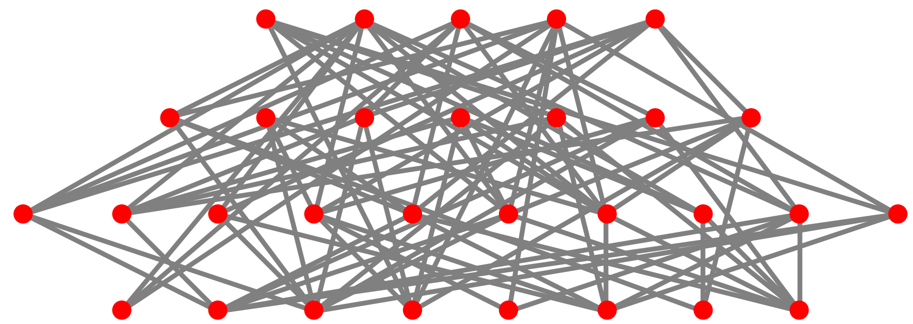
Quantum Dynamics of Causal Sets



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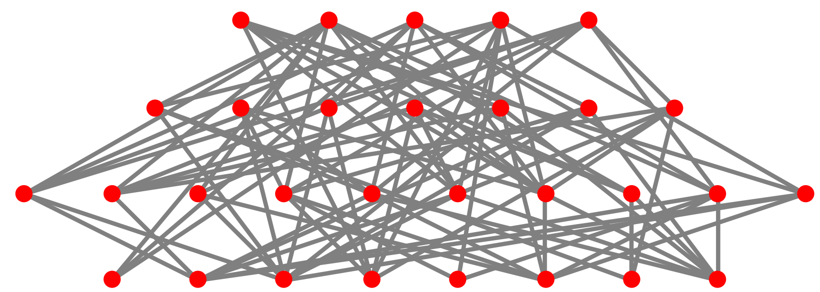
RRI

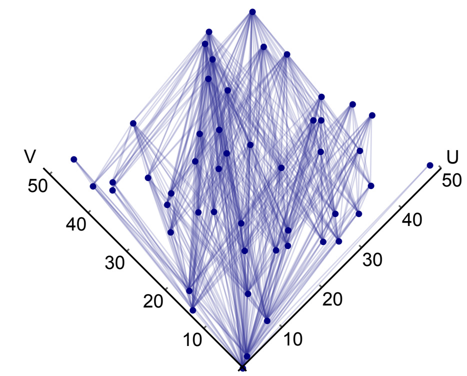
Algebraic, analytic, geometric structures emerging from quantum field theory
Sichuan University,
March, 2024

Outline



- Posets and Lorentzian geometry: an introduction

- Causal Sets: A route to quantising spacetime



- The quantum partition function: entropy versus action

- Sequential growth dynamics and quantum vector measures

Work done in collaboration with S. Carlip, P. Carlip, F. Dowker, S. Johnston, A. Mathur, A. A. Singh, S. Zalel

Posets and Lorentzian Geometry

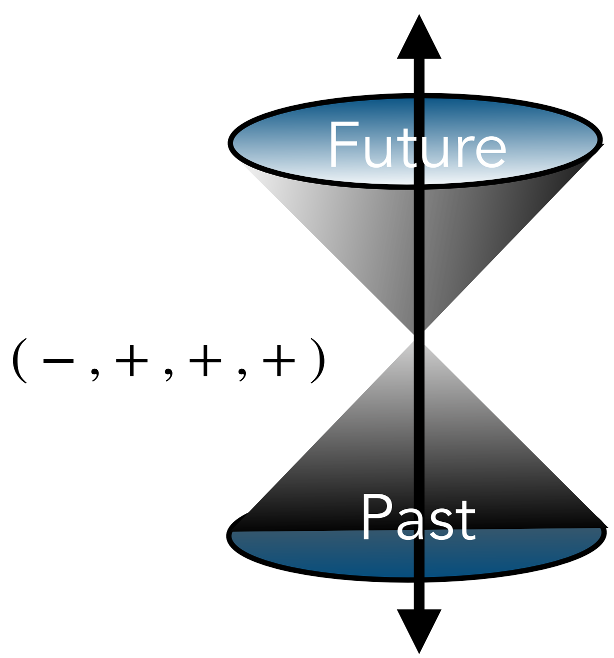
- Spacetime is a Lorentzian manifold (M, g), where g has signature $(-,+,+,+)$

- $ds^2 = g_{ab}dx^a dx^b$ can be positive, negative or zero

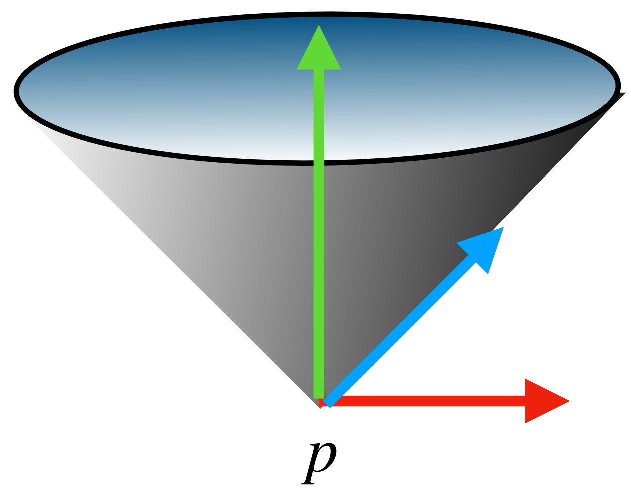
Example: Minkowski/Flat Spacetime

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

- At every point $p \in M$, the vectors in T_pM are arranged in ``lightcones" to the past and the future.



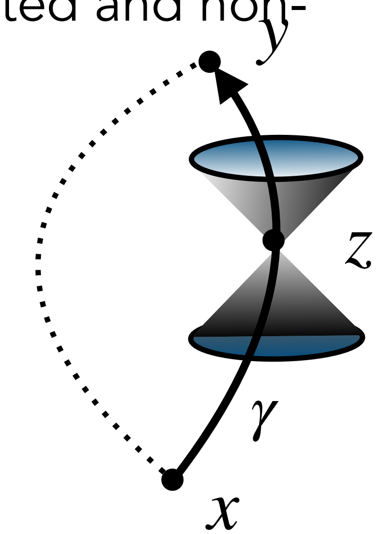
- Vectors are either
  - future or past timelike $(ds^2 < 0)$
  - future or past null/lightlike $(ds^2 = 0)$
  - spacelike $(ds^2 > 0)$



Posets and Lorentzian Geometry

Defines an order relations on M : $\prec$ **(causality relation) and** $\prec\!\!\prec$ **(chronology relation)**

- $x \prec y$ if there exists a curve γ from x to y whose tangent everywhere is future-directed and non-spacelike.
  - $\prec$ is **transitive:** if $x \prec z$ and $z \prec y \Rightarrow x \prec y$
  - If (M, g) is a **causal** spacetime, $\prec$ is **acyclic :** $x \prec y \Rightarrow y \nprec x$

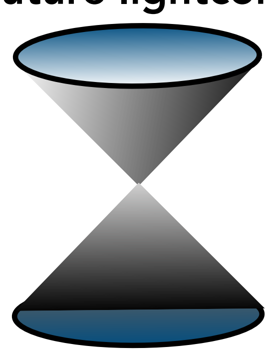


Principle of Causality

Acyclic: $x \prec y \Rightarrow y \nprec x$
Transitive: $x \prec y, y \prec z \Rightarrow x \prec z$

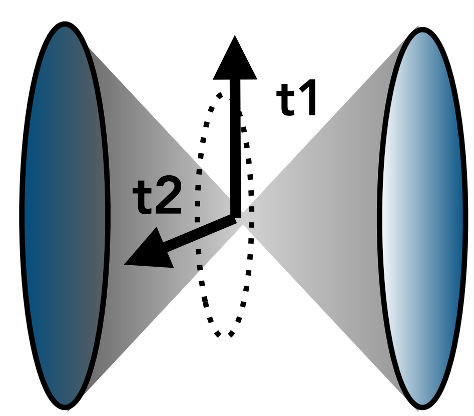
$\Rightarrow$ $(M, \prec)$ is a **partially ordered set**

Future lightcone



Past lightcone

$(-,-,+,+)$ has no associated poset



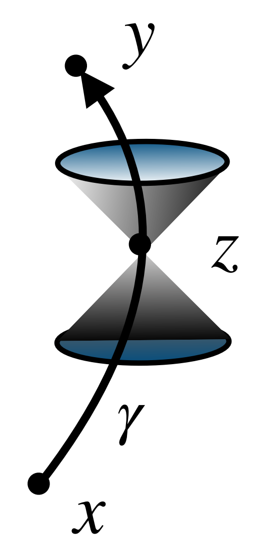
Robb, 1914, "A theory of time and space"

A quick review of terminology

▸ **Causal relation** $\prec$: Tangent to γ from x to y is everywhere either timelike or null

▸ **Chronological relation** $\ll$: Tangent to γ from x to y is everywhere timelike

▸ Both $(M, \prec)$ and $(M, \ll)$ are posets



- Causal Future and Past: $J^+(x) = \{y \in M \,|\, x \prec y\}, \quad J^-(x) = \{y \in M \,|\, y \prec x\}$

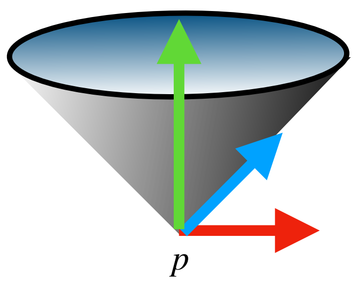
- Chronological Future and Past: $I^+(x) = \{y \in M \,|\, x \ll y\}, \quad I^-(x) = \{y \in M \,|\, y \ll x\}$

Open Sets

Requires g_{ab} to be at least C^2

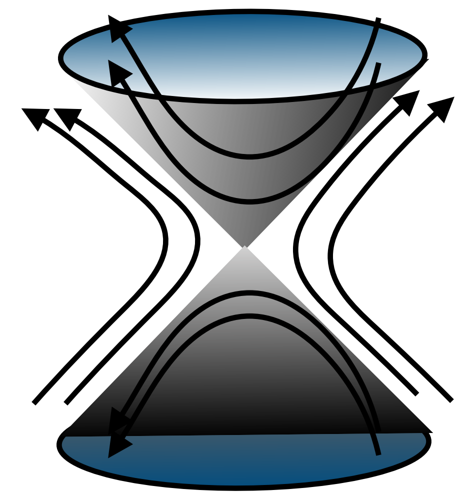
What part of (M, g) is $(M, \prec)$?

Under conformal transformations:



$$\tilde{g}_{ab} = \Omega^2 g_{ab},\ d\tilde{s}^2 = 0 = ds^2 \Rightarrow (M, \tilde{\prec}) = (M, \prec)$$

A Flat Spacetime Result



- $\mathbb{M}^d$: d dimensional Minkowski spacetime, $ds^2 = -dt^2 + \sum_{i=1}^{d-1} dx_i^2$

- Chronological Automorphism $f : \mathbb{M}^d \to \mathbb{M}^d, \quad x \ll y \Leftrightarrow f(x) \ll f(y), \forall\, x, y \in \mathbb{M}^d$.

- Conformal transformations: **Lorentz group + local dilatations.**

Theorem: The group of chronological automorphisms is isomorphic to the group of conformal transformations on $\mathbb{M}^\mathrm{d}$. ***--- Alexandrov and Ovchinnikova, 1953, Zeeman, 1964***

$(\mathbb{R}^d, \prec_{mink})$ determines ds^2_{mink} upto a conformal factor

Causal Structure as the ``Essence" of Lorentzian Geometry

- Let $(M_1, g_1), (M_2, g_2)$ be two causal spacetimes
- Let $(M_i, \prec\!\!\prec_i), (M_i, \prec_i)$ be their respective chronological and causal posets
- Chronological Bijection: $f:(M_1, \prec\!\!\prec_1) \to (M_2, \prec\!\!\prec_2), \quad f(x) \prec\!\!\prec_2 f(y) \Leftrightarrow x \prec\!\!\prec_1 y, \forall x, y \in M_1$
- Causal Bijection: $f:(M_1, \prec_1) \to (M_2, \prec_2), \quad f(x) \prec_2 f(y) \Leftrightarrow x \prec_1 y, \forall x, y \in M_1$
- Future and past distinguishing spacetimes: $I^+(x) = I^+(y)$ or $I^-(x) = I^-(y) \Rightarrow x = y$,

-- ***Hawking and Ellis, Penrose***

- Chronological Bijection $\Rightarrow$ Causal Bijection *if they are future and past distinguishing*

— ***Kronheimer and Penrose, 1967***

- Conformal Isometry : $F:(M_1, g_1) \to (M_2, g_2), \quad g_2 = \Omega^2 g_1$

Theorem: *If a chronological bijection exists between two future and past distinguishing spacetimes then they are conformally isometric*

--- ***Hawking, King, McCarthy, 1976, Malament, 1977***

Causal Structure as the ``Essence" of Lorentzian Geometry

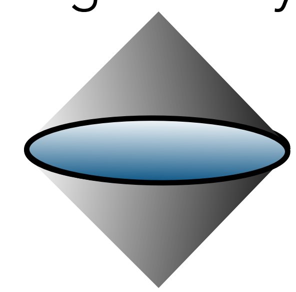
—Hawking, King, McCarthy: 1976
—Malament: 1977
-- Kronheimer and Penrose, 1967

HKMMKP theorem: (M, g) **=** $(M, \prec)$ **+** ϵ

Local Volume Element

``Causal Structure is 9/10th of the spacetime" ***-- Finkelstein, 1969***

- This decomposition of spacetime is unique to signature $(-,+,+,+,\ldots,+)$
- Suggests a non-Riemannian way of thinking about spacetime topology and geometry
  - **Alexandrov intervals** $I(x,y) \equiv I^+(x) \cap I^-(y)$ are open sets



  - Alexandrov interval topology = Manifold topology in strongly causal spacetimes
  - Chronological bijection ⇒ dimension and the topology (for <u>special</u> distinguishing spts) is the same. ***-- Malament, 1977, Parrikar and Surya, 2011***

Suggests a non-Riemannian order-theoretic route to quantising spacetime

Discretising the ``Essence":

*"To admit structures which can be very different from a manifold. The possibility arises, for example, of a **locally countable or discrete event-space equipped with causal relations macroscopically similar to those of a space-time continuum**."*

Axiomatic Approach to Causal Structure

"Extract from (M, g) its causal essence"

-- Kronheimer and Penrose 1967

Ideas of discrete Causal Structure

- ***Kronheimer and Penrose 1967***
- ***Finkelstein, 1969***
- ***Myrheim, 1978***
- ***'tHooft, 1979***
- ***Hemion, 1980***
- ***Bombelli, Lee, Meyer and Sorkin, 1987***

Discrete Posets or Causal Sets: A route to quantum spacetime

HKMMKP theorem: (M, g) = $(M, \prec)$ + ϵ

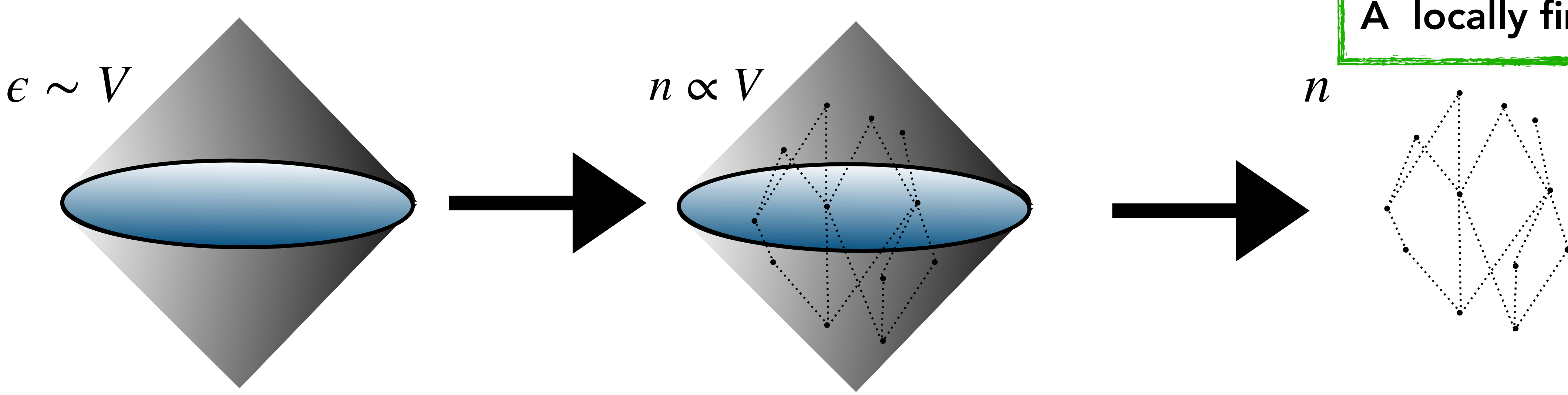
Acyclic: $x \prec y \Rightarrow y \nprec x$

Transitive: $x \prec y, y \prec z \Rightarrow x \prec z$

\+

Local Finiteness: $|\mathrm{Fut}(x) \cap \mathrm{Past}(y)| < \infty$

Causal Set:
A locally finite poset

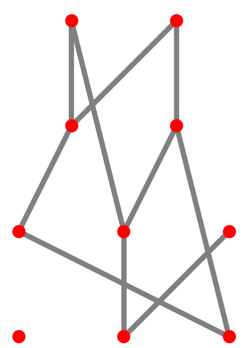


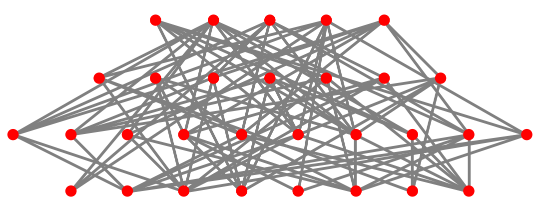
Continuum — Continuum-Discreteness correspondence — Discrete

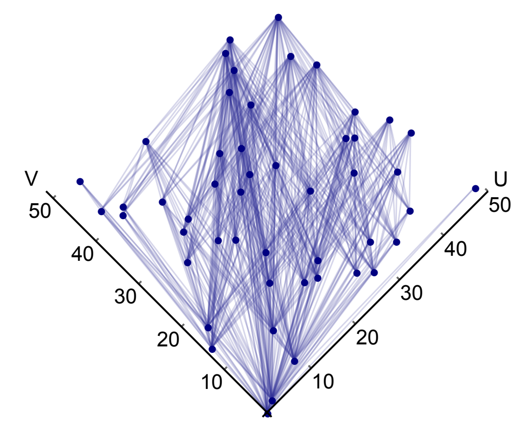
The Causal Set Hypothesis

-- Myrheim, 1978
-- Bombelli, Lee, Meyer and Sorkin,1987

1. Causal Sets are the fine grained structure of spacetime







2. Continuum Spacetime is an **approximation** of underlying causal sets

Order + Number ≈ Spacetime

$$C \approx (M, g)$$

Order ↔ Causal Order

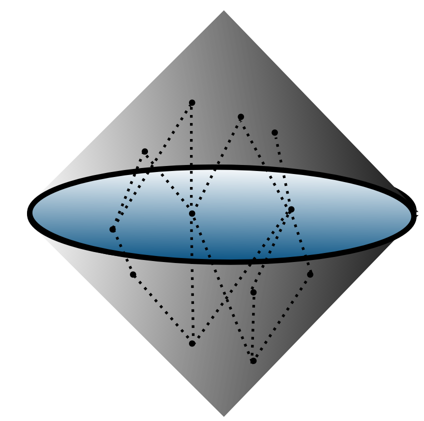
Number ↔ Spacetime Volume

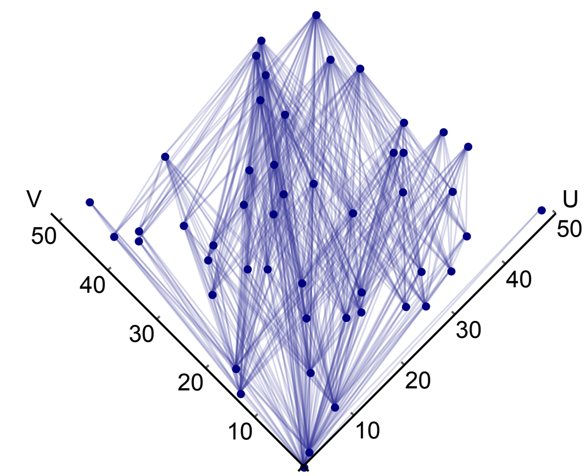
The continuum approximation

$(M,g) \approx C$

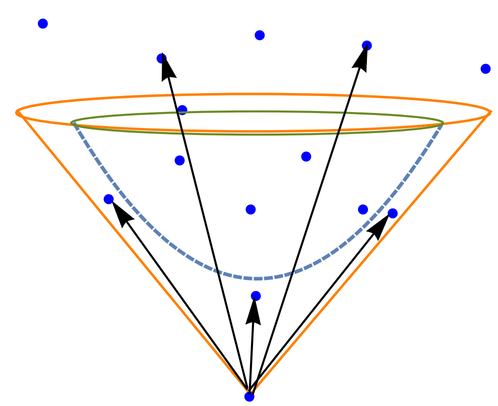
- Order $\prec$ $\leftrightarrow$ Causal Order, $\prec_g$
- Number, n $\leftrightarrow$ Volume, V

$n \propto V$





- $n \propto V$ cannot be implemented covariantly on a regular lattice
- $\langle n \rangle \propto V$, via a Poisson sprinkling: $P_V(n) = \frac{(\rho V)^n}{n!} e^{-\rho V}, \quad \rho^{-1} = V_c$
  - **Lorentz Invariance** is preserved for $C \approx \mathbb{M}^d$
  - **Non-locality**: resulting graph does not have a fixed/finite valency



$\Delta n = \sqrt{\rho V}$

Is Poisson optimal?

--Saravani and Aslanbeigi 2014

Very different from a regular lattice or even a Euclidean random lattice

Geometric Reconstruction: spacetime from causal sets

- Fundamental Conjecture: If $C \approx_\rho (M_1, g_1)$ and $C \approx_\rho (M_2, g_2)$, then $(M_1, g_1) \sim_\rho (M_2, g_2)$

$(M_1, g_1) \sim_\rho (M_2, g_2)$: When are two Lorentzian Spacetimes Close?

- Indirect evidence:
    - Timelike distance — ***Myrheim, Myer, Sorkin, Glaser, Surya, ..***
    - Spacetime dimension — ***Brightwell, Gregory***
    - Spatial homology —***Major, Rideout, Surya***
    - D'Alembertian — ***Sorkin, Henson, Benincasa, Dowker, Glaser***
    - Scalar curvature --***Benincasa, Dowker, Glaser***
    - **Einstein-Hilbert Action** --***Benincasa, Dowker, Glaser***
    - Gibbons-Hawking-York boundary terms — ***Buck, Dowker, Jubb & Surya***
    - Locality and Interval Abundance — ***Glaser & Surya***
    - Spatial and Spacelike Distance — ***Rideout & Wallden, Eichhorn, Mizera & Surya, Eichhorn, Surya, Versteegen***
    - Horizon area —***Dou & Sorkin, Barton, Counsell, Dowker, Gould & Jubb, ..***
    - Scalar Field theory —***Johnston, Sorkin, Dowker, Yazdi, Surya, Nomaan X, Jubb, Rejzner, et al.***
    - Entanglement Entropy —***Sorkin, Yazdi, Surya, Nomaan X***
    - ......

Quantum Dynamics for Causal Sets

``Causal Sets are the fine grained structure of spacetime"

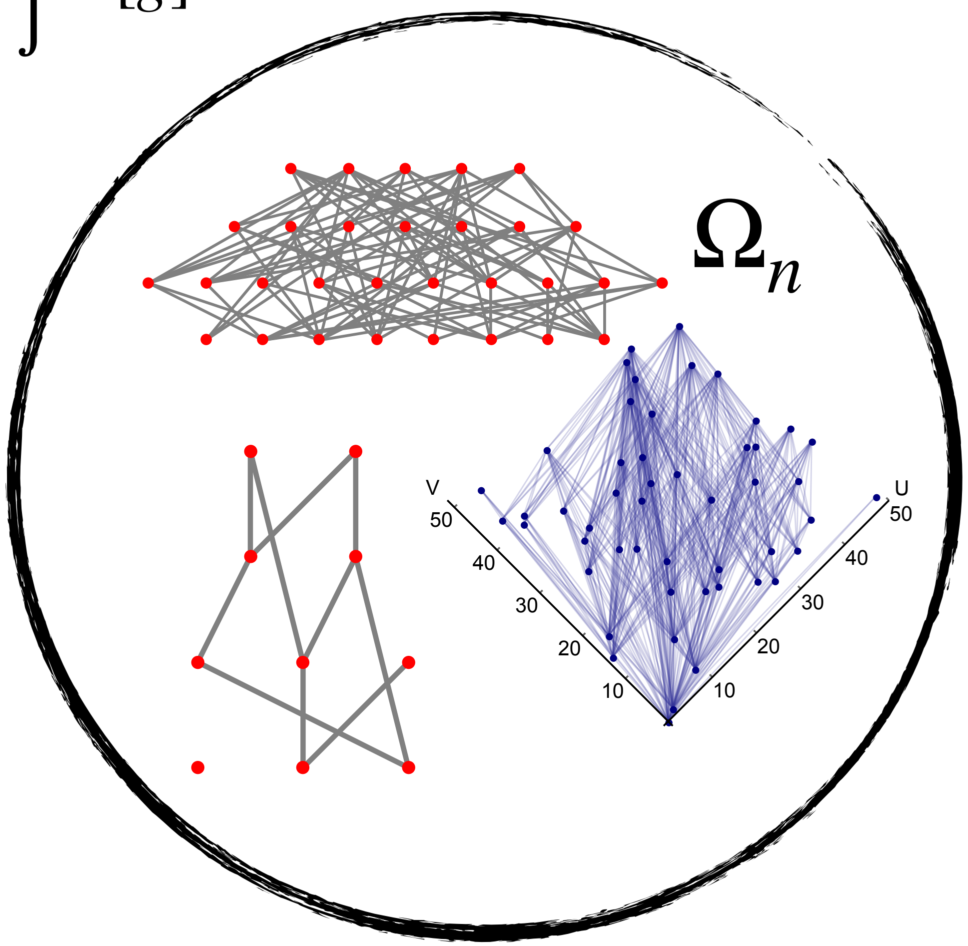
- In the continuum the gravitational path integral is: $Z = \int \mathcal{D}[g] e^{\frac{i}{\hbar} S_{EH}(g)}$

- Continuum replaced by discrete structure: ``$g \to c$ "

- Lorentzian Path Sum: $Z_n = \sum_{c \in \Omega_n} e^{\frac{i}{\hbar} S_{BDG}(c)}$

    - Ω_n is the sample space of all n element posets
    - $S_{BDG}(c)$ is the **Benincasa-Dowker-Glaser action**

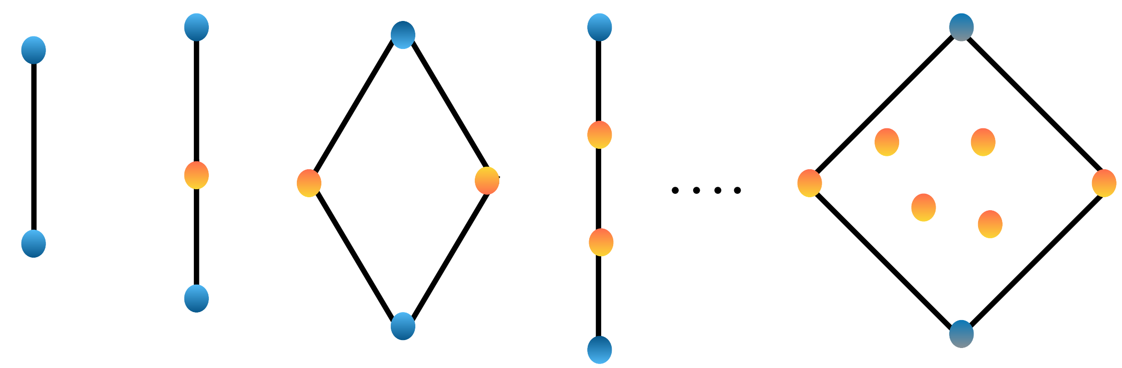
- Large n (thermodynamic limit): $Z = \lim_{n \to \infty} Z_n$



Discrete Einstein-Hilbert Action: The Benincasa-Dowker-Glaser Action(s)

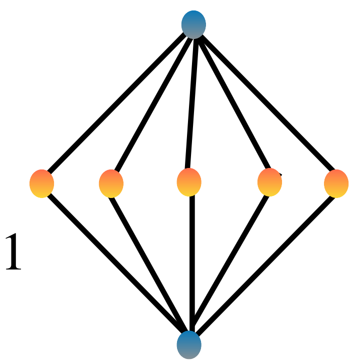
— Benincasa & Dowker, 2010,
— Dowker & Glaser, 2012,
— Glaser, 2014

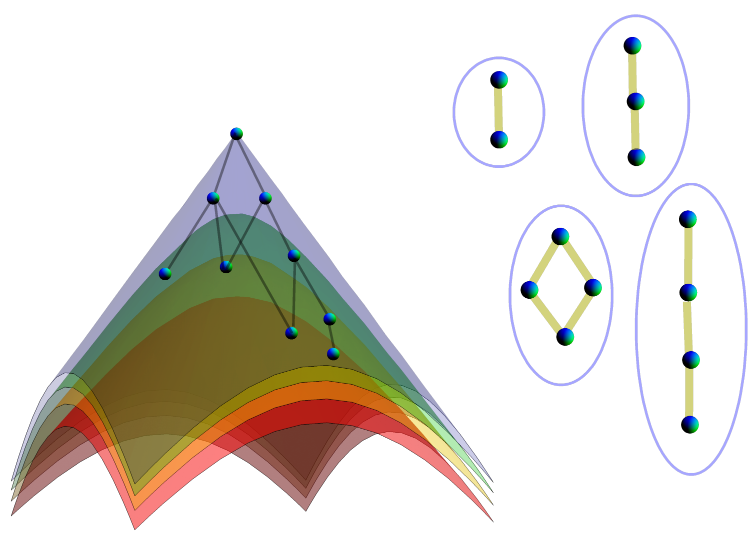
A discrete interval is an intersection of an up-set with a down-set



N_J = # of J-element intervals

example:
$N_0 = 10,\ N_5 = 1$





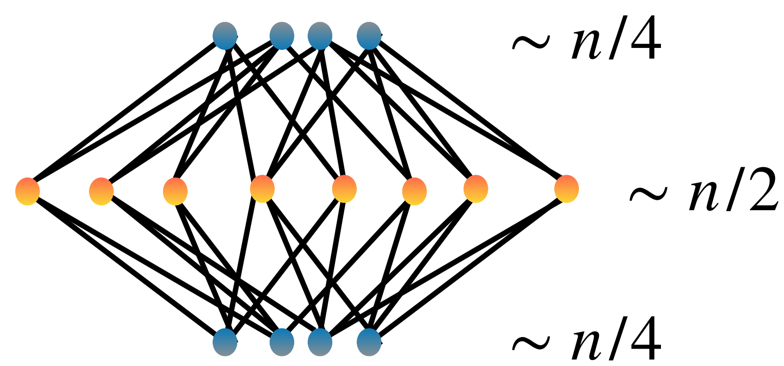
$J^-(p)$ is "foliated" by the the $J-$ element intervals to the past of $p \in \mathbb{M}^d$

- $$\frac{1}{\hbar} S^{(d)}_{BDG}(C) = -\alpha_d \left(\frac{\ell}{\ell_p}\right)^{d-2} \left(n + \frac{\beta_d}{\alpha_d} \sum_{J=0}^{J_{max}} C^d_J N_J\right)$$
- ℓ_p: Planck Length, ℓ: discreteness scale, α_d, β_d, C^d_J: known consts.
- Eg: For $d=4$, $\frac{1}{\hbar} S^{(4)}_{BDG} = \frac{4}{\sqrt{6}} \left(n - N_0 + 9N_1 - 16N_2 + 8N_3\right)$

$$\lim_{\rho \to \infty} \frac{\ell^2}{\ell_p^2} \langle S_{BDG} \rangle = S_{EH} + \text{bdry terms}$$

``Causal Sets are the fine grained structure of spacetime''

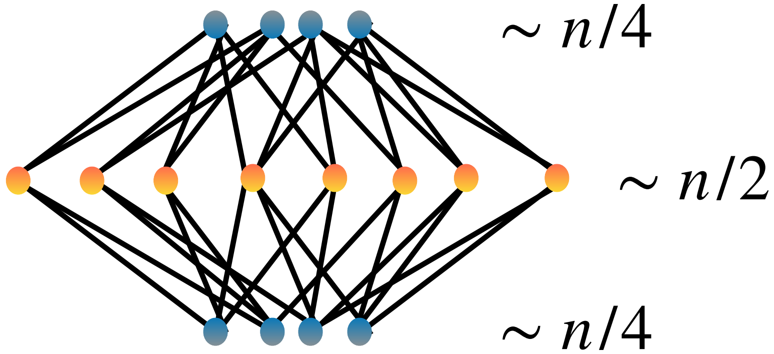
- Ω_n : sample space of all n-element causal sets
- $|\Omega_n| \sim 2^{\frac{n^2}{4}+\frac{3n}{2}+o(n)}$ — *- Kleitman and Rothschild, Trans AMS, 1975*
- Typical causal sets are **Kleitmann-Rothschild (KR)**:
    - 3 layers: $\mathbb{L}_k, \quad k=1,2,3, \quad |\mathbb{L}_{1,3}| \sim \frac{n}{4}, \, |\mathbb{L}_2| \sim \frac{n}{2}$
    - elements of $\mathbb{L}_k$ form an **antichain**
    - $\forall\, e \in \mathbb{L}_1, \, \exists \sim \frac{n}{4}$ no. of $e' \in \mathbb{L}_2$ such that $e \prec_* e'$,
    - $\forall\, e \in \mathbb{L}_3, \, \exists \sim \frac{n}{4}$ no. of $e' \in \mathbb{L}_2$ such that $e' \prec_* e$
    - $\forall\, e \in \mathbb{L}_1, \, e' \in \mathbb{L}_3, \quad e' \prec e$



- $|\Omega_{KR}| \sim 2^{\frac{n^2}{4}+\frac{3n}{2}+o(n)}$

Onset of asymptotic regime $n \sim 100$ — *-- J. Henson, D. Rideout, R. Sorkin and S.Surya, JEM, 2015*

A KR poset is not continuum-like



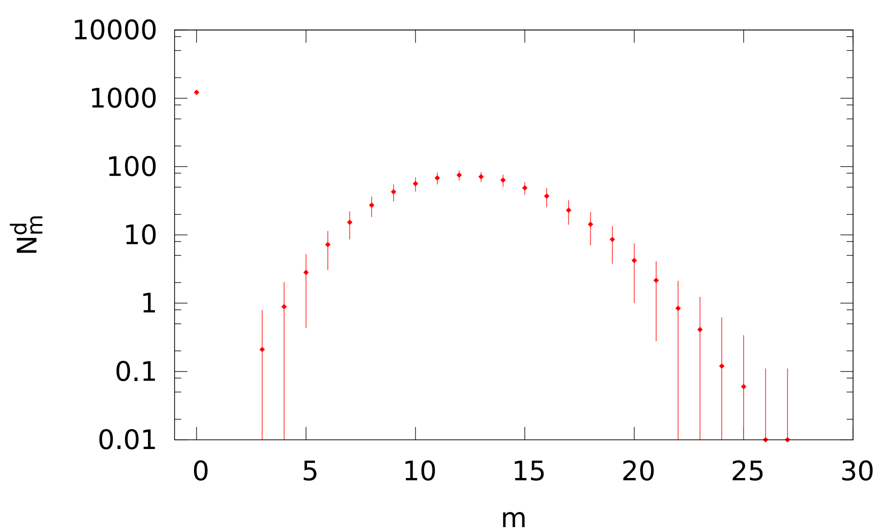
- Does not arise from a typical Poisson sprinkling into any continuum (M, g)

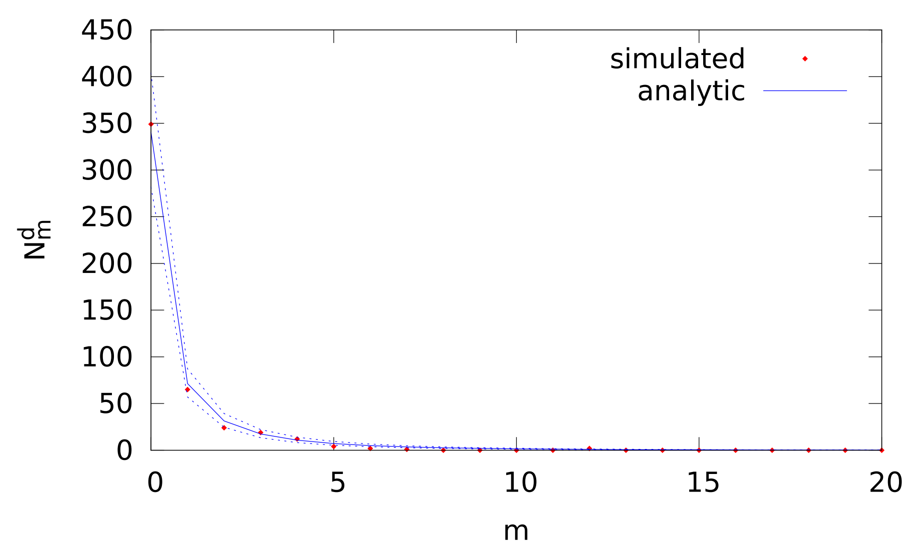
  - Myrheim-Myer Continuum Dimension is fractional :

$$\frac{\langle R \rangle}{n^2} = \frac{\Gamma(d+1)\Gamma(d/2)}{4\Gamma(3d/2)} \Rightarrow \frac{\Gamma(d_{KR}+1)\Gamma(d_{KR}/2)}{4\Gamma(3d_{KR}/2)} = \frac{3}{16} \Rightarrow d_{KR} \sim 2.5$$

  - Maximal time-like distance $H_{KR} = 3$

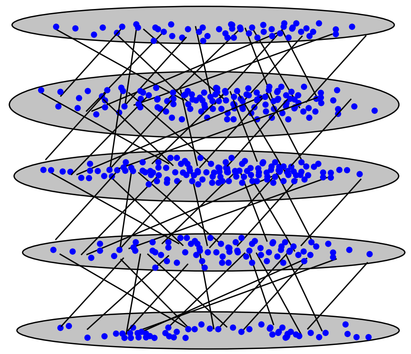
  - Interval Abundances are not like the continuum:





The layered hierarchy

-- D. Dhar, JMP, 1978
-- Promel, Steger, Taraz 2001



- K-layered poset: $C = \mathbb{L}_1 \sqcup \mathbb{L}_2 \ldots \mathbb{L}_K : e \prec e', e \in \mathbb{L}_k, e' \in \mathbb{L}_{k'} \Rightarrow k < k'$

- $|\Omega_n^{(K)}| \sim 2^{c(K)n^2 + o(n^2)}, \quad c(K) \leq 1/4,,$

- Dominant hierarchy: $|\Omega_n^{(3)}| > |\Omega_n^{(2)}| > |\Omega_n^{(4)}| > |\Omega_n^{(5)}| \ldots$

- For $K \ll n$ none of these are like continuumlike

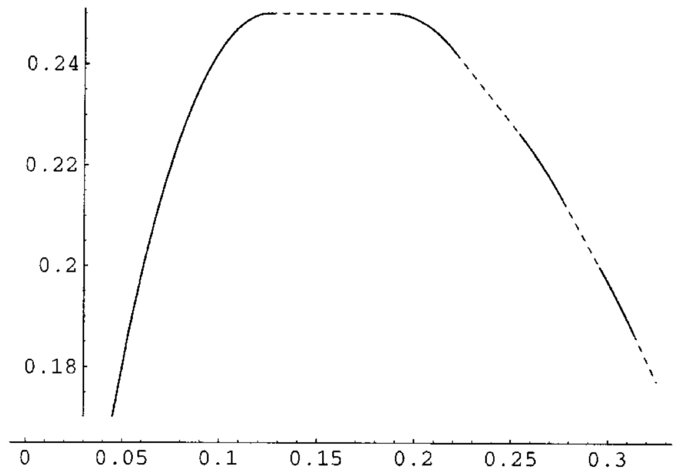
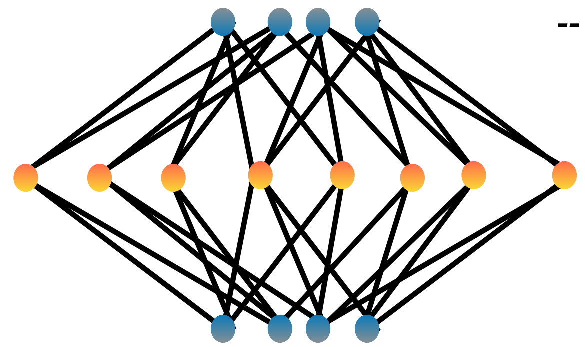


FIG. 5. c(d) in the range [0.05, 0.32].

The contribution from layered posets

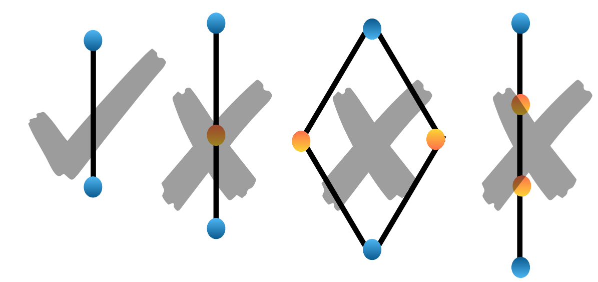
-- *Sorkin, unpublished,*
-- *Loomis and Carlip, 2017*
-- *A.Anand Singh, A.Mathur and Surya, 2021*
-- *P. Carlip, S. Carlip and S. Surya, 2022*
-- *P. Carlip, S. Carlip and S. Surya, 2023*



- $Z_{KR} = \sum_{c \in \Omega_{KR}} e^{iS_{BDG}(c)/\hbar}$
- Number of **links:** $N_0 = pN_0^{\max} \approx pn^2$
- $S^{(d)}_{\text{BDG}} = \mu p n^2 + o(n^2)$
- $\Rightarrow Z_{KR} = \sum_{N_0} \Gamma(N_0)\ e^{iS_L/\hbar}$
  - S_L: Link Action
  - $\Gamma(N_0)$: Density of states

Combinatorial Enumeration argument

$$S^{(4)}_{BDG} = \mu\left(n - N_0 + 9\cancel{N_1} - 16\cancel{N_2} + 8\cancel{N_3}\right)$$



- For all $K \ll n$ as well, $Z_K = \sum_{N_0} \Gamma(N_0)\ e^{iS_L/\hbar}$
- $\Gamma[N_0 = pN_0^{\max}] \propto h(p) = -p\ln p - (1-p)\ln(1-p)$ — **Dhar's Entropy function.**
- $Z_K \sim e^{i\mu n} \int_0^{1/2} \mathrm{d}p \exp\left[n^2\left(i\mu\lambda_0 p/2 + h(2p)/4\right) + o(n^2)\right]$

Bilayer calculation
—***Loomis and Carlip, 2017***

Theorem: If $\ell > \ell_{\min}$, the BDG action suppresses all K-layer orders when $K << n$, in any dimension.

- $\ell_{\min}$ is dimension dependent, $1.13 < (\ell_{\min}/\ell_p) < 2.33$, (Eg: $d = 4, \quad \ell_{\min} = 1.136\,\ell_p$)
- $Z_n \approx \sum_{c \in \Omega_n \backslash \Omega_n^{(K)}} e^{\frac{i}{\hbar} S_{\mathrm{BDG}}(c)}$

Bilayer calculation of
—*Loomis and Carlip, 2017*

- What about continuumlike causal sets?
- Ongoing work suggests that causal sets $\sim (M, g)$ are NOT suppressed..

What are other (non-layered) families of posets that are entropically sub-dominant?

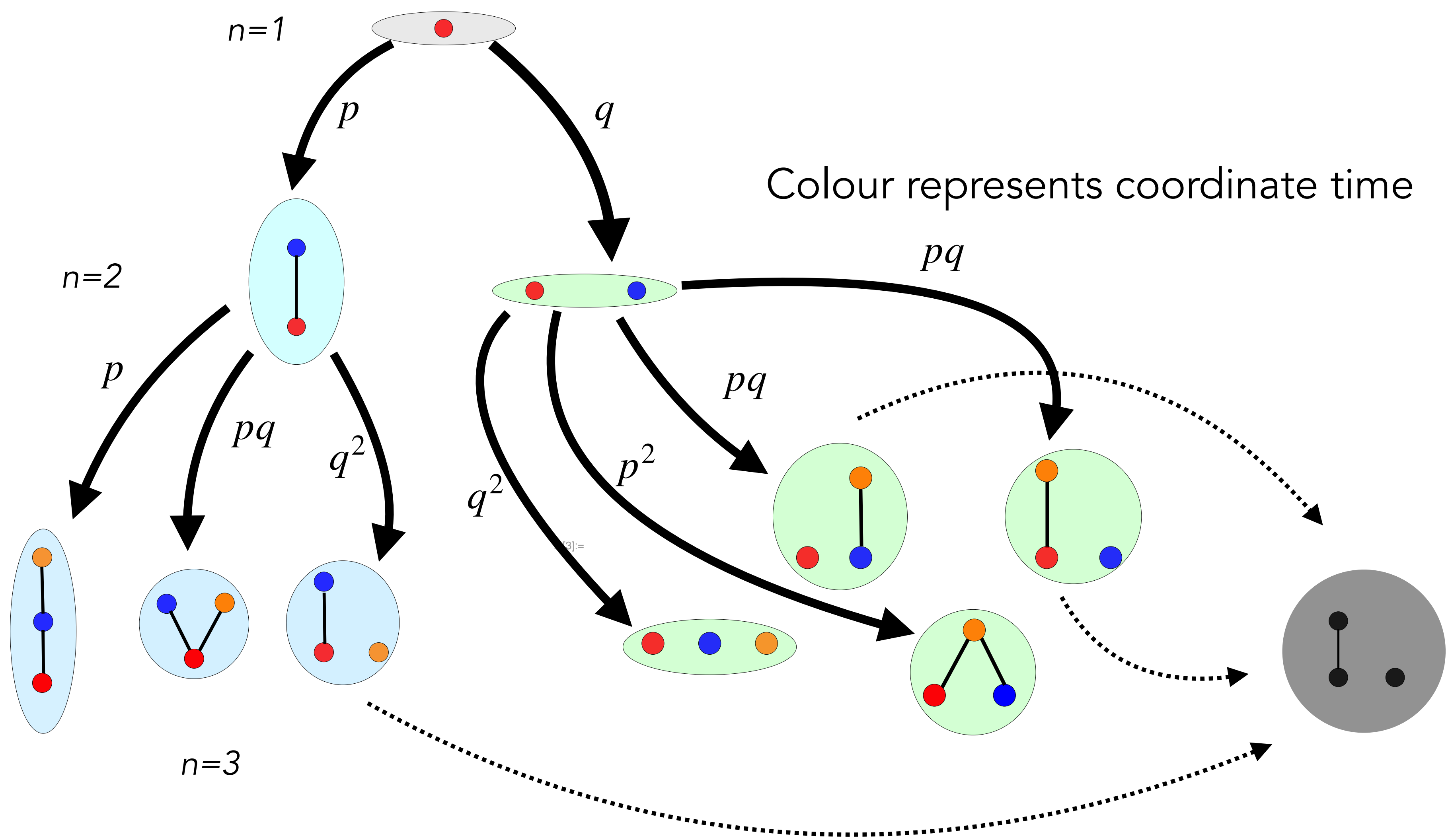
Can we find a more fundamental, ``principled'' model of causal set dynamics which is order theoretic?

The Sequential Growth Paradigm

— *Rideout & Sorkin,2000-2001*
— *O'Connor, Martin, Rideout & Sorkin, 2001*
— *Brightwell, Dowker, Garcia, Henson, Rideout & Sorkin, 2003*
— *Brightwell, Georgiou, 2010,*
— *Brightwell and Luczak, 2011-2012,*
— *Dowker, Johnston & Surya 2011,*
— *Surya & Zalel 2020*
— *Dowker, Imambaccus, Owens, Sorkin & Zalel, 2020....*

Stochastic triple $(\Omega, \mathfrak{A}, \mu)$

- Internal temporality: new element can never be to the past of an existing element
- $\mu(c_n^i) = \mu(c_n^j)$ if there is an order isomorphism between c_n^i and c_n^j
- ``Bell causality" or spectator independence

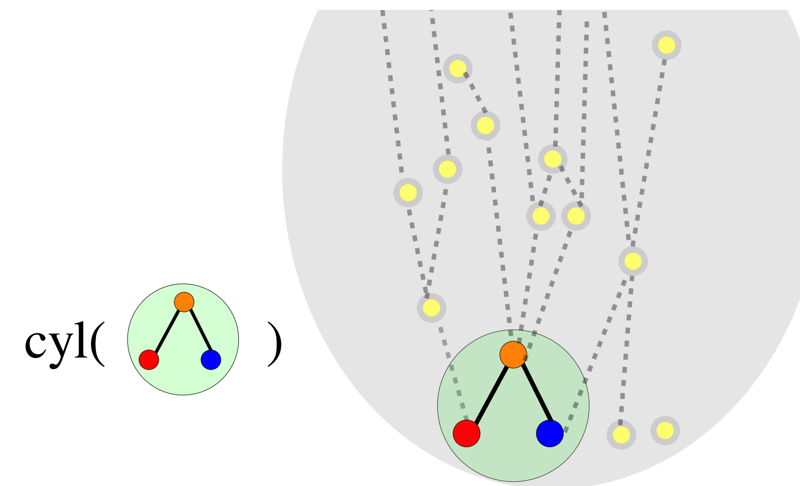


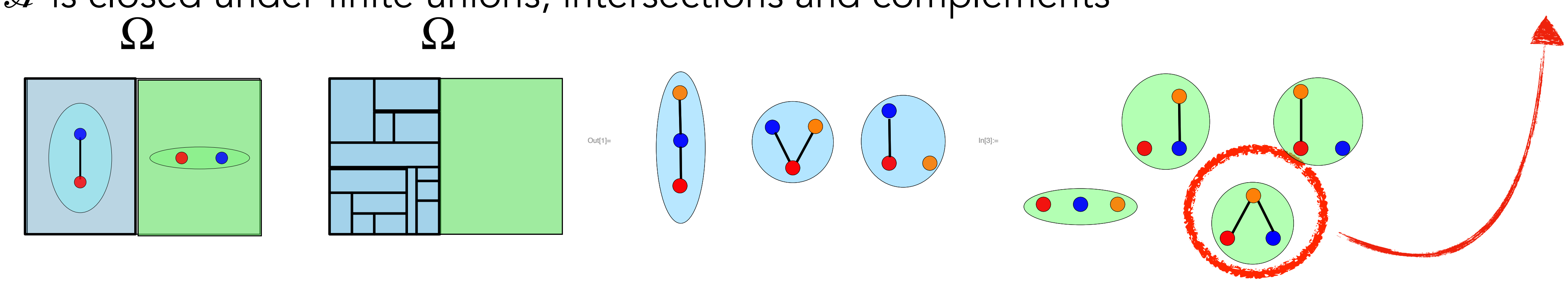
Classical Stochastic triple $(\Omega, \mathfrak{A}, \mu)$ ⟶ Quantum Stochastic triple $(\Omega, \mathfrak{A}, \mu_v)$

Dynamics as a Measure Space $(\Omega, \mathscr{A}, \mu)$

— Brightwell, Dowker, Garcia, Henson, Rideout & Sorkin

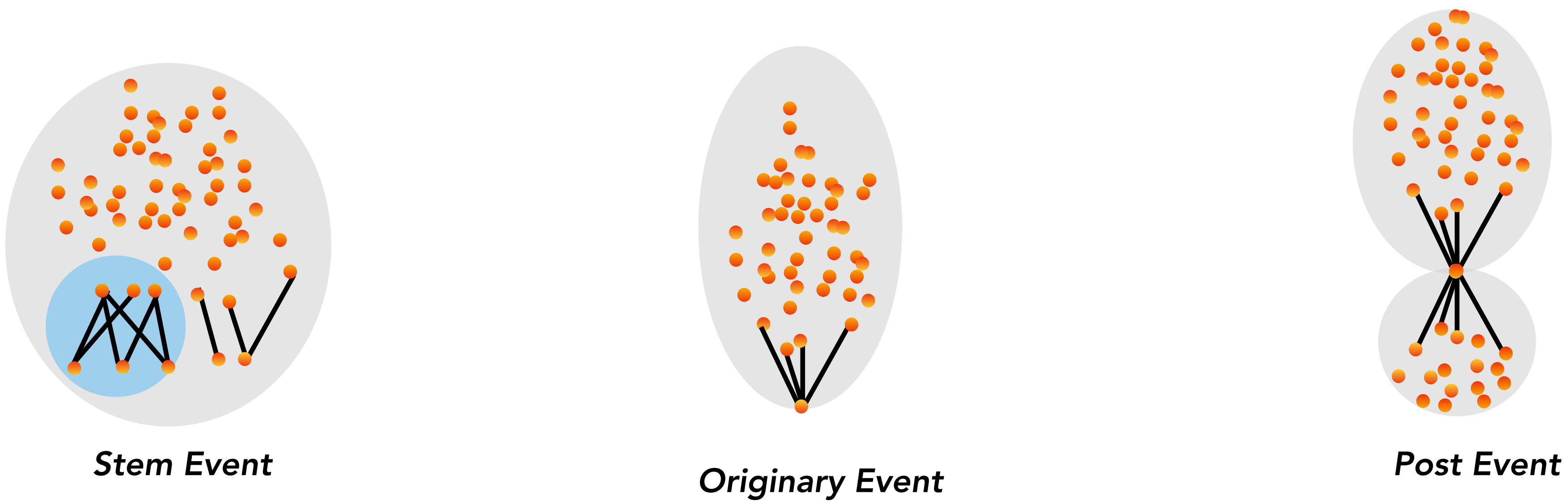
- Ω: Space of countable past finite causal sets
- Cylinder sets: $\mathrm{cyl}(c_n) \equiv \{c \in \Omega \,|\, c|_n = c_n\}$
- $\mathscr{A}$: **Event Algebra** generated by
- generate the finite time "labelled" algebra
- $\mathscr{A}$ is closed under finite unions, intersections and complements





- $\mu : \mathscr{A} \to X$ a ``colour'' invariant measure
- But $\mathscr{A}$ is not covariant!

Covariant Events in $\mathscr{A}$



- Analogues of the "return to the origin" event in the random walk
- Sigma algebra completion : $\mathscr{A} \to \mathfrak{S}(\mathscr{A})$
- Quotient sigma algebra $\widetilde{\mathfrak{S}} = \mathfrak{S}/\sim$ is covariant.
- Does μ extend to $\mathfrak{S}(\mathscr{A})$?
- If μ is a probability measure, then the Kolmogorov-Caratheodary-Hahn extension theorem

  guarantees this for any choice of μ

Quantum Sequential Growth

— Dowker, Johnston & Surya 2011,
---Surya & Zalel, 2020

Classical Stochastic triple $(\Omega, \mathfrak{A}, \mu)$ ⟶ Quantum Stochastic triple $(\Omega, \mathfrak{A}, \mu_v)$

$\mu_v : \mathscr{A} \to \mathscr{H}$

Vector Measure

- $\mu_v(\alpha) \equiv [\chi_\alpha] \in \mathscr{H}$
- $\langle \mu_v(\alpha), \mu_v(\beta) \rangle = D(\alpha, \beta)$
- $D(\alpha, \beta)$: decoherence functional

— Dowker, Johnston & Surya, 2010

$D(\alpha, \beta)$

- ***Hermitian,***
- ***Biadditive***
- ***Strongly Positive***

Histories Hilbert Space

- V: Free vector space : $f : \mathscr{A} \to \mathbb{C}$
- $\langle f, g \rangle \equiv \sum_{\alpha, \beta \in \mathfrak{A}} f^*(\alpha) g(\beta) D(\alpha, \beta)$
- $\lim_{i \to \infty} || f_i - g_i ||_V = 0 \Rightarrow \{f_i\} \sim \{g_i\}$
- $\mathscr{H} \equiv V/\sim, \quad [\{f_i\}] \in \mathscr{H}$

— Dowker, Johnston & Sorkin,2010

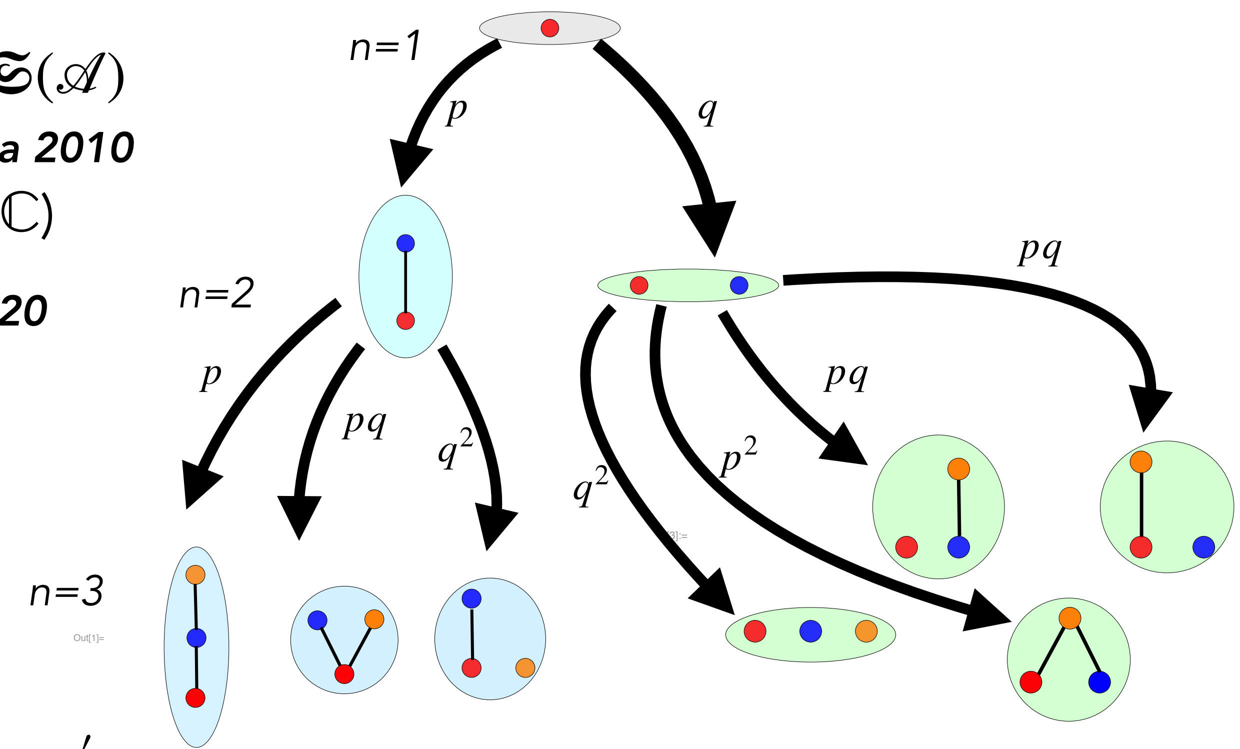
Caratheodary-Hahn-Kluvnek theorem:
μ_v extends to $\mathfrak{S}(\mathscr{A})$ only under special convergence conditions

Experiments with Quantum Sequential Growth

- $p \in \mathbb{C}$, $D(c_1, c_2) = A^*(c_1)A(c_2)$ does not extend to $\mathfrak{S}(\mathscr{A})$

  — Dowker, Johnston, Surya 2010
- Simple examples of Commutative Dynamics ($\mathscr{H} \simeq \mathbb{C}$) which extend to $\mathfrak{S}(\mathscr{A})$ ***— Surya & Zalel, 2020***
- Non-Commutative Dynamics, $\mathscr{H} \simeq \mathbb{C}^m, m > 1$
  - Transfer Operators:

    $$|c^i_{n+1}\rangle = \hat{A}^i_n |c_n\rangle, \quad \Rightarrow \sum_{i \in I(c_n)} \hat{A}^i_n = \hat{\mathrm{I}}$$
  - Color independence: $\hat{A}_\gamma = \hat{A}_{\gamma'}$, if $\gamma \sim \gamma'$, where $\hat{A}_\gamma$ is a product of a sequence of transfer operators ***— ongoing work***



How can one characterise this algebra of transfer operators?
What are the representations of this algebra?

In Conclusion..

- Causal set theory is strongly motivated by Lorentzian geometry
- Continuum spacetime can be reconstructed from causal sets : lots of evidence!
- There is an intimate interplay between causality/order and dynamics
- Causal set theory is a discrete playground for mathematicians to explore!
    - Combinatorics: enumeration of (sub)^K-dominant posets ?
    - Quantum Stochastic Growth:
        - Transfer Operator Algebras
        - Vector Measure and its Extension
        - Recent comparison of Classical SG with Hopf-Algebras
    - .....

--Yates, Zalel, 2023

Thank you!