Renormalisation of enhanced quartic tensor field theories

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joint work with Joseph Ben Geloun (Univ. Paris 13)

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Algebraic, analytic, geometric structures emerging from quantum field theory
Chengdu, Sichuan, China, March 2024

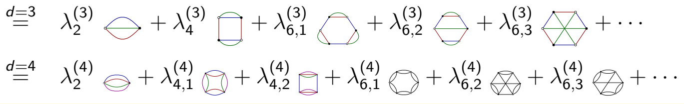
Tensor field theories

$$\mathcal{Z} = \int \mathcal{D}\phi \mathcal{D}\bar{\phi}\ e^{-(S^{\text{kinetic}} + S^{\text{interaction}})},$$

where ϕ, $\bar{\phi}$ are order-d tensor fields $\phi : G^d \to \mathbb{C}$, and

$$S^{\text{kinetic}}[\phi, \bar{\phi}] = \mu \mathrm{Tr}_2(\phi^2) + \mathrm{Tr}_2(\bar{\phi} \cdot K \cdot \phi)$$

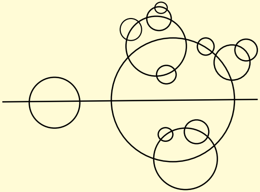
$$S^{\text{interaction}}[\phi, \bar{\phi}] = \sum_{\mathcal{B}} \lambda_{\mathcal{B}} \mathrm{Tr}_{2n_{\mathcal{B}}}(\bar{\phi}^{n_{\mathcal{B}}} \cdot \mathcal{V}_{\mathcal{B}} \cdot \phi^{n_{\mathcal{B}}})$$



- After Wick contraction, it generates $(d+1)$-edge-colored Feynman graphs.
- $(d+1)$-edge-colored graphs (also, called graph encoding manifolds (GEM)) are dual to simplicial triangulations of piecewise linear (PL) d-dimensional pseudo-manifolds [Bandieri, Gagliardi 1982; Ferri, Gagliardi, Grasselli 1986].
- Relevant for random geometric (path integral) approach to quantum gravity in dimensions $d \geq 3$.
- Lower dimensional counterpart, matrix models generate the Brownian sphere at criticality and are rigorously proven to be equivalent to 2-dimensional Liouville quantum gravity [Le Gall, Miermont 2011; Miller, Scheffield 2015].

tensor models

Melons dominate and they are branched polymers.



[V. Bonzom, R. Gurau, A. Riello, V. Rivasseau "Critical behavior of colored tensor models in the large N limit," Nucl. Phys. B 853, 174 (2011)]

[R. Gurau, J Ryan "Melons are branched polymers," Annales Henri Poincare 15, no. 11, 2085 (2014).]

The melonic 2-point function admits the following expansion:

$$G_{\mathrm{melonic}}(t) = \sum_{p=0}^{\infty} t^p \, FC_p^{(d+1)} \,,$$

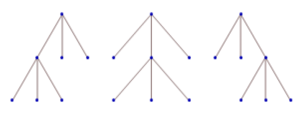
where Fuss-Catalan numbers

$$FC_n^{(d+1)} = \frac{1}{(d+1)n+1} \binom{(d+1)n+1}{n} \,.$$

tensor models

Fuss-Catalan numbers $FC_n^{(d+1)}$ ($d = 1$ is Catalan) correspond to

- the number of planar $(d+1)$-ary trees with n vertices and with $dn+1$ leaves.
- the number of non-crossing partitions of the set $\{1, 2, \cdots, d\,n\}$ that contain only subsets of size d, etc



($d = 2$, $n = 2$)

Enhanced tensor models

[V.Bonzom, T. Delepouve, V. Rivasseau, "Enhancing non-melonic triangulations: A tensor model mixing melonic and planar maps," Nucl. Phys. B 895, 161 (2015)]

Introduced a non-melonic interaction (necklace) properly scaled in N along with a melonic interaction, and recovered the string suceptibility exponent of pure 2-dimensional gravity $\gamma = -1/2$, $\gamma = 1/2$ (trees/branched polymers), and $\gamma = 1/3$ (a proliferation of baby universes).

Goal: enhance non-melons.

Tensor field theory models

- Consider a field theory defined by a complex field $\phi : G^d \to \mathbb{C}$, where $G = \mathrm{U}(1)^D$.
- The Fourier transform of ϕ yields an order-d complex tensor $\phi_{\mathbf{P}}$[1], with $\mathbf{P} = (p_1, p_2, \ldots, p_d)$ a multi-index, where $p_1, p_2, \ldots, p_d$ are also multi-indices $p_s = (p_{s,1}, p_{s,2}, \ldots, p_{s,D})\,,\ \ p_{s,i} \in \mathbb{Z}\,.$
- $\bar\phi_{\mathbf{P}}$ denotes its complex conjugate.

The action

$$S[\bar\phi, \phi] = S^{\text{kinetic}}[\bar\phi, \phi] + S^{\text{interaction}}[\bar\phi, \phi]\,,$$

is given by convolutions of tensors

$$S^{\text{kinetic}}[\bar\phi, \phi] = Z_{\mathbf{K}} \mathrm{Tr}_2(\bar\phi \cdot \mathbf{K} \cdot \phi) + \mu\, \mathrm{Tr}_2(\phi^2)$$

with

$$\mathrm{Tr}_2(\phi^2) = \sum_{\mathbf{P}} \bar\phi_{\mathbf{P}} \phi_{\mathbf{P}}\,, \qquad \mathrm{Tr}_2(\bar\phi \cdot \mathbf{K} \cdot \phi) = \sum_{\mathbf{P},\,\mathbf{P}'} \bar\phi_{\mathbf{P}}\, \mathbf{K}(\mathbf{P}; \mathbf{P}')\, \phi_{\mathbf{P}'}\,,$$

where $\mathrm{Tr}_{n_{\mathcal{B}}}$ are sums over all indices $p_{s,i}$ of $\mathbf{P}$ on $n_{\mathcal{B}}$ tensors ϕ and $\bar\phi$.

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[1] Considering $\phi_{\mathbf{P}}$ as a tensor is a slight abuse because the modes $p_{s,i}$ range up to infinity. We cut off at N, then $\phi_{\mathbf{P}}$ transforms under the fundamental representation of $\mathrm{U}(N)^{D\times d}$, and hence a tensor.

where the kinetic term kernel can be simply given by

$$\mathbf{K}(\mathbf{P};\mathbf{P}') = \boldsymbol{\delta}_{\mathbf{P};\mathbf{P}'}\mathbf{P}^{2b}\,,$$

with $\boldsymbol{\delta}_{\mathbf{P};\mathbf{P}'} = \prod_{s=1}^{d}\prod_{i=1}^{D}\delta_{p_{s,i},p'_{s,i}}$, $\mathbf{P}^{2b} = \sum_{s=1}^{d}\sum_{i=1}^{D}|p_{s,i}|^{2b}$.

Then, denote $\mathrm{Tr}_2(\bar{\phi}\cdot\mathbf{K}\cdot\phi) = \mathrm{Tr}_2(p^{2b}\phi^2)$.

Remark

In ordinary QFT on $\mathbb{R}^d$, the restriction $b \leq 1$ ensures the Osterwalder-Schrader positivity axiom (to satisfy Wightman axioms on Minkowski), however, here a priori we have no such restriction but we still restrict b to be a positive real number.

Partition function is given by

$$Z = \int d\nu_C(\bar{\phi},\phi)\; e^{-S^{\mathrm{interaction}}[\bar{\phi},\phi]}\,,$$

where $d\nu_C(\bar{\phi},\phi)$ is a Gaussian measure with covariance C given by the inverse of the kinetic term:

$$C(\mathbf{P};\mathbf{P}') = \frac{1}{Z_b\mathbf{P}^{2b}+\mu}\,\boldsymbol{\delta}_{\mathbf{P},\mathbf{P}'}\,.$$

Our enhanced quartic models

$(D, d, a, b) \in \mathbb{N} \times \mathbb{N} \times \mathbb{R}_+ \times \mathbb{R}_+$. ϕ is a tensor field with d number of indices.

$$S^{\text{kinetic}} = Z_b \mathrm{Tr}_2(p^{2b}\phi^2) + \mu \mathrm{Tr}_2(\phi^2)$$

- model $+$

$$S_{+}^{\text{interation}}[\bar{\phi}, \phi] = \frac{\lambda}{2} \mathrm{Tr}_4(\phi^4) + \frac{\lambda_+}{2} \mathrm{Tr}_4(p^{2a}\,\phi^4) + Z_a \mathrm{Tr}_2(p^{2a}\phi^2)$$

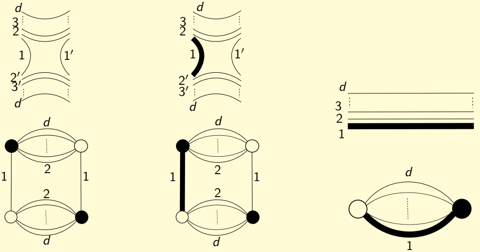
- model $\times$

$$S_{\times}^{\text{interation}}[\bar{\phi}, \phi] = \frac{\lambda}{2} \mathrm{Tr}_4(\phi^4) + \frac{\lambda_\times}{2} \mathrm{Tr}_4([p^{2a}p'^{2a}]\,\phi^4) + \sum_{\xi=a,2a} Z_\xi \mathrm{Tr}_2(p^{2\xi}\phi^2)$$

where

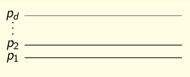
$$\mathrm{Tr}_4(\phi^4) := \sum_{p_s, p'_s \in \mathbb{Z}^D} \phi_{12\ldots d}\, \bar{\phi}_{1'23\ldots d}\, \phi_{1'2'3'\ldots d'}\, \bar{\phi}_{12'3'\ldots d'} + \mathrm{Sym}(1 \to 2 \to \cdots \to d)\,,$$
$$\mathrm{Tr}_4(p^{2a}\,\phi^4) := \sum_{p_s, p'_s \in \mathbb{Z}^D} |p_1|^{2a} \phi_{12\ldots d}\, \bar{\phi}_{1'23\ldots d}\, \phi_{1'2'3'\ldots d'}\, \bar{\phi}_{12'3'\ldots d'} + \mathrm{Sym}(1 \to 2 \to \cdots \to d),$$
$$\mathrm{Tr}_4([p^{2a}p'^{2a}]\,\phi^4) := \sum_{p_s, p'_s \in \mathbb{Z}^D} \Big(|p_1|^{2a}|p'_1|^{2a}\Big) \phi_{12\ldots d}\, \bar{\phi}_{1'23\ldots d}\, \phi_{1'2'3'\ldots d'}\, \bar{\phi}_{12'3'\ldots d'} + \mathrm{Sym}\,(1 \to 2 \to \cdots \to d)\,.$$

Enhanced model +

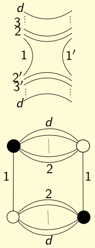


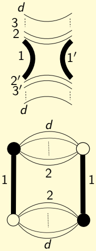
$$S_+^{\mathrm{int}}[\bar\phi,\phi] = \frac{\lambda}{2}\,\mathrm{Tr}_4(\phi^4) + \frac{\lambda_+}{2}\,\mathrm{Tr}_4(p^{2a}\,\phi^4) + Z_a\mathrm{Tr}_2(p^2\phi^2)\,,$$

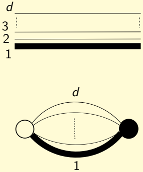
$$S_+^{\mathrm{kinetic}}[\bar\phi,\phi] = Z_b\mathrm{Tr}_2(p^{2b}\phi^2) + \mu\mathrm{Tr}_2(\phi^2)\,,$$



Enhanced model ×

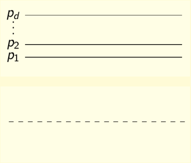






$$S_\times^{\mathrm{int}}[\bar\phi,\phi] = \frac{\lambda}{2}\,\mathrm{Tr}_4(\phi^4) + \frac{\lambda_\times}{2}\,\mathrm{Tr}_4([p^{2a}p'^{2a}]\,\phi^4) + \sum_{\xi=a,2a} Z_\xi\,\mathrm{Tr}_2(p^{2\xi}\phi^2)$$

$$S_\times^{\mathrm{kinetic}}[\bar\phi,\phi] = Z_b\mathrm{Tr}_2(p^{2b}\phi^2) + \mu\mathrm{Tr}_2(\phi^2)\,,$$

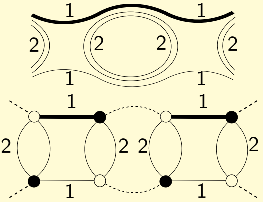


Advertisement

Amplitudes for illustration $d = 3$, in the enhanced model $+$,

A non-melonic Feynman graph

A melonic Feynman graph



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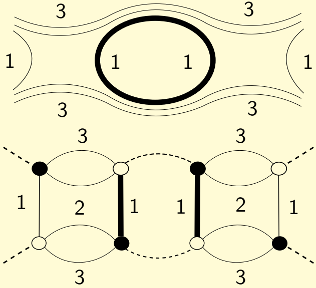


Power counting theorems

The amplitude of a Feynman graph $\mathcal{G}(\mathcal{V}, \mathcal{L})$ with a set of vertices $\mathcal{V}$ and a set of propagator lines $\mathcal{L}$, in perturbation theory:

$$A_{\mathcal{G}}(\{p_{\rm ext}\}) = \sum_{\mathbf{P}_v} \prod_{l \in \mathcal{L}} C_{\bullet, l}(\mathbf{P}_v, \mathbf{P}'_{v'}) \prod_{v \in \mathcal{V}} (-\mathbf{V}_v(\mathbf{P}_v))$$

where $C_{\bullet,l}$ is a propagator with line index l, $\mathbf{V}_v(\mathbf{P}_v)$ is a given vertex weight that contains a coupling constant but also a momentum weight if the vertex v is enhanced. Superficial degrees of divergence are given by,

- model $+$

$$\begin{aligned}\omega_{\rm d;+}(\mathcal{G}) = & -\Omega(\mathcal{G}) - D(C_{\partial\mathcal{G}} - 1) \\ & -\tfrac{1}{2}\left[(D\,(d-1) - 2b)N_{\rm ext} - 2D\,(d-1)\right] \\ & +\tfrac{1}{2}\left[-2D\,(d-1) + (D\,(d-1) - 2b)n\right] \cdot V \\ & +2a\rho_{+} + 2a\rho_{2;a} + 2b\rho_{2;b}\,.\end{aligned}$$

- model $\times$

$$\begin{aligned}\omega_{\rm d;\times}(\mathcal{G}) \;=\; & -\Omega(\mathcal{G}) - D(C_{\partial\mathcal{G}} - 1) \\ & -\tfrac{1}{2}\left[(D\,(d-1) - 2b)N_{\rm ext} - 2D\,(d-1)\right] \\ & +\tfrac{1}{2}\left[-2D\,(d-1) + (D\,(d-1) - 2b)n\right] \cdot V + 2a\rho_{\times} + \textstyle\sum_{\xi=a,2a,b} 2\xi\rho_{2;\xi}\,.\end{aligned}$$

We will focus on enhanced model $+$.

Power counting theorem for model $+$

Theorem

The enhanced model $+$

$$S^{\text{kinetic}} = Z_b \mathrm{Tr}_2(p^{2b}\phi^2) + \mu \mathrm{Tr}_2(\phi^2)$$
$$S_+^{\text{interation}}[\bar{\phi},\phi] = \frac{\lambda}{2}\mathrm{Tr}_4(\phi^4) + \frac{\lambda_+}{2}\mathrm{Tr}_4(p^{2a}\,\phi^4) + Z_a \mathrm{Tr}_2(p^{2a}\phi^2)$$

with parameters $a = \frac{1}{2}D(d-2)$, $b = \frac{1}{2}D(d-\frac{3}{2})$ *for arbitrary order* $d \geq 3$ *and dimension* $D > 0$ *is just-renormalisable at all orders of perturbation theory.*

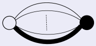
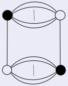
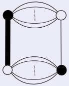
	$d=3$	$d=4$
$D=1$	$a=\frac{1}{2}$, $b=\frac{3}{4}$	$a=1$, $b=\frac{5}{4}$
$D=2$	$a=1$, $b=\frac{3}{2}$	$a=2$, $b=\frac{5}{2}$
$D=3$	$a=\frac{3}{2}$, $b=\frac{9}{4}$	$a=3$, $b=\frac{15}{4}$
$D=4$	$a=2$, $b=3$	$a=4$, $b=5$

Values of a and b for potentially just-renormalisable theories with $d \leq 4$ and $D \leq 4$.
(Impose $\omega_{\mathrm{d};+}(\mathcal{G})|_{N_{\mathrm{ext}}\geq 6} < 0$ and $\omega_{\mathrm{d};+}(\mathcal{G})$ is independent of numbers of 4-pt vertices with $\omega_{\mathrm{d};+}(\mathcal{G}^{\mathrm{non-melon}})|_{N_{\mathrm{ext}}=4} = 0$)

Power counting theorem for model +

Proposition (List of divergent graphs for the model +)

The enhanced model + with parameters $a = \frac{1}{2}D(d-2)$, $b = \frac{1}{2}D(d-\frac{3}{2})$ *for two integers* $d > 2$ *and* $D > 0$*, has divergent graphs*

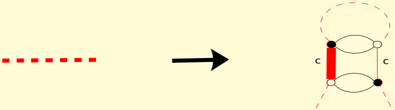




$\text{class}_{\mathcal{G}}$		N_{ext}	$V_{2;a}$	V_4	ρ_+	$\Omega(\mathcal{G})$	$\omega_{\text{d};+}(\mathcal{G})$
	(4-pt λ)	4	0	0	$V_{+;4}$	1	0
I	*(mass)*	2	0	0	$V_{+;4}$	1	$D/2$
II	*(2-pt Z_a)*	2	0	0	$V_{+;4}-1$	0	$D/2$
III	*(mass)*	2	0	1	$V_{+;4}$	0	$D/2$
IV	*(mass)*	2	1	0	$V_{+;4}$	1	0
V	*(2-pt Z_a)*	2	1	0	$V_{+;4}-1$	0	0
VI	*(mass)*	2	1	1	$V_{+;4}$	0	0

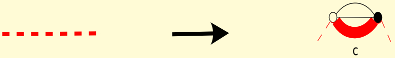
List of divergent graphs of the enhanced model +. $\Omega = 0$ is melonic, and $\Omega = 1$ is nonmelonic.

Moves (enhanced model $+$)

- An enhanced melonic insertion has $\triangle\omega_{d;+} = 0$.

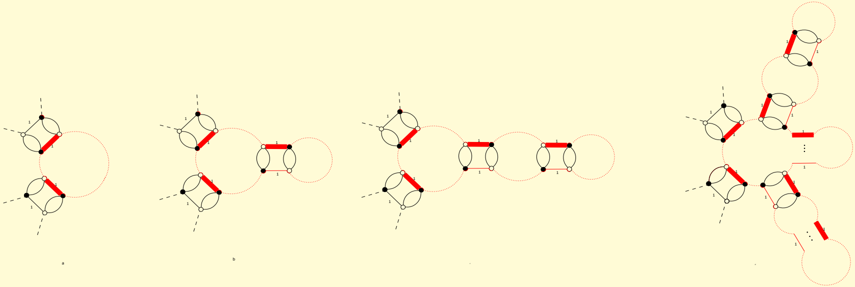


- An enhanced d-dipole insertion has $\triangle\omega_{d;+} = -\frac{D}{2}$.

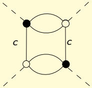


Divergent graphs for 4-pt coupling λ (model $+$)

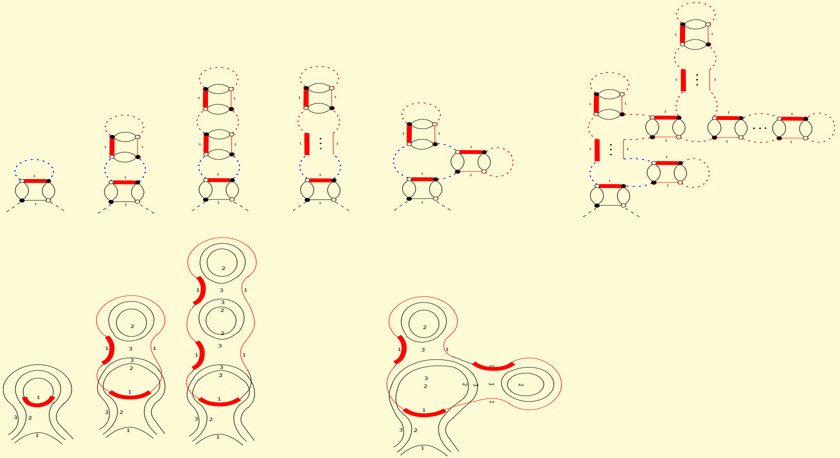
- 4-point divergent graphs (non-melonic graphs) with $\omega_{d;+} = 0$. They renormalise 4-pt coupling $\lambda\,\mathrm{Tr}_4(\phi^4)$.



- their boundary graph is

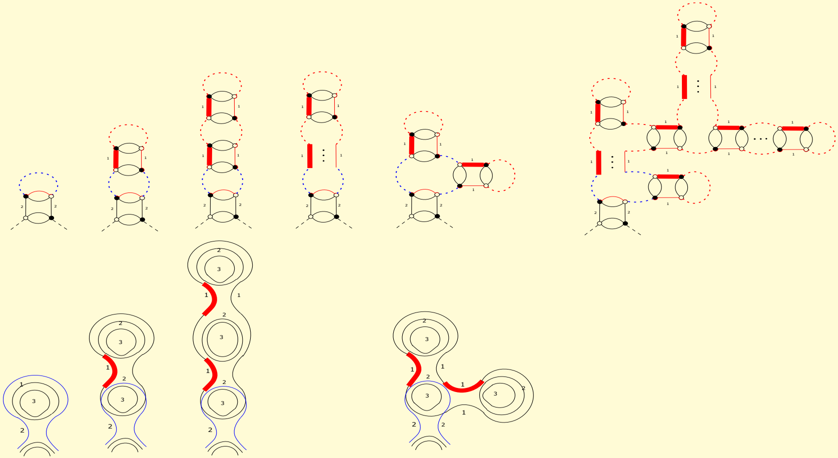


Divergent graphs (nonmelonic) for mass (model +)



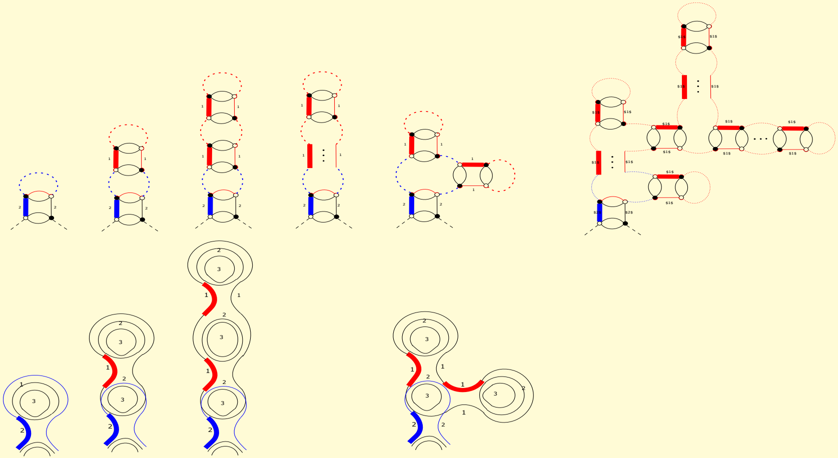
Renormalise mass $\mu \mathrm{Tr}_2(\phi^2)$. 2-pt divergent graphs with $\omega_{d;+} = \frac{D}{2}$. Class I (non-melonic graphs). We can insert *one* d-dipole anywhere on a propagator; one d-dipole with either color 1 enhanced on a blue dotted propagator, or one d-dipole with any color 1, 2, or 3 enhanced on a red dotted propagator. Then $\omega_{d;+} = 0$ and they belong to the class IV (non-melonic graphs) and renormalise mass.

Divergent graphs (melonic) for mass (model +)



Renormalise mass $\mu \mathrm{Tr}_2(\phi^2)$. 2-pt divergent graphs with $\omega_{d;+} = \frac{D}{2}$. Class III (melonic graphs). We can insert *one* d-dipole anywhere on a propagator; one d-dipole with either colors 1 or 3 enhanced on a blue dotted propagator, or one d-dipole with any color 1, 2, or 3 enhanced on a red dotted propagator. Then, $\omega_{d;+} = 0$ and they belong to the class VI (melonic graphs) and renormalise mass.

Divergent graphs for 2-pt coupling Z_a (model +)



Renormalise $Z_a \mathrm{Tr}_2(p^{2a}\phi^2)$. 2-pt divergent graphs with $\omega_{d;+} = \frac{D}{2}$. Class II (melonic graphs). We can insert *one* d-dipole anywhere on a propagator; one d-dipole with either colors 1 or 3 enhanced on a blue dotted propagator, or one d-dipole with any color 1, 2, or 3 enhanced on a red dotted propagator. Then, $\omega_{d;+} = 0$ and they belong to the class V (melonic graphs) and renormalise $Z_a \mathrm{Tr}_2(p^{2a}\phi^2)$.

Effective Action via multiscale analysis

We slice our covariance in a discrete sum of contributions, each corresponding to an energy sector (scale), i a nonnegative integer,

$$C(\mathbf{P};\mathbf{P}') = \tilde{C}(\mathbf{P})\,\delta_{\mathbf{P},\mathbf{P}'}\,, \qquad \tilde{C}(\mathbf{P}) = \frac{1}{\mathbf{P}^{2b}+\mu} = \sum_{i=0}^{\infty} \tilde{C}_i(\mathbf{P})\,,$$

with $M > 1$ positive real number, and in Schwinger parametrisation,

$$\tilde{C}_i(\mathbf{P}) = \int_{M^{-2bi}}^{M^{-2b(i-1)}} d\alpha\, e^{-\alpha(\mathbf{P}^{2b}+\mu)}\,, \quad \tilde{C}_0(\mathbf{P}) = \int_1^{\infty} d\alpha\, e^{-\alpha(\mathbf{P}^{2b}+\mu)}\,,$$

$\rightarrow$ Start by integrating out the fields at high scales $> i$ and include their effects in the effective action W^i.

$$Z = \int d\nu_{C_{\leq i}}(\bar{\phi}_{\leq i}, \phi_{\leq i})\, e^{-S^{interaction}(\bar{\phi}_{\leq i},\, \phi_{\leq i})}\,, \quad \text{where} \quad C_{\leq i}(\mathbf{P};\mathbf{P}') = \delta_{\mathbf{P},\mathbf{P}'} \sum_{j \leq i} \tilde{C}_j(\mathbf{P})\,.$$

Effective Action

$\rightarrow$ Continue integrating out another layer down to lower scale $i-1$ (Wilsonian renormalisation group). Decompose covariance $C_{\leq i} = C_i + C_{\leq i-1}$ and the corresponding fields $\phi_{\leq i} = \psi_i + \phi_{\leq i-1}$ ($\bar{\phi}_{\leq i} = \bar{\psi}_i + \bar{\phi}_{\leq i-1}$).

$$Z = \int d\nu_{C_{\leq i-1}}(\bar{\phi}_{\leq i-1}, \phi_{\leq i-1}) e^{-W^{i-1}(\bar{\phi}_{\leq i-1}, \phi_{\leq i-1})},$$

where the effective action at scale $i-1$ is given by

$$-W^{i-1}(\bar{\phi}_{\leq i-1},\, \phi_{\leq i-1}) = \log \int d\nu_{C_i}(\bar{\psi}_i, \psi_i) e^{-S^{interaction}(\bar{\psi}_i + \bar{\phi}_{\leq i-1},\, \psi_i + \phi_{\leq i-1})}.$$

If the theory is renormalisable, one can assert the effective action at any scale takes the same form as the initial bare action, therefore

$$-W^{i-1}(\bar{\phi}_{\leq i-1}, \phi_{\leq i-1}) = \mathrm{Tr}_2(\bar{\phi}_{\leq i-1} \cdot \Sigma \cdot \phi_{\leq i-1}) + \frac{1}{2}\mathrm{Tr}_4(\phi^4_{\leq i-1} \cdot \Gamma_4) + R(\phi_{\leq i-1}),$$

- $\Sigma(\{p\})$ is the sum over all amputated 1PI 2-pt graphs,
- $\Gamma_4(\{p\})$ is the sum of 1PI 4-pt graphs following the pattern of $\mathrm{Tr}_4(\phi^4)$,
- and $R(\phi_{\leq i-1})$ is the rest of the terms containing 1PR graphs (they do not contribute to the iteration process) and the finite terms.

Effective 2-pt function (model +)

Expand the 2-pt function contribution,

$$\Sigma(\{p\}) = \Sigma(\{0\}) + \sum_c |p_c|^{2b} \partial_{|p_c|^{2b}} \Sigma\Big|_{\{p\}=0} + \sum_c |p_c|^{2a} \partial_{|p_c|^{2a}} \Sigma\Big|_{\{p\}=0} + \cdots$$

- mass renormalisation $\Sigma(\{0\})$ is divergent with $\omega_{d;+} = D/2$ (classes I and III) and $\omega_{d;+} = 0$ (classes IV and VI).
- $\partial_{|p_c|^{2b}} \Sigma\Big|_{\{p\}=0} = 0$.
- $\partial_{|p_c|^{2a}} \Sigma|_{\{p\}=0} \equiv \mathbf{\Gamma}_2^{(c)}(\{\mathbf{0}\})$ is divergent.

  $|p_c|^{2a} \mathbf{\Gamma}_2^{(c)}(\{\boldsymbol{p}\})$ is the sum of all amputated 1PI 2pt-functions following the pattern of $\mathrm{Tr}_2(p_c^{2a} \phi^2)$ on their boundary graphs as dictated by class II ($\omega_{d;+} = D/2$) and class V ($\omega_{d;+} = 0$).
- $\cdots$ are finite.

$a = \frac{1}{2} D(d-2)$, $b = \frac{1}{2} D(d - \frac{3}{2})$

Effective 4-pt function (model $+$)

Similarly, expand the 4-point function contribution,

$$\Gamma_4(\{p\}) = \sum_c \left\{ \Gamma_4^{(c)}(\{0\}) + |p_c|^{2a} \partial_{|p_c|^{2a}} \Gamma_4^{(c)} \Big|_{\{p\}=0} + |p_c|^{2b} \partial_{|p_c|^{2b}} \Gamma_4^{(c)} \Big|_{\{p\}=0} \right\} + \cdots ,$$

- $\sum_c \Gamma_4^{(c)}(\{0\}) \equiv \mathbf{\Gamma_4(\{0\})}$ is the sum of all amputated 1PI 4pt-functions following the pattern of $\mathrm{Tr}_4(\phi^4)$ on their boundary graphs, and is divergent ($\omega_{d;+} = 0$).
- $\partial_{|p_c|^{2a}} \Gamma_4^{(c)} \Big|_{\{p\}=0} \equiv \mathbf{\Gamma_{4;+}^{(c)}(\{0\})}$ are all amputated 1PI 4pt-functions following the pattern of $\mathrm{Tr}_{4;c}(p^{2a}\phi^4)$ having a boundary with external $|p|^{2a}$-enhancement. In fact, there is only the leading order $\mathcal{O}(\lambda_+)$ contribution in $\mathbf{\Gamma_{4;+}^{(c)}(\{0\})}$ and there are no contributions from higher orders in perturbation theory in λ_+.
- $\partial_{|p_c|^{2b}} \Gamma_4^{(c)} \Big|_{\{p\}=0}$ is finite.
- $\cdots$ are finite.

Effective Gaussian measure (model +)

The effective Gaussian measure at the scale $i-1$ is given by

$$d\nu_{\tilde{C}^{i-1}(\phi_{\leq i-1})} \exp\left[\Sigma_{i-1}(\{0\})\mathrm{Tr}_2(\phi^2_{\leq i-1}) + \sum_c (\partial_{|p_c|^{2b}}\Sigma|_{\{p\}=0})_{i-1}\mathrm{Tr}_2(p_c^{2b}\phi^2_{\leq i-1})\right],$$

with actually $\partial_{|p_c|^{2b}}\Sigma\big|_{\{p\}=0} = 0$. The new covariance for the above Gaussian measure will be

$$\frac{1}{Z_{b,\,i-1}}\int_{M^{-2b(i-1)}}^{M^{-2b(i-2)}} d\alpha\, e^{-\alpha(|p|^{2b}+\mu_{\mathrm{ren},i-1})} = \frac{1}{Z_{b,\,i-1}}\tilde{C}^{i-1}(p)\,.$$

- the wave function renormalisation $Z_{b,\,i-1} \equiv 1 + (\partial_{|p_c|^{2b}}\Sigma\big|_{\{p\}=0})_{i-1}\,.$
- the renormalised mass $\mu_{\mathrm{ren},i-1} = \frac{1}{Z_{b,\,i-1}}(\mu_{i-1} - \Sigma_{i-1}(\{0\}))$.

With a field rescaling $\phi_{\leq i-1} \to \sqrt{Z_{b,\,i-1}}\phi_{\leq i-1}$ (actually here $Z_{b,\,i-1} = 1$) the effective theory for $\phi_{\leq i-1}$ can be recast:

Effective action (model +)

$$\int d\nu_{\tilde{C}_{i-1}}(\phi_{\leq i-1})$$

$$\exp\Big[\sum_c \frac{\Gamma^{(c)}_{2,i-1}(\{0\})}{Z_{b,i-1}} \mathrm{Tr}_{2;c}(p^{2a}\phi^2_{\leq i-1}) + \sum_c \frac{\Gamma^{(c)}_{4,i-1}(\{0\})}{2Z^2_{b,i-1}} \mathrm{Tr}_{4;c}(\phi^4_{\leq i-1})$$

$$+ \sum_c \frac{\Gamma^{(c)}_{4;+,i-1}(\{0\})}{2Z^2_{b,i-1}} \mathrm{Tr}_{4;c}(p^{2a}\phi^4_{\leq i-1}) + \tilde{R}(\sqrt{Z_{b,\,i-1}}\phi_{\leq i-1})\Big] .$$

Now we can identify the effective couplings at scale $i-1$,

$$Z_{a,i-1} = -\frac{\Gamma^{(c)}_{2,i-1}(\{0\})}{Z_{b,i-1}} , \quad \lambda^{(c)}_{i-1} = -\frac{\Gamma^{(c)}_{4,i-1}(\{0\})}{Z^2_{b,i-1}} , \quad \lambda^{(c)}_{+;\,i-1} \quad = \quad -\frac{\Gamma^{(c)}_{4;+,i-1}(\{0\})}{Z^2_{b,i-1}} .$$

So, compute $\Gamma^{(c)}_{2,i-1}(\{0\})$, $\Gamma^{(c)}_{4,i-1}(\{0\})$, $\Gamma^{(c)}_{4;+,i-1}(\{0\})$, and $\Sigma_{i-1}(\{0\})$, and let us fix the group dimension $D = 1$ in the following.

Renormalisation of model +

We write the β-function for a coupling g as

$$\beta_g(k) = k\partial_k g(k) = \partial_t g(t)\,,$$

where k is a momentum scale, and $t = \log(k/k_0)$.

The momentum scale must be compared to the multiscale slice range $k/k_0 \sim M^i$, $M > 1$.

β-functions in dimensionless couplings (model $+$)

- Scaling dimension $\{g\}$ of a coupling g can be read from the degree of divergence as the coefficient of the corresponding vertex number V (e.g., Peskin and Schroeder).

- Recall the degree of divergence for model $+$,

$$\omega_{\mathrm{d};+}(\mathcal{G})|_{D=1} = -\Omega(\mathcal{G}) - (C_{\partial\mathcal{G}} - 1) - \frac{1}{2}\left[((d-1)-2b)N_{\mathrm{ext}} - 2(d-1)\right] + \frac{1}{2}\left[-2(d-1) + ((d-1)-2b)n\right]\cdot V_n + 2a\rho_+ + 2a\rho_{2;a} + 2b\rho_{2;b}\,,$$

where $n \cdot V = 4V_4 + 4V_{+;4} + 2V_{2;a} + 2V_2$.

$$\begin{array}{rcllcl} \{\lambda_+\} & = & 0 & \text{(marginal)} & \to & \lambda_+ = \widetilde{\lambda}_+ \\ \{\lambda\} & = & d-2 & \text{(relevant)} & \to & \lambda = k^{d-2}\widetilde{\lambda} \\ \{\mu\} & = & d-\dfrac{3}{2} & \text{(relevant)} & \to & \mu = k^{d-\frac{3}{2}}\widetilde{\mu} \\ \{Z_a\} & = & \dfrac{1}{2} & \text{(relevant)} & \to & Z_a = k^{\frac{1}{2}}\widetilde{Z}_a \end{array}$$

where we dressed with $\widetilde{\ }$ dimensionless quantities (they do not have scaling behavior in t nor k).

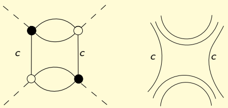
$a = \frac{1}{2}D(d-2)$, $b = \frac{1}{2}D(d-\frac{3}{2})$ and $d \geq 3$ for tensors

β-function of 4-pt coupling λ (model $+$)

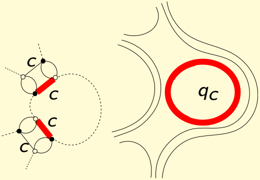
$$\Gamma_4^{(c)}(\{p\}) = \sum_{\mathcal{G}_{\mathbf{4},\iota}^{(c)}} K_{\mathcal{G}_{\mathbf{4},\iota}^{(c)}} \; S_{\mathcal{G}_{\mathbf{4},\iota}^{(c)}}(\{p\}) \, ,$$

where $K_{\mathcal{G}_{\mathbf{4},\iota}^{(c)}}$ is a combinatorial factor and $S_{\mathcal{G}_{\mathbf{4},\iota}^{(c)}}(\{p\})$ is a formal amplitude sum.

The sum over $\mathcal{G}_{4,\iota}^{(c)}$ runs over a list of 4pt-graphs obeying the multiscale power counting analysis. Up to one-loop, we have the following two graphs:



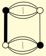
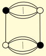
Zero-loop divergent graph at $d = 3$.



One-loop divergent graph, $n_4^{(c)}$ at $d = 3$ contributing to the flow of λ. $K_{n_{\mathbf{4}}^{(c)}} = 2$,

$$S_{n_{\mathbf{4}}^{(c)}}(\{\mathbf{p}, \mathbf{p}'\}) = \frac{1}{2!}\left(\frac{-\lambda_+^{(c)}}{2}\right)^2 \sum_{q_c} \frac{|q_c|^{\mathbf{2}a}}{(|\mathbf{p}_{\check{c}}|^{\mathbf{2}b}+|q_c|^{\mathbf{2}b}+\mu)} \frac{|q_c|^{\mathbf{2}a}}{(|\mathbf{p}'_{\check{c}}|^{\mathbf{2}b}+|q_c|^{\mathbf{2}b}+\mu)}$$

β-functions of 4-pt couplings λ and λ_+ (model $+$)

 λ_+  λ

The β-functions of the 4-pt couplings up to one-loop are (set all couplings to $\lambda^{(c)} = \lambda$, and $\lambda_+^{(c)} = \lambda_+$ to simplify),

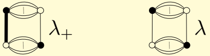
$$\lambda_{+,\mathrm{ren}} = \lambda_+ \,,$$
$$\lambda_{\mathrm{ren}} = \lambda - \frac{1}{4}(\lambda_+)^2 S_0 \,, \qquad\qquad S_0 = \sum_q \frac{|q|^{4a}}{(|q|^{2b} + \mu_i)^2} \; > \; 0 \,.$$

Observations

- λ_+ does not run! and defines a fixed point at all orders of perturbation.
- λ and λ_+ never coincide and could not be set at an equal value (leads to inconsistency).

Now, we are going to compute...

β-functions of λ and λ_+ in multiscale analysis (model $+$)



In the multiscale analysis with discrete scale i, the system can be written as

$$\lambda_{+,i-1} = \lambda_{+,i}\,, \qquad \lambda_{i-1} = \lambda_i - \frac{1}{4}\lambda_{+,i}^2\, S_{0,i}\,,$$

where using Schwinger parameterisation,

$$S_{0,i} \;=\; \sum_q |q|^{4a} \int_{M^{-2bi}}^{M^{-2b(i-1)}} d\alpha e^{-\alpha(|q|^{2b}+\mu_i)} \int_{M^{-2bi}}^{M^{-2b(i-1)}} d\alpha' e^{-\alpha'(|q|^{2b}+\mu_i)}\,,$$

and explicit dimensions (with $\tilde{}$ dimensionless quantities),

$$q = k\tilde{q}\,, \qquad \widetilde{q} \in \mathbb{Z}$$
$$\alpha = k^{-2b}\,\widetilde{\alpha}\,.$$

Then, using Euler-Maclauren formula

$$S_{0,i} = k^{4a+1}k^{-4b}\widetilde{S}_{0,i} = \widetilde{S}_{0,i} \;=\; \frac{1}{b}\log\frac{(M^{2b}+1)^2}{4M^{2b}} + \mathcal{O}(M^{-2bi}\log(M^{-2bi}))$$
$$> \;0 \qquad (\text{recall } M>1)\,.$$

β-function of λ (model +)

In the multiscale formulation, rewriting in a suggestive way

$$-(\lambda_{i-1} - \lambda_i) = \frac{\partial \lambda_i}{\partial i} \quad = \quad \frac{1}{4} \lambda_{+,i}^2 \, \widetilde{S}_{0,i} \, .$$

We write the β-function for a coupling g as $\beta_g(k) = k\partial_k g(k) = \partial_t g(t)$, where k is a momentum scale, and $t = \log(k/k_0)$ (or $k = k_0 e^t$). The momentum scale must be compared to the slice range $k/k_0 \sim M^i$. So, $t = \log(k/k_0) \sim i \log M$.

β-function of λ is

$$\frac{\partial \lambda_i}{\partial((\log M) i)} = \partial_t \lambda(t) \quad = \quad \beta_\lambda \, \lambda_+^2 \, , \qquad \beta_\lambda = \frac{1}{4b} \frac{\log \frac{(M^{2b}+1)^2}{4M^{2b}}}{\log(M)} > 0 \, ,$$

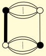
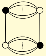
where λ_+ does not run.

Recalling that λ is dimensionful, $\lambda = k^{d-2}\widetilde{\lambda} = k_0 e^{(d-2)t}\widetilde{\lambda}$, integrate and

$$\widetilde{\lambda}(t) \quad = \quad c_0 \, \beta_\lambda \, \lambda_+^2 \, e^{-(d-2)t}(t - t_0) + \widetilde{\lambda}(t_0) \, e^{-(d-2)(t-t_0)} \, ,$$

where the initial condition was set at some IR scale $e^{t_0} \ll \Lambda/k_0 = e^t$ with c_0 some constant.

Running of $\widetilde{\lambda}$ in enhanced model + and its discussion

 λ_+  λ

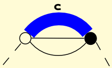
$$
\begin{aligned}
\lambda_+ &= \textit{const}. \\
\widetilde{\lambda}(t) &= c_0\, \beta_\lambda\, \lambda_+^2\, e^{-(d-2)t}(t - t_0) + \widetilde{\lambda}(t_0)\, e^{-(d-2)(t-t_0)}
\end{aligned}
$$

- There is no pole in the solution at first order (no Landau pole, not like ordinary ϕ^4_4 model).
- - In the UV ($t \to \infty$), $\widetilde{\lambda}(t)$ is suppressed by the exponential factors (ordinary behavior of a relevant coupling). One may be tempted to conclude to asymptotic freedom, however, λ_+ is constant and does not run to zero.
  - In the IR ($t \to -\infty$), $\widetilde{\lambda}$ grows; we may expect phase transition in this regime.
- It differs from the ϕ^4-TFT model with only ordinary λ coupling, where a different class of graphs (i.e., melons) dominate yielding asymptotic freedom.
- At $d = 2$, something special happens; it is an (enhanced) matrix model which enhances some planar graphs. This deserves a full-fledged investigation.

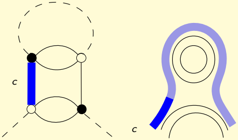
Renormalisation of 2-pt coupling Z_a (model $+$)

$$|p_c|^{2a}\Gamma_2^{(c)}(\{p\}) \quad = \quad \sum_{\mathcal{G}^{(c)}_{\mathbf{2};a;\iota}} K_{\mathcal{G}^{(c)}_{\mathbf{2};a;\iota}} S_{\mathcal{G}^{(c)}_{\mathbf{2};a;\iota}}(\{p\})\,,$$

where the sum is over all amputated 1PI 2-pt graphs at 1-loop whose boundaries are in the form of $\mathrm{Tr}_{2;(c)}(p^{2a}\phi^2)$. Up to the first order in perturbation theory, we have $\mathcal{G}^{(c)}_{2;a;\iota} \in \{z_a^{(c)}, m_e^{(c)}\}$,



$z_a^{(c)}$



$m_e^{(c)}$

$$Z^{(c)}_{a,\mathrm{ren}} \quad = \quad -\Gamma_2^{(c)}(\{0\}) = Z_a^{(c)} + \lambda_+^{(c)} \sum_{\{q_{\check{c}}\}} \frac{1}{(|\mathbf{q}_{\check{c}}|^{2b} + \mu)}\,.$$

Furthermore, set the couplings to be independent of colors.

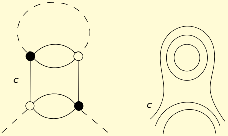
Renormalisation of self energy and mass (model +)

Compute the self energy,

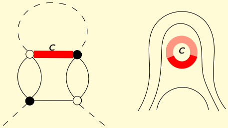
$$\Sigma_b(\{p\}) = \sum_{c=1}^{d} \sum_{\mathcal{G}^{(c)}_{\mathbf{2},\iota}} K_{\mathcal{G}^{(c)}_{\mathbf{2},\iota}} S_{\mathcal{G}^{(c)}_{\mathbf{2},\iota}}(\{p\}) \,,$$

where $\mathcal{G}^{(c)}_{2,\iota} \in \{m^{(c)}, n^{(c)}\}_{c=1,2,\ldots,d}$ up to one loop.

$\Sigma_b(\{p\})$ corresponds to the part $\Sigma(\{0\}) + \sum_c |p_c|^{2b} \partial_{|p_c|^{2b}} \Sigma\big|_{\{p\}=0}$ of total self-energy function $\Sigma(\{p\})$. However, $\partial_{|p_c|^{2b}} \Sigma\big|_{\{p\}=0} = 0$, we only focus on the contribution $\Sigma(\{0\})$, namely the contribution to the mass renormalisation.



The graph $m^{(c)}$ in the case $d = 3$. The degree of divergence $\omega_{d;+}(m^{(c)}) = \frac{D}{2}$.



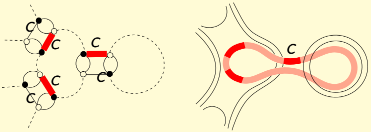
The Feynman graph $n^{(c)}$ for $d = 3$. $\omega_{d;+}(n^{(c)}) = \frac{D}{2}$.

Summary of perturbative renormalisation β-functions for model + up to first order

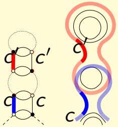
$\partial_t \lambda_+ = 0$	$\lambda_+ = const.$
$\partial_t \widetilde{\lambda}(t) = -(d-2)\widetilde{\lambda}(t) + c_0\|\beta_\lambda\|\lambda_+^2 e^{-(d-2)t}$	$\widetilde{\lambda}(t) = e^{-(d-2)t}(c_0\|\beta_\lambda\|\,\lambda_+^2\, t + const.)$
$\partial_t \widetilde{Z}_a(t) = -\frac{1}{2}\widetilde{Z}_a(t) - \|\beta_{Z_a}\|\,\lambda_+$	$\widetilde{Z}_a(t) = c_1\, e^{-t/2} - 2\|\beta_{Z_a}\|\,\lambda_+$
$\partial_t \widetilde{\mu}(t) = -(d-\frac{3}{2})\widetilde{\mu}(t) - \|\beta_{\mu,1}\|\,\widetilde{\lambda}(t)$ $-\|c_0\beta_{\mu,2}\|\,\lambda_+ e^{-(d-2)t}$	$\widetilde{\mu}(t) = 2\, e^{-(d-2)t}\Big(-\|\beta_1\|\, t + const.\, e^{-(d-\frac{3}{2})t} + \gamma\Big)$ $\gamma = \|\beta_1\|(t_0+2) - (\|\beta_{\mu,1}\|\tilde{\lambda}(t_0)e^{(d-2)t_0} + \|c_0\beta_{\mu,2}\|\,\lambda_+)$

- λ_+. It does not run and is constant (nonzero).
- $\widetilde{\lambda}(t)$. There is no pole in the solution at first order (no Landau pole, not like ordinary ϕ_4^4 model). In the IR, $\widetilde{\lambda}$ grows. In the UV, $\widetilde{\lambda}(t)$ is suppressed by the exponential factors (ordinary behavior of a relevant coupling).
- $\widetilde{\mu}$. Common behavior of any relevant mass coupling; In the UV, $\widetilde{\mu}$ goes to 0 whereas in the IR the mass exponentially increases.
- $\widetilde{Z}_a(t)$. Ordinary behavior of a relevant coupling; decreases exponentially in the UV and suppressed up until it reaches a constant. In the IR, it blows up.

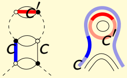
Higher order corrections for model +



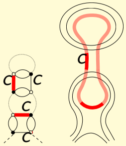
$\omega = 0$. 4-pt λ renorm.



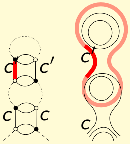
$\omega = D/2$, class II, 2-pt Z_a renorm.



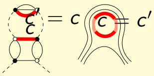
$\omega = 0$, class V, 2-pt Z_a renorm.



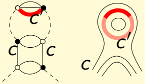
$\omega = D/2$, class I, mass renorm.



$\omega = D/2$, class III, mass renorm.



$\omega = 0$, class IV, mass renorm.



$\omega = 0$, class VI, mass renorm.

to all orders in perturbation (model +)

- No diverging amplitudes contributing to the renormalisation of λ_+ at all orders in perturbation. *λ_+ is constant at all orders.*
- To arbitrary n^{th} order,

$$\begin{aligned}
\partial_t \lambda(t) &= P_n(\lambda_+)\,, \\
\widetilde{\lambda}(t) &= e^{-(d-2)t}\big(t\,P_n(\lambda_+) + const.\big),
\end{aligned}$$

$$\begin{aligned}
\partial_t Z_a(t) &= e^{t/2} Q_{1;n}(\lambda_+) + t\,Z_a(t)\,Q_{2;n}(\lambda_+)\,, \\
\partial_t \mu(t) &= e^{t/2}\Big(\lambda(t)\,R_{1;n}(\lambda_+) + R_{2;n}(\lambda_+)\Big) \\
&\quad + t\,Z_a(t)\Big(\lambda(t)\,R_{3;n}(\lambda_+) + R_{4;n}(\lambda_+)\Big)\,,
\end{aligned}$$

  where $P_n(\lambda_+) = \beta_\lambda \lambda_+^2 + \ldots,\;\; R_{i;n}(\lambda_+)$, and $Q_{j;n}(\lambda_+)$ are all polynomials in λ_+ and some constants.
- Still, $\widetilde{\lambda}(t)$ vanishes in the UV.
- Apart from $\widetilde{\lambda}$, the solution of these equations requires more knowledge before interpreting the asymptotic behavior of the model. For instance, the behavior of $\widetilde{Z}_a$ strongly depends on $Q_{2;n}$ that is yet unknown.

Conclusions

We have explicitly computed the one-loop β-functions of the couplings of two enhanced TFTs, the model $+$ and the model $\times$, at first order of perturbation theory.

The system of RG flow equations can be explicitly solved. Both models $+$ and $\times$ have a constant wave function renormalization ($Z_b = 1$). Nevertheless, we have obtained some nontrivial RG flows of the couplings.

For the model $+$,

- To first order in perturbation, one marginal direction $\lambda_+ = \theta$ is kept fixed and there are three relevant operators (λ, μ, Z_a) with dimensionless counterparts $(\widetilde{\lambda}, \widetilde{\mu}, \widetilde{Z}_a)$ flowing to $(0, 0, c\,\theta)$ in UV. Suggests asymptotic safety.
- To all orders in perturbation, still $\lambda_+ = \theta$ is kept fixed but apart from $\tilde{\lambda}$, asymptotic behaviors of the other couplings are unknown, but possibly can be resummable and computed.

Summary of perturbative β-functions for model $\times$

We give a summary of the 1-loop RG flow equations for the model $\times$ and their solutions.

$\partial_t \lambda_\times = 0$	$\lambda_\times = c_1$
$\partial_t \widetilde{\lambda}(t) = -2\,\widetilde{\lambda}(t)$	$\widetilde{\lambda}(t) = c_2\, e^{-2t}$
$\partial_t \widetilde{\mu}(t) = -2\,\widetilde{\mu}(t) - \beta_{\mu,1}\,\widetilde{\lambda}(t)$	$\widetilde{\mu}(t) = \big(-c_1\,\beta_{\mu,1}\,t + c_3\big)\,e^{-2t}$
$\partial_t \widetilde{Z}_a(t) = -\widetilde{Z}_a(t) - c_0\,\beta_{Z_a}\,\lambda_\times\, e^{-t}$	$\widetilde{Z}_a(t) = c_0\big(-\beta_{Z_a}\,\lambda_\times\,t + c_4\big)\,e^{-t}$
$\partial_t Z_{2a}(t) = -\beta_{Z_{2a}}\,\lambda_\times$	$Z_{2a}(t) = -\beta_{Z_{2a}}\,\lambda_\times\,t + c_5$

c_1, c_2, c_3, c_4, c_5 are all constants.

$$\beta_{\mu,1} = 2d\pi > 0\,, \qquad \beta_{Z_a} = 2 > 0\,, \qquad \beta_{Z_{2a}} = 2\pi > 0\,.$$

- The mass, the 4-point coupling λ, and the 2-point coupling Z_a are ordinary relevant couplings, exponentially decaying to a constant, zero and zero respectively in the UV.
- The enhanced 4-point coupling $\lambda_\times$ does not run.
- Z_{2a} in the model $\times$ grows linearly in t in its magnitude.

Actually, to all orders in perturbation theory, such behaviors persist.

Conclusions, continued

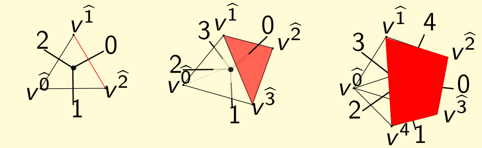
Finally, comparing the RG flow equations between the conventional models and these enhanced TFT, $+$ or $\times$ models, shows drastic differences. In the present context, they are simple enough to exhibit explicit solutions.

- These models may not give rise to quantum gravity, but possibly a new kind of exhotic ϕ^4 models.
- Solve for higher orders. The model may be resummable.

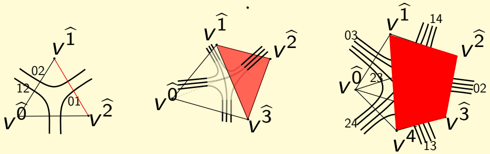
the end

Illustration below of d-simplices in $d = 2, 3, 4$ dimensions, where we embedded $(d+1)$-edge-colored graphs.

$d = 2$ $d = 3$ $d = 4$



(Colored representation.)



(Stranded representation.)