The FBSDE Approach to sine-Gordon up to 6π

Sarah-Jean Meyer | University of Oxford

based on arXiv:2401.13648, joint work with M. Gubinelli

- Osterwalder-Schrader reconstruction theorem ('75):

Quantum Field Theory		Euclidean Quantum Field Theory
Defined on $\mathbb{M}_d \sim \mathbb{R}^1 \times \mathbb{R}^{d-1}$	$\overset{t \mapsto it}{\longleftrightarrow}$ Wick rotation	Defined on $\mathbb{R}^d$
$\lvert (x_0, x_{1:d-1}) \rvert_{\mathbb{M}}^2 = -x_0^2 + x_1^2 + \cdots + x_{d-1}^2$		$\lvert x \rvert_{\mathbb{R}}^2 = x_0^2 + x_1^2 + \cdots + x_{d-1}^2$

- EQFT: Certain **Probability measures** ν on the space of **distributions** $\mathcal{S}'(\mathbb{R}^d)$.

$$\text{“}\mathbb{E}_\nu[\mathcal{O}] = \int_{\mathcal{S}'(\mathbb{R}^d)} \mathcal{O}(\varphi)\nu(\mathrm{d}\varphi) = \frac{1}{\text{norm.}} \int_{\mathcal{S}'(\mathbb{R}^d)} \mathcal{O}(\varphi) e^{-V(\varphi)} \mu(\mathrm{d}\varphi)\text{”}$$

where “$\mu(\mathrm{d}\varphi) = \exp(-Q(\varphi,\varphi))\mathrm{d}\varphi$” is the Gaussian free field,

$$Q(\varphi,\varphi) = \frac{1}{2}\int_{\mathbb{R}^d} (m^2|\varphi(x)|^2 + |\nabla\varphi(x)|^2)\mathrm{d}x$$

positive quadratic form

$$V(\varphi) = \lambda \int_{\mathbb{R}^d} U(\varphi(x))\mathrm{d}x$$

$U : \mathbb{R}^d \longrightarrow \mathbb{R}$ bounded from below

Simplest case: Gaussian Free Field

For $\lambda = 0$,

$$\text{“}\mu(\mathrm{d}\varphi) = \mathrm{e}^{-S_{\mathrm{free}}(\varphi)}\mathrm{d}\varphi\text{”},$$

$$S_{\mathrm{free}}(\varphi) = Q(\varphi, \varphi) = (\varphi, (m^2 - \Delta)\varphi) = \frac{1}{2}\int_{\mathbb{R}^d} (m^2|\varphi(x)|^2 + |\nabla\varphi(x)|^2)\mathrm{d}x,$$

formally corresponds to a **Gaussian measure** on $\mathcal{S}'(\mathbb{R}^d)$ with

$$\mathrm{Cov}(\mu) = (m^2 - \Delta)^{-1},$$

and $\mathrm{supp}(\mu) \subset H^\alpha(\mathbb{R}^d)$ for any $\alpha < \frac{2-d}{2}$

⟶ only for $d = 1$ supported on functions.

⟶ Starting point for more interesting EQFTs

Gibbsian pertubations of the GFF

$$\text{“}\nu(\mathrm{d}\varphi) = \frac{1}{\text{norm.}} \mathrm{e}^{-V(\varphi)} \mu(\mathrm{d}\varphi)\text{”} \quad \text{where} \quad V(\varphi) = \lambda \int_{\mathbb{R}^d} U(\varphi(x))\,\mathrm{d}x$$

Some possible starting points to obtain non-Gaussian models:

- in $d = 2$:

$$U(x) = x^{2p} + \sum_{\ell}^{2p-1} a_\ell x^\ell \quad \text{for any } p > 0,$$
$$U(x) = \exp(\beta x),$$
$$U(x) = \cos(\beta x),$$

- in $d = 2, 3$:

$$U(x) = x^4 - b x^2.$$

Goal: Make sense of

$$\text{“}\,\nu(\mathcal{O}) = \frac{1}{\text{norm.}}\int_{\mathcal{S}'(\mathbb{R}^d)} \mathcal{O}(\varphi)e^{-S(\varphi)}\mathrm{d}\varphi = \frac{1}{\text{norm.}}\int_{\mathcal{S}'(\mathbb{R}^d)} \mathcal{O}(\varphi)e^{-\lambda\int_{\mathbb{R}^d}U(\varphi(x))\mathrm{d}x}\mu(\mathrm{d}\varphi)\text{”},$$

with μ the Gaussian free field, some $U: \mathbb{R} \longrightarrow \mathbb{R}$ and

$$S(\varphi) := \int_{\mathbb{R}^d} \frac{1}{2}(|\nabla\varphi(x)|^2 + m^2|\varphi(x)|^2) + U(\varphi(x))\mathrm{d}x.$$

Problems:

Large Scales: No decay in space: $S(\varphi) = \infty$ at best (non-sense at worst)

Small Scales: μ not supported on function spaces but only on **distributions**

$\longrightarrow U(\varphi(x))$ ill-defined

Approximate Measures

With $V(\varphi) = \int_{\mathbb{R}^d} U(\varphi(x))\mathrm{d}x$, define approximations

$$e^{-V(\varphi)}\mu(\mathrm{d}\varphi) \quad \approx \quad e^{-V^{\rho,\varepsilon}(\varphi)}\mu^{\varepsilon}(\mathrm{d}\varphi),$$

Large Scale Problem	**Small Scale Problem**
$\int_{\mathbb{R}^d} U(\varphi(x))\mathrm{d}x = \infty?$	$\varphi(x) = ???$ for $\varphi \in H^{(2-d)/2-}(\mathbb{R}^d)$
cut-off in space $\rho \in C_c^{\infty}(\mathbb{R}^{\mathrm{d}})$:	Regularise the measure:
	$\mu^{\varepsilon} \longrightarrow \mu$,
$V^{\rho}(\varphi) = \int_{\mathbb{R}^d} \rho(x) U(\varphi(x))\mathrm{d}x$	μ^{ε} supported on a **function** space
	Additionally:
	Choose V^{ε} depending on ε

Question: Can we recover a EQFT?

$$\nu^{\rho,\varepsilon}(\mathcal{O}) = \frac{1}{\text{norm.}} \int_{\mathcal{S}'(\mathbb{R}^d)} \mathcal{O}(\varphi) e^{-V^{\rho,\varepsilon}(\varphi)} \mu^{\varepsilon}(d\varphi)$$

$$\xrightarrow{???}$$

$$\text{“ } \nu(\mathcal{O}) = \frac{1}{\text{norm.}} \int_{\mathcal{S}'(\mathbb{R}^d)} \mathcal{O}(\varphi) e^{-V(\varphi)} \mu(d\varphi) \text{ ”.}$$

Problem: ν is not absolutely continuous w.r.t. the Gaussian free field μ

$\longrightarrow$ Move to different characterisations for $\nu^{\rho,\varepsilon}$ that do not rely on absolute continuity

Stochastic Quantisation: Basic Idea

Starting point:

Given a Gaussian (here μ) and two cut-offs ρ, $\varepsilon > 0$ we can construct $\nu^{\rho,\varepsilon}$

(namely as the Gibbsian perturbation of the GFF)

Think of a map

$$\text{“ } \Phi^{\rho,\varepsilon} \colon \mu \longmapsto \nu^{\rho,\varepsilon} \text{ ”} \qquad \rho \in C_c^\infty(\mathbb{R}^2), \;\; \varepsilon > 0.$$

Idea: Study the maps $\Phi^{\rho,\varepsilon}$ to learn about the measures $\nu^{\rho,\varepsilon}$ and (ideally) remove both regularisations ρ, ε.

Examples: Parabolic $\mathscr{L} = \partial_t - \Delta$ or elliptic $\mathscr{L} = -\Delta$ SQ

$$\mathscr{L}\varphi = \xi - V'(\varphi) \quad \text{for a space time white noise } \xi$$

[G. Parisi, Y. Wu · Perturbation theory without gauge fixing · Sci. Sin. · 1981]

Input:

- a **scale decomposition** $(\int_0^t \dot{G}_s \mathrm{d}s)_{t\geqslant 0} =: (G_t)_{t\geqslant 0}$ of the covariance of the GFF $G_\infty = (m^2 - \Delta)^{-1}$
- a cylindrical Brownian motion $(B_t)_{t\geqslant 0}$ on $L^2(\mathbb{R}^d)$

Output:

Mod suitable UV and IR regularisations, a measure

$$\nu(\mathrm{d}\varphi) = \frac{1}{\text{norm.}} \mathrm{e}^{-\lambda V(\varphi)} \mu(\mathrm{d}\varphi), \quad \text{where} \quad \mu := \mathrm{Law}\left(\int_0^\infty \dot{G}_t^{1/2} \mathrm{d}B_t \right) \sim \mathrm{GFF}.$$

as $\nu := \mathrm{Law}(\varphi_\infty)$, where φ is the solution to

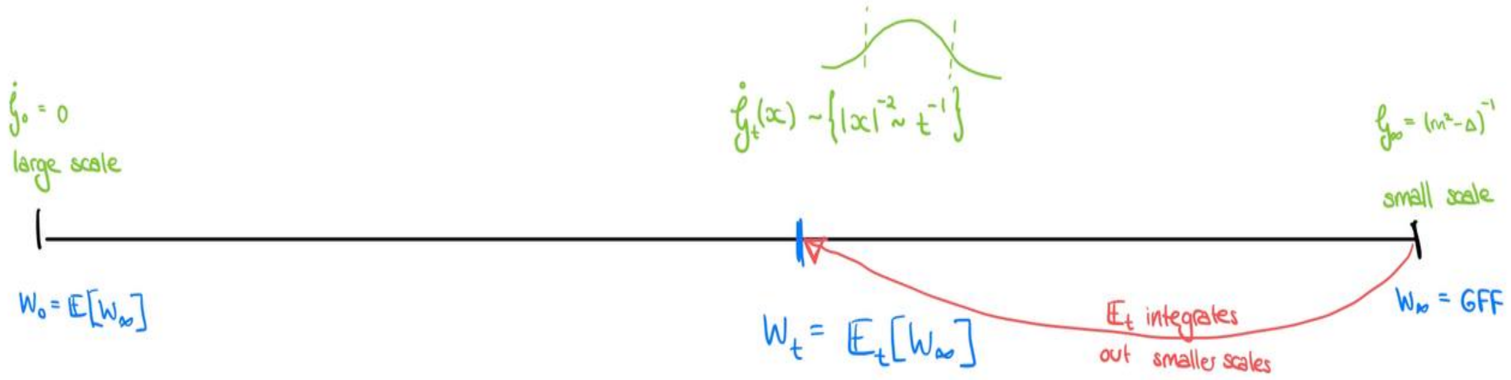
$$\varphi_t = \int_0^t \dot{G}_s \mathbb{E}_s[-\mathrm{D}V(\varphi_\infty)] \mathrm{d}s + W_t, \qquad W_t = \int_0^t \dot{G}_s^{1/2} \mathrm{d}B_s, \qquad t \in [0, \infty]. \tag{1}$$

For $V \equiv 0$,

$$\varphi_t^{\text{free}} = W_t = \int_0^t \dot{G}_s^{1/2} \mathrm{d} B_s, \qquad \mu = \text{Law}(W_\infty)$$

is a martingale: Given the **small scale** description W_∞, we can recover an **effective** description

$$W_t = \mathbb{E}_t[W_\infty] \quad \text{for any scale } t \geqslant 0.$$



Interpretation as RG-flow: Interacting case

For $V \neq 0$,

$$\varphi_t^V = \int_0^t \dot{G}_s \mathbb{E}_s[-\mathrm{D}V(\varphi_\infty)]\mathrm{d}s + \int_0^t \dot{G}_s^{1/2} \mathrm{d}B_s,$$

is no longer a martingale under $\mathbb{P}$, but $\varphi_t \hat{\in} \mathscr{F}_t$ for any $t \geqslant 0$.

That is

$$(Z_t^V, W_t) := \left(\int_0^t \dot{G}_s \mathbb{E}_s[-\mathrm{D}V(\varphi_\infty^V)]\mathrm{d}s, \int_0^t \dot{G}_s^{1/2} \mathrm{d}B_s \right),$$

provides a **scale-by-scale** coupling and an **effective description** of the field φ_∞^V at any scale $t \geqslant 0$.

$\longrightarrow$ The FBSDE describes the dynamics of changing between scales.

Why this approach; aka why listen to me talk?

- Better (physical) interpretability: Explicit description *at every scale* without reference to a limiting procedure (c.f. [BCG · Wilson-Ito fields… arXiv:2307.11580])

  \+ Pathwise, scale-by-scale coupling of the GFF and the EQFT

- No reference to the potential V required, sufficient to study the force $F = -\mathrm{D}V$
- precise control over ν from μ (e.g. decay of correlations, singularity, OS-axioms, LDP, etc)
- Fully non-perturbative: Approximate solutions to the renormalisation flow equation

$$\partial_s F_s + \frac{1}{2}\mathrm{Tr}(\dot{G}_s \mathrm{D}^2 F_s) + \mathrm{D}F_s \dot{G}_s F_s = 0, \quad F_\infty = -\mathrm{D}V_\infty,$$

  are sufficient (**if** you can control the resulting FBSDE).

A case study: sine-Gordon in the first few regions

Stochastic Quantisation Equation: Mod suitable renormalisation, the FBSDE,

$$\varphi_t = \int_0^t \dot{G}_s \mathbb{E}_s[F(\varphi_\infty)]\mathrm{d}s + W_t, \qquad W_t = \int_0^t \dot{G}_s^{1/2} \mathrm{d}B_s, \tag{2}$$

with the cosine interaction (in $d = 2$)

$$V(\varphi) = \lambda \int_{\mathbb{R}^2} \cos(\beta\varphi(x))\mathrm{d}x; \qquad F(\varphi) := -\mathrm{D}V(\varphi) = \lambda\beta \sin(\varphi(x)).$$

is a stochastic quantisation equation for "$\nu_{\mathrm{SG}}(\mathrm{d}\varphi) = \frac{\mathrm{e}^{-\lambda V(\varphi)}}{\text{norm.}}\mu(\mathrm{d}\varphi)$" for $\beta^2 < 6\pi < 8\pi$, $\lambda \in \mathbb{R}$ on the full space $\mathbb{R}^2$ via

$$\mathrm{Law}(\varphi_\infty) = \nu_{\mathrm{SG}}.$$

Moreover: The FBSDE (2) can be effectively used to study properties of the infinite volume measure ν_{SG}.

Effect of $\beta^2 \in [0, 8\pi)$

Formally,

$$[\![\cos(\beta\varphi_\infty)]\!] = [\![\cos(\beta(Z_\infty + W_\infty))]\!] = [\![\cos(\beta W_\infty)]\!]\cos(\beta Z_\infty) - [\![\sin(\beta W_\infty)]\!]\sin(\beta Z_\infty),$$

and

$$\begin{matrix} [\![\cos(\beta W_\infty)]\!] \in C^{-\beta^2/4\pi} \\ \sin(\beta Z_\infty) \in C^{2-\beta^2/4\pi} \end{matrix} \implies [\![\cos(\beta(\varphi_\infty))]\!] \text{ well-defined only for } \beta^2 < 4\pi.$$

treshholds	
$\beta^2 < 4\pi$	Da Prato-Debussche regime
$\beta^2 < 6\pi$	what this work covers
$\beta^2_\infty = 8\pi$	criticality

$\longleftarrow$ c.f. [N.Barashkov · A stochastic control… arXiv:2203.06626]

$\longleftarrow$ c.f. [M.Gubinelli–SJ.M.· The FBSDE approach…arXiv:2401.13648]

Problems to overcome:

1. **Regularisation:** for the large scales $\rho \in C_c^\infty(\mathbb{R}^d)$ and the small scales $T \in [0,\infty)$ and instead study a family of approximate fields $(\varphi_t^{\rho,T})_{t\in[0,T]}$ and pass to the limit.

   ($\longrightarrow$ Will be hidden in this talk; sorry)

2. **Dependence on φ_∞:** The equation at scale $t<\infty$ depends on *all* small-scale information φ_∞,

$$\varphi_t = \int_0^t \dot{G}_s \mathbb{E}_s[F(\varphi_\infty)]\mathrm{d}s + W_t, \qquad W_t = \int_0^t \dot{G}_s^{1/2} \mathrm{d}B_s.$$

   In other words, we are looking for a solution (φ, Y, η) to an **FBSDE**

$$\begin{cases} \varphi_t = \int_0^t \dot{G}_s Y_s \mathrm{d}s + W_t, & \text{solved } \textit{forward} \text{ from } \varphi_0 = 0 \\ Y_t = F(\varphi_\infty) - \int_t^\infty \eta_s \mathrm{d}W_s, & \text{solved } \textit{backwards} \text{ from } Y_\infty = F(\varphi_\infty) \end{cases}$$

Why is 2. a problem?

$$\varphi_t = \int_0^t \dot{G}_s \mathbb{E}_s[F(\varphi_\infty)]\mathrm{d}s + W_t, \qquad W_t = \int_0^t \dot{G}_s^{1/2} \mathrm{d}B_s. \tag{3}$$

Recall $\varphi_\infty \sim W_\infty \in B^\alpha(\mathbb{R}^d)$ for $\alpha < -\frac{d-2}{2} \leqslant 0$ is a **distribution**

i.e. $F(\varphi_\infty)(x) = \lambda\beta \sin(\beta\varphi_\infty(x))$ makes no sense/ will diverge

However: $\mathbb{E}_s[F(\varphi_\infty)] \in \mathcal{F}_s^W$ lives at scale $s < \infty$, meaning it should be a **function**.

Upshot: Need a way to bring down the scale of the force $F(\varphi_\infty)$ from $\infty \rightsquigarrow s$.

How to bring down the scale?

$$\varphi_t = \int_0^t \dot{G}_s \mathbb{E}_s[F(\varphi_\infty)]\mathrm{d}s + W_t, \qquad W_t = \int_0^t \dot{G}_s^{1/2} \mathrm{d}B_s.$$

For a generic **scale dependent** functional $(F_t)_{t\geqslant 0}$ by Ito's formula,

$$F_\infty(\varphi_\infty) = F_t(\varphi_t) + \int_t^\infty \left(\partial_s F_s + \frac{1}{2}\mathrm{Tr}(\dot{G}_s \mathrm{D}^2 F_s) + \mathrm{D}F_s \dot{G}_s F_s\right)(\varphi_s)\mathrm{d}s + \int_t^\infty \mathrm{D}F_s(\varphi_s)\mathrm{d}W_s$$

Therefore if $(F_t)_{t\geqslant 0}$ is a solution to the (backward) RG-flow equation,

$$H_s^F := \partial_s F_s + \frac{1}{2}\mathrm{Tr}(\dot{G}_s \mathrm{D}^2 F_s) + \mathrm{D}F_s \dot{G}_s F_s \equiv 0, \qquad F_\infty = F$$

we find

$$\mathbb{E}_s[F(\varphi_\infty)] = \mathbb{E}_s[F_\infty(\varphi_\infty)] = F_s(\varphi_s).$$

and instead of an **FBSDE** we have the **SDE**

$$\varphi_t = \int_0^t \dot{G}_s F_s(\varphi_s)\mathrm{d}s + W_t.$$

There's a catch…

$$\partial_s F_s + \frac{1}{2}\mathrm{Tr}(\dot{G}_s \mathrm{D}^2 F_s) + \mathrm{D}F_s \dot{G}_s F_s \equiv 0 \qquad F_\infty = F$$

is a **nonlinear** and **infinite dimensional** PDE for a function

$$F_t : \mathcal{S}'(\mathbb{R}^d) \longrightarrow \mathcal{S}'(\mathbb{R}^d), \qquad t \in [0, \infty].$$

Asking for **exact** solutions may be too much!

Instead: Can we find **approximate solutions** $(F_t)_{t\in[0,\infty]}$ such that $F_\infty = F$,

$$H_s^F = \partial_s F_s + \frac{1}{2}\mathrm{Tr}(\dot{G}_s \mathrm{D}^2 F_s) + \mathrm{D}F_s \dot{G}_s F_s \approx 0,$$

is small (in some sense)?

The approximation error

Given **any** $(F_t)_{t\in[0,\infty]}$ define

$$H_s^F := \partial_s F_s + \frac{1}{2}\mathrm{Tr}(\dot{G}_s \mathrm{D}^2 F_s) + \mathrm{D}F_s \dot{G}_s F_s, \quad F_\infty = F$$

and

$$R_t := \mathbb{E}_t[F_\infty(\varphi_\infty) - F_t(\varphi_t)] \quad \text{with } R_\infty = 0.$$

to obtain the FBSDE

$$\begin{cases} \varphi_t = \int_0^t \dot{G}_s(F_s(\varphi_s) + R_s)\mathrm{d}s + W_t \\ R_t = \mathbb{E}_t \int_t^\infty H_s^F(\varphi_s)\mathrm{d}s + \mathbb{E}_t \int_t^\infty \mathrm{D}F_s(\varphi_s)\dot{G}_s R_s \mathrm{d}s. \end{cases}$$

Note: $H_s^F \equiv 0 \Longrightarrow R \equiv 0$, and we recover the SDE as before; but R allows more freedom in the choice of $(F_t)_{t\in[0,\infty)}$.

Two related sub-problems

Reformulated equation: $\varphi_t = Z_t + W_t$

$$\begin{cases} Z_t = \int_0^t \dot{G}_s(F_s(\varphi_s) + R_s)\mathrm{d}s \\ R_t = \mathbb{E}_t \int_t^\infty H_s^F(\varphi_s)\mathrm{d}s + \mathbb{E}_t \int_t^\infty \mathrm{D}F_s(\varphi_s)\dot{G}_s R_s \mathrm{d}s, \end{cases} \qquad t \in [0,\infty] \tag{4}$$

Now the problem comes down to two (related) steps

1. Find a "good enough" approximation $(F_t)_{t\in[0,\infty)}$ to the flow equation which allows for good estimates of $\|F_s(\varphi_s)\|$ uniformly in the (surpressed) regularisation.
2. Show well-posedness for the FBSDE (4) and obtain uniform a-priori estimates which allow to remove the regularisations.

Theorem 1. *Let* $\beta^2 < 6\pi$*. There is a choice* $(F_t)_{t\in[0,\infty)}$ *such that* $F_t(W_t) = \lambda\rho\beta[\![\sin(W_t)]\!] + O(\lambda^2)$ *and there is a solution* $(Z, R) \in L^\infty(\mathrm{d}\mathbb{P}; L^\infty(\mathbb{R}^2)) \times L^\infty(\mathrm{d}\mathbb{P}, L^\infty(\mathbb{R}^2))$ *to the FBSDE*

$$\begin{cases} Z_t = \int_0^t \dot{G}_s(F_s(\varphi_s) + R_s)\mathrm{d}s \\ R_t = \mathbb{E}_t \int_t^\infty H_s^F(\varphi_s)\mathrm{d}s + \mathbb{E}_t \int_t^\infty \mathrm{D}F_s(\varphi_s)\dot{G}_s R_s \mathrm{d}s, \end{cases} \qquad t \in [0, \infty].$$

Moreover,

- *If* $|\lambda| \ll 1$ *or* $\rho \in C_c^\infty(\mathbb{R}^2)$*, then this solution is unique.*
- *Regularity: For any* $\varepsilon > 0$*, and* $p \in [1, \infty]$ *there is a version of the unregularised solution s.t.*

$$Z = Z_\infty \in L^\infty(\mathrm{d}\mathbb{P}; B_{p,p}^\alpha(\langle x\rangle^{-3})), \quad \alpha = 2 - \beta^2/4\pi - \varepsilon,$$

- *Stochastic quantisation: It holds that*

$$\nu_{\mathrm{SG}} = \mathrm{Law}(Z_\infty + W_\infty) = \mathrm{Law}(\varphi_\infty).$$

Here $[\![\cdot]\!]$ *denotes the usual Wick-ordering and* $B_{p,p}^\alpha(w)$ *are the usual weigthed Besov spaces.*

Upshot

The FBSDE can **define** $\nu_{\mathrm{SG}} = \mathrm{Law}(Z_\infty + W_\infty)$ on the full space for $\beta^2 < 6\pi$ without reference to a limiting procedure

$$\begin{cases} Z_t = \int_0^t \dot{G}_s(F_s(\varphi_s) + R_s)\mathrm{d}s, \\ R_t = \mathbb{E}_t \int_t^\infty H_s(\varphi_s)\mathrm{d}s + \mathbb{E}_t \int_t^\infty \mathrm{D}F_s(\varphi_s)\dot{G}_s R_s \mathrm{d}s, \end{cases}$$

and $\boldsymbol{Z_t + W_t}$ is an **effective description** at scale t.

$\longrightarrow$ Can try to transport properties of the GFF $W_\infty \sim \mu$ to the sine-Gordon field $Z_\infty + W_\infty \sim \nu_{\mathrm{SG}}$ along the FBSDE.

- **OS-Axioms:** Verify OS-Axioms from FBSDE (3) + the limit is not Gaussian
- **Decay of correlations:** For two compactly supported observables $\mathcal{O}_1, \mathcal{O}_2$,

$$\mathrm{Cov}_\nu(\mathcal{O}_1, \mathcal{O}_2) \lesssim \mathrm{e}^{-m\gamma\mathfrak{l}}, \quad \mathfrak{l} := d(\mathrm{supp}(\mathcal{O}_1), \mathrm{supp}(\mathcal{O}_2)),$$

  where the implicit constant depends only on regularity constants of $\mathcal{O}_1, \mathcal{O}_2$ and $\gamma = \gamma(m)$.
- **Singularity:** For $\beta^2 \geqslant 4\pi$, ν_{SG}^ρ is *not absolutely continuous* w.r.t. the GFF even in finite volume.
- **Variational description & LDP:** The Laplace-transform of the **infinite volume** measure satisfies a Boue-Dupuis type variational problem (& LDP as a result)

$$\mathcal{W}(f) := -\log \int_{\mathcal{S}'(\mathbb{R}^d)} \mathrm{e}^{-f(\varphi)} \nu_{\mathrm{SG}}(\mathrm{d}\varphi)$$

$$= \inf_{v \in \mathbb{H}^2(\langle x \rangle^n)} \mathbb{E}[f(\bar{\varphi}_\infty(\bar{r} + v)) + \mathscr{H}_s(\varphi(v + \bar{r}), \varphi(\bar{r})) + \mathcal{E}(v, \bar{r})].$$

The effective force: Approximate solutions to the flow equation

Reformulated equation: $\varphi_t = Z_t + W_t$

$$\begin{cases} Z_t = \int_0^t \dot{G}_s (F_s(\varphi_s) + R_s) \mathrm{d}s \\ R_t = \mathbb{E}_t \int_t^\infty H_s^F(\varphi_s) \mathrm{d}s + \mathbb{E}_t \int_t^\infty \mathrm{D}F_s(\varphi_s) \dot{G}_s R_s \mathrm{d}s, \end{cases} \qquad t \in [0, \infty)$$

$$H_t^F := \partial_t F_t + \frac{1}{2} \mathrm{Tr}(\dot{G}_t \mathrm{D}^2 F_t) + \mathrm{D}F_t \dot{G}_t F_t, \tag{5}$$

Now the problem comes down to two steps

1. **Find a “good enough” approximation F to the flow equation (5) which allows for good estimates of $\|F_s(\varphi_s)\|_{L^\infty}$ uniformly in the UV and IR cut-offs.**
2. Show well-posedness for the FBSDE and obtain uniform a-priori estimates which allow to remove the UV and IR regularisations.

Aim: Systematic way to find "good" approximate solutions to

$$\partial_t F_t + \frac{1}{2}\mathrm{Tr}(\dot{G}_t \mathrm{D}^2 F_t) + \mathrm{D}F_t \dot{G}_t F_t \approx 0, \quad F_\infty(\phi) = F(\phi) = \lambda\beta[\![\sin(\beta\phi)]\!].$$

Ansatz: Picard iterations starting from $F^{[0]} \equiv 0$, define inductively

$$\partial_t F_t^{[\ell]} + \frac{1}{2}\mathrm{Tr}(\dot{G}_t \mathrm{D}^2 F_t^{[\ell]}) = \sum_{\ell'+\ell''=\ell} \mathrm{D}F_t^{[\ell']} \dot{G}_t F_t^{[\ell'']}, \qquad \ell > 0,$$

with *terminal* conditions $F_\infty^{[1]} = F$ and $F_\infty^{[\ell]} = 0$ for the levels $\ell > 1$.

Then, with $F_t := F_t^{[\leq \ell^*]} = \sum_{\ell=0}^{\ell^*} F_t^{[\ell]}$ we compute the remainder

$$H_t^F = H_t^{[\leq \ell^*]} = \sum_{\substack{\ell', \ell'' < \ell^* \\ \ell'+\ell'' > \ell^*}} \mathrm{D}F_t^{[\ell']} \dot{G}_t F_t^{[\ell'']}.$$

Heuristics: What bounds to expect

For concreteness, suppose that $\dot{G}_t = t^{-2} e^{-(m^2-\Delta)/t}$ so that

$$\|\dot{G}_t\|_{L^1} \lesssim \langle t \rangle^{-2}.$$

Heuristically, the bounds

$$\|F_t^{[\ell]}(\varphi_t)\|_{L^\infty} + \|\mathrm{D}F_t^{[\ell]}(\varphi_t)\|_{L^\infty} \lesssim \langle t \rangle^{-\delta\ell+1} \quad \text{where } \delta = 1 - \frac{\beta^2}{8\pi},$$

propagate: they are compatible with

$$\partial_t F_t^{[\ell]} + \frac{1}{2}\mathrm{Tr}(\dot{G}_t \mathrm{D}^2 F_t^{[\ell]}) = \sum_{\ell'+\ell''=\ell} \mathrm{D}F_t^{[\ell']} \dot{G}_t F_t^{[\ell'']},$$

since

$$\|F_t^{[\ell]}(\varphi_t)\|_{L^\infty} \leqslant \sum_{\ell'+\ell''=\ell} \int_t^\infty \mathrm{d}s \, \|\mathrm{D}F_t^{[\ell']}(\varphi_t)\|_{L^\infty} \|\dot{G}_t\|_{L^1} \|F_t^{[\ell'']}(\varphi_t)\|_{L^\infty} \lesssim \int_t^\infty \mathrm{d}s \, \langle s \rangle^{-\delta\ell} \overset{!}{\lesssim} \langle t \rangle^{-\delta\ell+1}.$$

Heuristics: Why should this approximation work?

Assuming for the moment,

$$\|F_t^{[\ell]}(\varphi_t)\|_{L^\infty} + \|\mathrm{D}F_t^{[\ell]}(\varphi_t)\|_{L^\infty} \lesssim \langle t\rangle^{-\delta\ell+1} \quad \text{where } \delta = 1 - \frac{\beta^2}{8\pi}, \tag{6}$$

since $R_t = \mathbb{E}_t \int_t^\infty H_s^{[\leqslant \ell^*]}(\varphi_s)\mathrm{d}s + \mathbb{E}_t \int_t^\infty \mathrm{D}F_s^{[\leqslant \ell^*]} \dot{G}_s R_s \mathrm{d}s$ we require

$$\|H_t^{[\leqslant \ell^*]}(\varphi_t)\|_{L^\infty} \lesssim \sum_{\ell' + \ell'' > \ell^*} \|\mathrm{D}F_t^{[\ell']}(\varphi_t)\|_{L^\infty} \|\dot{G}_t\|_{L^1} \|F_t^{[\ell'']}(\varphi_t)\|_{L^\infty} \lesssim \langle t\rangle^{-\delta(\ell^*+1)} \overset{!}{\in} L^1(\mathbb{R}_+) \Longleftrightarrow \beta^2 \leqslant \beta_{\ell^*}^2 := \frac{\ell^*}{\ell^*+1} 8\pi$$

Problems to overcome:

- for $\ell < \ell^*$, we cannot propagate the bounds along the flow from ∞,

  $\longrightarrow$ this is solved by *localisation+renormalisation*

- The bounds on $F_t^{[\ell]}$ and $H_t^{[\leqslant \ell^*]}$ in general depend polynomially on $\nabla\varphi$ (degree depending on ℓ, ℓ^*).

Starting point: $F_t^{[1]}(\varphi) = -\lambda_t \beta \sin(\beta\varphi(x)) = \frac{\lambda_t}{2i}[e^{i\beta\varphi(x)} - e^{-i\beta\varphi(x)}]$ and thus

$$\partial_t F_t^{[\ell]} + \mathrm{Tr}(\dot{G}_t D^2 F_t^{[\ell]}) = \sum_{\ell'+\ell''=\ell} DF_t^{[\ell']}\dot{G}_t F_t^{[\ell'']}, \qquad \ell > 0, \quad F_\infty^{[1]} = F \text{ and } F_\infty^{[\ell]} = 0 \text{ for the levels } \ell > 1.$$

Naturally reproduces functionals of the form

$$F_t^{[\ell]}(\varphi)(x_1) = \sum_{\sigma_1,\ldots\sigma_\ell \in \{\pm 1\}} \int dx_2 \ldots \int dx_\ell f_t^{[\ell]}(x_1,\ldots,x_\ell) e^{i\sigma_1\beta\varphi(x_1)} \ldots e^{i\sigma_\ell\beta\varphi(x_\ell)}.$$

$\longrightarrow$ flow equation for the coefficients $f^{[\ell]}$, try to estimate these kernels in a suitable norm and recover estimates for $F^{[\ell]}$.

Fully inductive procedure: produces bounds in the full subcritical regime.

Where's the catch?

Finding approximate solutions to the flow equation is only the first step:

One still needs to show well-posedness for the resulting FBSDE,

$$\begin{cases} \varphi_t = \int_0^t \dot{G}_s(F_s^{[\leqslant \ell^*]}(\varphi_s) + R_s)\mathrm{d}s + \int_0^t \dot{G}_s^{1/2}\mathrm{d}B_s, \\ R_t = \mathbb{E}_t \int_t^\infty H_s^{[\leqslant \ell^*]}(\varphi_s)\mathrm{d}s + \mathbb{E}_t \int_t^\infty \mathrm{D}F_s^{[\leqslant \ell^*]}\dot{G}_s R_s \mathrm{d}s. \end{cases}$$

In general, the argument described yields, for $\delta = 1 - \beta^2/8\pi$

$$\|F_t(\varphi)\|_{L^\infty} \lesssim \sum_{\ell \leqslant \ell^*} \langle t \rangle^{1-\ell\delta} (1 + \langle t \rangle^{-1} \|\nabla\varphi\|^2 + \langle t \rangle^{-1} \|\nabla\varphi\|)^\ell,$$

which becomes more and more nonlinear as $\beta^2 \longrightarrow 8\pi$, and $\ell^* = \ell^*(\beta) \longrightarrow \infty$.

Some interesting problems: sine-Gordon

Can we cover a wider parameter range $\beta \in [0, \beta^*)$ for some $(\beta^*)^2 > 6\pi$?

- For $\beta^2 \in (0, 8\pi)$: model is known to be renormalisable but with **infinitely many thresh-holds** requiring additional renormalisations. [G. Benfatto, G. Gallavotti, F. Nicoló, et al.· On the massive sine-Gordon equation in {the first few/ higher/ all} regions of collapse · Comm. math. phys. {1982/ 1983/ 1986}]
- Many **partial** results, valid for different ranges of λ, β or finite/infinite volume. **But:** No works covering the full subcritical regime on the full space for all λ.

Smaller questions

Can we remove some of the smallness assumptions on the coupling constant λ?

i.e. Variational problem for any λ? OS-Axioms for any λ?

More generally:

- Can we make this approach work also for other models, e.g. φ^4_3?

  $\longrightarrow$ Requires **global** solutions for the resulting FBSDE and thus strong a priori estimates for FBSDEs

- The FBSDE only uses the **force** $F = -\mathrm{D}V$; we never need to reference the **potential** V,

  $\longrightarrow$ Can we consider non-scalar models, where the potential does not make sense?

- General solution theory for these kinds of FBSDEs? Are there more general conditions about the existence/uniqueness of solutions?

Variational description on $\mathbb{R}^2$:

The Laplace transform

$$\mathcal{W}(f) := -\log \int_{\mathcal{S}'(\mathbb{R}^d)} \mathrm{e}^{-f(\varphi)} \nu_{\mathrm{SG}}(\mathrm{d}\varphi),$$

satisfies

$$\mathcal{W}(f) = \inf_{v \in \mathbb{H}^2(\langle x \rangle^n)} \mathbb{E}\Big[f(\bar{\varphi}_\infty(\bar{r}+v)) + \int_0^\infty [\mathscr{H}_s(\bar{\varphi}_s(\bar{r}+v)) - \mathscr{H}_s(\bar{\varphi}_s(v))] \mathrm{d}s + \mathcal{E}(v, \bar{r}) \Big],$$

where

- $\mathcal{E}$ is a quadratic form on $L^2(\mathbb{R}^2)$,
- $\mathscr{H}$ is a function of V, formally given by

$$\mathscr{H}_t(\varphi) = \left(\partial_t V_t + \mathrm{Tr}(\dot{G}_t \mathrm{D} V_t) + \frac{1}{2} \mathrm{D} V_t \dot{G}_t \mathrm{D} V_t \right)(\varphi).$$

- $\bar{\varphi}(v)$ is the solution to

$$\bar{\varphi}_t(v) = \int_0^t \mathrm{d}s \, \dot{G}_s(F_s(\bar{\varphi}_s(v))) + \int_0^t \mathrm{d}s \, Q_s v_s + W_t,$$

- $\bar{r}_t = \dot{G}_t^{1/2} R_t$ where R_t is the solution to the FBSDE.

Laplace principle for $\hbar \longrightarrow 0$:

Let $\mu^{\hbar}$ be Gaussian with $\mathrm{Cov}(\mu) = \hbar(m^2 - \Delta)^{-1}$ and (formally) define

$$\nu_{\mathrm{SG}}^{\hbar}(\mathrm{d}\varphi) = \frac{\exp(\hbar^{-1} V(\varphi))}{\text{norm.}} \mu^{\hbar}(\mathrm{d}\varphi).$$

Then, $\nu_{\mathrm{SG}}^{\hbar}$ satisfies a Laplace principle with good rate function

$$I(\varphi) := \begin{cases} \lambda \int_{\mathbb{R}^2} (\cos(\beta\varphi) - 1) + \frac{1}{2} \int_{\mathbb{R}^2} \varphi (m^2 - \Delta) \varphi, & \text{for } \varphi \in H^1(\mathbb{R}^2) \\ \infty & \text{otherwise.} \end{cases}$$

i.e.

$$\lim_{\hbar \longrightarrow 0} - \hbar \log \nu_{\mathrm{SG}}^{\hbar}(\mathrm{e}^{-\hbar^{-1} f}) = \inf_{\varphi \in H^1} \{ f(\varphi) + I(\varphi) \}.$$