Quantization and Index Theory

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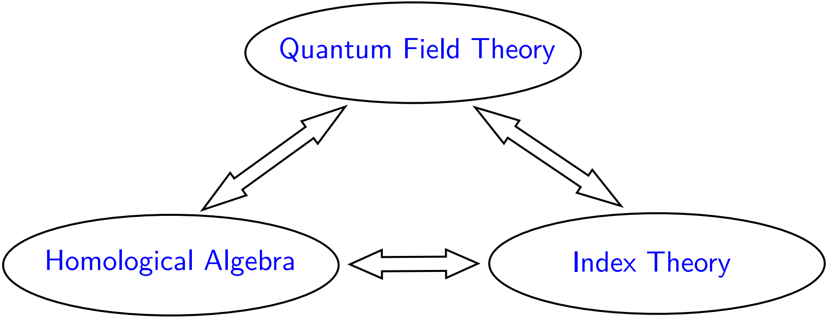
2024 Algebraic, analytic, geometric structures emerging from quantum field theory @ Chengdu


Quantum Field Theory
Homological Algebra
Index Theory

infinite dim geometry $\Longrightarrow$ finite dim geometry

QFT *Math*

Typically, one starts with a path integral in quantum field theory

$$\int_{\mathcal{E}} e^{iS/\hbar}$$

In good situations (e.g. when supersymmetry exists), the problematic path integral is localized to a well-defined integral

$$\int_{\mathcal{E}} e^{iS/\hbar} = \int_{\mathcal{M}} (-)$$

over a finite dim $\mathcal{M} \subset \mathcal{E}$. $\mathcal{M}$ is some interesting moduli space.

Example: Topological QM leads to a path integral on loop space

$$\int_{\mathrm{Map}(S^1,X)} e^{-S/\hbar} \quad \overset{\hbar\to 0}{\Longrightarrow} \quad \int_X (\text{curvatures})$$

Topological nature implies the exact semi-classical limit $\hbar \to 0$, which localizes the path integral to constant loops.

- LHS= the analytic index
- RHS= the topological index

This is the physics "derivation" of Atiyah-Singer Index Theorem (**Alvarez-Gaumé, Friedan-Windey, Witten**).

Witten's "Index Theorem" on loop space

Replace S^1 by an elliptic curve E:

$$\int_{\mathrm{Map}(E,X)} e^{-S/\hbar} \quad \overset{\hbar\to 0}{\Longrightarrow}$$

Intuitively, if we view

$$\mathrm{Map}(E, X) = \mathrm{Map}(S^1, LX)$$

as defining a quantum mechanics on LX, then this leads to **Witten**'s proposal for index of dirac operators on loop space.

We will illustrate a homological algebraic aspect of this in this talk.

Let us first explain how some notions of homological algebra in noncommutative geometry arise naturally in quantum field theory.

Hochschild-Kostant-Rosenberg = Renormalization Group Flow

A: associative algebra. Hochschild chain complex

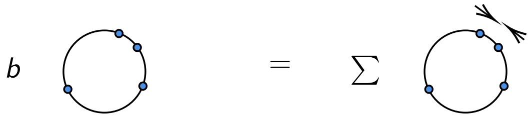
$$(C_\bullet(A), b) = \cdots C_p(A) \xrightarrow{b} C_{p-1}(A) \to \cdots \to C_1(A) \xrightarrow{b} C_0(A)$$

where

$$C_p(A) := A^{\otimes p+1}$$

The Hochschild differential b is

$$b(a_0 \otimes \cdots \otimes a_p) = a_0 a_1 \otimes \cdots \otimes a_p - a_0 \otimes a_1 a_2 \otimes \cdots \otimes a_p + \cdots + (-1)^{p-1} a_0 \otimes a_1 \otimes \cdots \otimes a_{p-1} a_p + (-1)^p a_p a_0 \otimes \cdots \otimes a_{p-1}.$$



Hochschild Homology $HH_\bullet(A) = H_\bullet(C_\bullet(A), b)$

Hochschild-Kostant-Rosenberg (HKR) Theorem

Let $A = k[x_1, \cdots, x_n]$. The HKR Theorem says

$$HH_\bullet(A) = \Omega^\bullet(A) = k[x_i][dx_i].$$

It can be realized by an explicit HKR map of chain complexes

$$\sigma : (C_\bullet(A), b) \to (\Omega^\bullet(A), 0)$$
$$a_0 \otimes a_1 \otimes \cdots \otimes a_p \to a_0 da_1 \wedge \cdots \wedge da_p$$

In general, Hochschild chain complex describes the analogue of differential forms in noncommutative geometry.

Quantized HKR map

Let us consider the Weyl algebra

$$W_{2n}^{\hbar} := \left(\mathbb{R}[x^i, p_i][\hbar], \star\right) \qquad \star = \text{Moyal-Weyl product}$$

The canonical quantization condition $[x^i, p_i]_\star = \hbar$ holds.

$$\mathbb{R}[x^i, p_i] \quad \xleftarrow{0 \leftarrow \hbar} \quad W_{2n}^{\hbar} \quad \xrightarrow{\hbar \to 1} \quad \mathbb{R}\left\langle x^i, \partial_{x^i} \right\rangle$$

Quantized HKR: there exists a quasi-isomorphism

$$\sigma^{\hbar} : (C_\bullet(W_{2n}^{\hbar}), b) \to (\Omega_{2n}^\bullet, \hbar\Delta)$$

where $\Omega_{2n}^\bullet = \mathbb{R}[x^i, p_i, dx^i, dp_i]$. $\Delta = \mathcal{L}_\Pi$ is the Lie-derivative w.r.t.

$$\Pi = \sum_i \partial_{x^i} \wedge \partial_{p_i}.$$

Q: What is an explicit formula of $\sigma^{\hbar}$?

Algebraic Index Theorem

Given a deformation quantization $\mathcal{A}_\hbar(M) = (C^\infty(M)[\![\hbar]\!], \star)$ on a symplectic manifold (M, ω), there exists a unique linear map

$$\mathrm{Tr} : \mathcal{A}_\hbar(M) \to \mathbb{C}(\!(\hbar)\!)$$

satisfying a normalization condition and the trace property

$$\mathrm{Tr}(f \star g) = \mathrm{Tr}(g \star f).$$

Then

$$\mathrm{Tr}(1) = \int_M e^{\omega_\hbar/\hbar} \hat{A}(M).$$

This is the algebraic index theorem which was first formulated by **Fedosov** and **Nest-Tsygan** as the algebraic analogue of Atiyah-Singer index theorem.

Topological Quantum Mechanics (TQM)

Consider the locally ringed space

$$S^1_{dR} = (S^1, \quad \mathcal{O} = \Omega^\bullet_{S^1})$$

We consider the space of maps

$$\varphi : S^1_{dR} \to \mathbb{R}^{2n}$$

Each φ can be described by $2n$ functions on S^1_{dR}

$$\varphi = (\mathbb{X}^1, \cdots, \mathbb{X}^n, \mathbb{P}_1, \cdots, \mathbb{P}_n), \qquad \mathbb{X}^i, \mathbb{P}_i \in \mathcal{O}(S^1_{dR}) = \Omega^\bullet_{S^1}.$$

Define the action functional of free TQM

$$S[\varphi] = \sum_i \int_{S^1} \mathbb{X}^i d\mathbb{P}_i$$

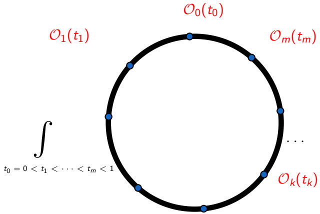
For any $\mathcal{O} \in \mathbb{R}[x^i, p_i]$, we define (using $\varphi : S^1_{dR} \to \mathbb{R}^{2n}$)

$$\varphi^* \mathcal{O} = \mathcal{O}(\mathbb{X}^i, \mathbb{P}_i) = \mathcal{O}^{(0)}(t) + \mathcal{O}^{(1)}(t) dt.$$

Consider the correlation via configuration space integral

$$\langle \mathcal{O}_0 \otimes \mathcal{O}_1 \cdots \otimes \mathcal{O}_m \rangle_{1d} \qquad \mathcal{O}_i \in \mathbb{R}[x^i, p_i]$$

$$:= \int_{t_0 = 0 < t_1 < \cdots < t_m < 1} \left\langle \mathcal{O}_0^{(0)}(t_0) \mathcal{O}_1^{(1)}(t_1) \cdots \mathcal{O}_m^{(1)}(t_m) \right\rangle_{free}$$



Theorem (**Gui-L-Xu**, CMP 2021)

The correlation map in TQM intertwines

$$\begin{array}{rcl} \langle - \rangle_{1d} : C_\bullet(W_{2n}^{\hbar}) & \to & \Omega^\bullet_{2n}((\hbar)) \\ b & \to & \hbar\Delta = \hbar\mathcal{L}_\Pi \\ B & \to & d_{2n} \end{array}$$

It particular, it gives an explicit formula of quantized HKR map.

Such map can be glued on a symplectic manifold X, leading to

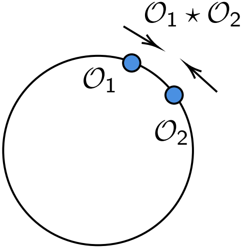
- a trace map on deformation quantized algebra (**Feigin-Felder-Shoikhet** formula).
- TQM $\Longrightarrow$ algebraic index theorem (**Grady-Li-L**, **Gui-L-Xu**)
- Symplectic orbifolds (**L-Peng**)

Topological Quantum Mechanics proves Algebraic Index!

2d Chiral CFT and elliptic chiral index

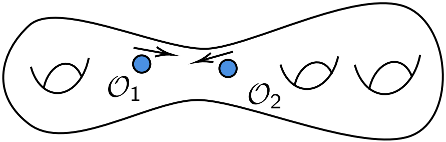
1d TQM	2d Chiral CFT
S^1	Σ
Associative algebra	Vertex operator algebra

Associative product



Operator product expansion

$$\mathcal{O}_1(z)\mathcal{O}_2(w) \sim \sum_n \frac{\mathcal{O}_{1(n)}\mathcal{O}_2(w)}{(z-w)^{n+1}}$$



Example: $\beta\gamma - bc$ system

The VOA $\mathcal{V}^{\beta\gamma-bc}$ of $\beta\gamma - bc$ system is the chiral analogue of Weyl/Clifford algebra.

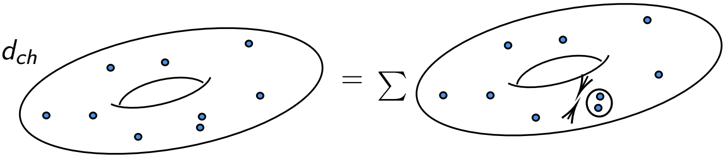
$$\beta(z)\gamma(w) \sim \frac{1}{z-w} + \text{reg.} \qquad b(z)c(w) \sim \frac{1}{z-w} + \text{reg.}.$$

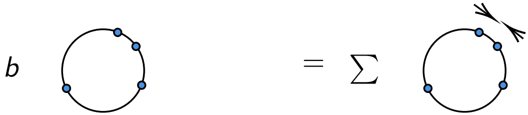
It gives rise to a chiral algebra (in the sense of Beilinson and Drinfeld) $\mathcal{A}^{\beta\gamma-bc} = \mathcal{V}^{\beta\gamma-bc} \otimes_{\mathcal{O}_x} \omega_X$ on a Riemann surface $X = \Sigma$.

Elliptic chiral homology

- In [**Zhu**, 1994], **Zhu** studied the space of genus 1 conformal block (the 0-th elliptic chiral homology) and establish the modular invariance for certain class of VOA.
- **Beilinson** and **Drinfeld** define the chiral homology for general algebraic curves using the Chevalley-Cousin complex.
- Recently, [**Ekeren-Heluani**,2018,2021]: an explicit complex expressing the 0th and 1st elliptic chiral homology.

Intuitively, the chiral differential in the chiral complex looks like a 2d chiral analogue of the Hochschild differential b.





Theorem (Gui-L, 2021)

Let $E_\tau = \mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}\tau$. We can construct an explicit map

$$\langle - \rangle_{2d} : C^{\mathrm{ch}}(E_\tau, \mathcal{A}^{\beta\gamma-bc}) \to \mathcal{A}_{2d}(\!(\hbar)\!)$$

which intertwines the chiral differential d_{ch} with $\hbar\Delta$

$$\langle \mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_n \rangle_{2d} := \fint_{E_\tau^n} \langle \mathcal{O}_1(z_1) \cdots \mathcal{O}_n(z_n) \rangle .$$

- $\mathcal{A}_{2d}$ are functions on zero modes (=copies of $H^\bullet(E_\tau, \mathcal{O}_{E_\tau})$).
- $\langle - \rangle_{2d}$ is a quasi-isomorphism. Chiral analogue of HKR.
- $\langle \mathcal{O}_1(z_1) \cdots \mathcal{O}_n(z_n) \rangle$ is local correlation (via Feynman rules).
- $\fint$ is the regularized integral introduced by [**L-Zhou**, CMP 2021]. Geometric renormalization method for 2d chiral QFT.
- The BV trace map leads to Witten genus.

The issue of singular integral and renormalization

We need to understand the integral of local correlators

$$\int_{\Sigma^n} \langle \mathcal{O}_1(z_1) \cdots \mathcal{O}_n(z_n) \rangle \text{''} \stackrel{?}{=} \text{''}$$

Unlike the situation in topological field theory, $\langle \mathcal{O}_1(z_1) \cdots \mathcal{O}_n(z_n) \rangle$ is very singular along diagonals and there is no way to extend it to certain compactification of $\mathrm{Conf}_n(\Sigma)$.

Regularized integral (L-Zhou 2020)

Let us first consider the integral of a 2-form ω on Σ with meromorphic poles of arbitrary orders along a finite subset $D \subset \Sigma$. Locally we can write $\omega = \frac{\eta}{z^n}$ where η is smooth 2-form and $n \in \mathbb{Z}$.

We can decompose ω into

$$\omega = \alpha + \partial\beta$$

where α is a 2-form with at most logarithmic pole along D, β is a $(0,1)$-form with arbitrary order of poles along D, and $\partial = dz\frac{\partial}{\partial z}$ is the holomorphic de Rham. We define the regularized integral

$$\boxed{\fint_\Sigma \omega := \int_\Sigma \alpha + \int_{\partial\Sigma} \beta}$$

This does not depend on the choice of the decomposition.

The regularized integral can be viewed as a "homological integration" by the holomorphic de Rham ∂

$$\fint_{\Sigma} \partial(-) = \int_{\partial\Sigma} (-).$$

The $\bar{\partial}$-operator intertwines the residue

$$\fint_{\Sigma} \bar{\partial}(-) = \mathrm{Res}(-).$$

In general, we can define

$$\fint_{\Sigma^n} (-) := \fint_{\Sigma} \fint_{\Sigma} \cdots \fint_{\Sigma} (-) \,.$$

This gives a rigorous and intrinsic definition of

$$\langle \mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_n \rangle_{2d} := \fint_{\Sigma^n} \langle \mathcal{O}_1(z_1) \cdots \mathcal{O}_n(z_n) \rangle .$$

Elliptic chiral index (after Douglas-Dijkgraaf)

The partition function of a chiral deformation by a chiral lagrangian $\mathcal{L}$ is given by

$$\left\langle e^{\frac{1}{\hbar}\int_\Sigma \mathcal{L}} \right\rangle_{2d}.$$

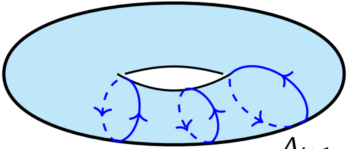
If we quantize the theory on the elliptic curve E_τ,

$$\boxed{\lim_{\bar\tau\to\infty} \left\langle e^{\frac{1}{\hbar}\int_{E_\tau} \mathcal{L}} \right\rangle_{2d} = \mathrm{Tr}_{\mathcal{H}}\, q^{L_0-\frac{c}{24}} e^{\frac{1}{\hbar}\oint dz\mathcal{L}}, \quad q = e^{2\pi i\tau}}$$

where the operation $\lim_{\bar\tau\to\infty}$ sends

almost holomorphic modular forms $\Longrightarrow$ quasi-modular forms.

This can be viewed as a chiral algebraic index. The regularized integral [**L-Zhou**] precisely explains $\bar\tau \to \infty$.



Theorem (L-Zhou, CMP 2021)

Let $\Phi(z_1, \cdots, z_n; \tau)$ be a meromorphic elliptic function on $\mathbb{C}^n \times \mathbf{H}$ which is holomorphic away from diagonals. Let $A_1, \cdots, A_n$ be n disjoint A-cycles on E_τ. Then the regularized integral

$$\fint_{E_\tau^n} \left(\prod_{i=1}^n \frac{d^2 z_i}{\operatorname{im} \tau} \right) \Phi(z_1, \cdots, z_n; \tau) \quad \textit{lies in} \quad \mathcal{O}_{\mathbf{H}}[\frac{1}{\operatorname{im} \tau}] \quad \textit{and}$$

$$\boxed{\lim_{\bar\tau \to \infty} \fint_{E_\tau^n} \left(\prod_{i=1}^n \frac{d^2 z_i}{\operatorname{im} \tau} \right) \Phi(z_1, \cdots, z_n; \tau) = \frac{1}{n!} \sum_{\sigma \in S_n} \int_{A_1} dz_{\sigma(1)} \cdots \int_{A_n} dz_{\sigma(n)} \Phi(z_1, \cdots, z_n; \tau)}.$$

In particular, $\fint_{E_\tau^n}$ gives a geometric modular completion for quasi-modular forms arising from A-cycle integrals. Higher order terms in $\frac{1}{\operatorname{im} \tau}$ can be obtained via holomorphic anomaly equation.

Algebraic Index vs Elliptic Chiral Index

1d TQM	2d Chiral CFT
Associative algebra	Vertex operator algebra
Hochschild homology	Chiral homology
QME: $(\hbar\Delta + b)\langle - \rangle_{1d} = 0$	QME: $(\hbar\Delta + d_{ch})\langle - \rangle_{2d} = 0$
$\langle \mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_n \rangle_{1d} =$integrals on the compactified configuration spaces of S^1	$\langle \mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_n \rangle_{2d} =$regularized integrals of singular forms on Σ^n
Algebraic Index	Elliptic Chiral Algebraic Index

Joint work with **Zhengping Gui**. arXiv:2112.14572 [math.QA]

Application: Higher genus mirror symmetry

Quantum B-twisted topological string-field theory on general Calabi-Yau is formulated by **Costello-L** (2012, 2015, 2016) generalizing **Bershadsky-Cecotti-Ooguri-Vafa**'s Kodaira-Spencer gravity on Calabi-Yau 3-folds. We call it

quantum BCOV theory.

It is conjectured to be mirror to higher genus Gromov-Witten invariants in the holomorphic limit.

It has an extension by coupling with holomorphic Chern-Simons theory in the large N limit, leading to open-closed BCOV theory.

Ref: [**Costello-L**, ATMP 2020]

Quantum BCOV theory on elliptic curves is completely solved (**L**, JDG 2023) by the chiral deformation of free chiral boson

$$S = \int \partial\phi \wedge \bar{\partial}\phi + \sum_{k \geq 0} \int \eta_k \frac{W^{(k+2)}(\partial_z \phi)}{k+2}$$

where

$$W^{(k)}(\partial_z \phi) = (\partial_z \phi)^k + O(\hbar)$$

are the bosonic realization of the $W_{1+\infty}$-algebra.

Higher genus mirror symmetry on elliptic curves

- The chiral index of quantum BCOV theory is

$$\mathrm{Ind}^{\mathrm{BCOV}}(E_\tau) = \mathrm{Tr}\, q^{L_0 - \frac{1}{24}} e^{\frac{1}{\hbar}\sum_{k\geq 0}\oint_A \eta_k \frac{W^{(k+2)}}{k+2}}$$

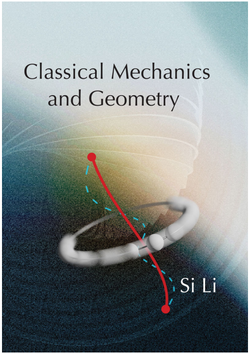
- The chiral index coincides with the stationary Gromov-Witten invariants on the mirror elliptic curve computed by **Dijkgraaf** and **Okounkov-Pandharipande**.

$$\boxed{\mathrm{Ind}^{\mathrm{BCOV}}(E_\tau) = \langle \mathrm{Stationary} \rangle_E^{\mathrm{GW}}}$$

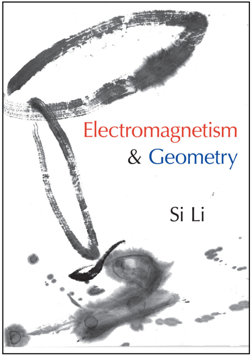
In this case, we find [**L**, JDG 2023]

Quantum Mirror Symmetry=Boson-Fermion Correspondence.

Available at: https://sili-math.github.io/



This is a drawing by my daughter expressing herself with her brother. I found it interesting as it illustrates precisely the essence of electromagnetism on the coupling of electric and magnetic fields as well as the topological nature of Maxwell's equations.



Thank you!