrak{H})$.

Solving the Equation

- Renormalised products appearing in the Stochastic Quantisation equations (of superrenormalisable theories) are always contained in $\mathscr{A}(\mathfrak{H})$, correspond to unbounded operators affiliated with $(\mathcal{A}(\mathfrak{H}), \omega)$.
- Equation can be lifted from being naïvely $\mathcal{A}(\mathfrak{H})$-valued to $\mathscr{A}(\mathfrak{H})$.
- Solving equation in $\mathscr{A}(\mathfrak{H})$ equivalent to solving equation in $\mathscr{A}_n(\mathfrak{H}) := \mathscr{A}(\mathfrak{H}) / \ker \|\bullet\|_n$.
- For each $n \in \mathbb{N}$ obtain maximal local existence time T_n. These can be pieced together to a stopped solution in $\mathscr{A}(\mathfrak{H})$.

Open Problems

- Find method to prove global in time existence, Pauli Principle?
- Find robust methods to show correspondence with unbounded operators affiliated to original CAR algebra, Non-Commutative L^p-Spaces?
- Use with more models.

Thank You!

References

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