From combinatorial species to spaces of generalized independences

Yannic VARGAS

Algebraic, analytic and geometric structures emerging from quantum field theory
π, Chengdu, 2024

If $n \in \mathbb{N}$, let $[n] := \{1, 2, \ldots, n\}$.

Let I be a finite set.

A **composition** of I is a sequence

$$F = (F_1, \ldots, F_k) = F_1 | \cdots | F_k$$

of disjoint non-empty sets such that their reunion is I. We write $F \vDash I$.

For example,

$$2|569|3|1478 \vDash [10].$$

Let I be a finite set.

Let I be a finite set.

Let $\Sigma[\mathrm{I}]$ the vector space generated by all compositions of I and let $\mathfrak{S}_{\mathrm{I}}$ be the group of permutations on I.

Let I be a finite set.

Let $\Sigma[\mathrm{I}]$ the vector space generated by all compositions of I and let $\mathfrak{S}_\mathrm{I}$ be the group of permutations on I.

The vector space $\Sigma[\mathrm{I}]$ is a right $\mathfrak{S}_\mathrm{I}$-module, where the action permutes the elements of I.

Let I be a finite set.

Let $\Sigma[\mathrm{I}]$ the vector space generated by all compositions of I and let $\mathfrak{S}_{\mathrm{I}}$ be the group of permutations on I.

The vector space $\Sigma[\mathrm{I}]$ is a right $\mathfrak{S}_{\mathrm{I}}$-module, where the action permutes the elements of I.

This action extends to a (covariant) functor

$$\Sigma : \mathsf{FinSet} \longrightarrow \mathsf{Vect},$$

where

- $\mathrm{I} \mapsto \Sigma[\mathrm{I}]$;
- $(\mathrm{I} \xrightarrow{\sigma} \mathrm{J}) \ \mapsto \ (\Sigma[\mathrm{I}] \xrightarrow{\Sigma[\sigma]} \Sigma[\mathrm{J}])$.

Let I be a finite set.

Let $\Sigma[\mathrm{I}]$ the vector space generated by all compositions of I and let $\mathfrak{S}_{\mathrm{I}}$ be the group of permutations on I.

The vector space $\Sigma[\mathrm{I}]$ is a right $\mathfrak{S}_{\mathrm{I}}$-module, where the action permutes the elements of I.

This action extends to a (covariant) functor

$$\Sigma : \mathsf{FinSet} \longrightarrow \mathsf{Vect},$$

where

- $\mathrm{I} \mapsto \Sigma[\mathrm{I}]$;
- $(\mathrm{I} \xrightarrow{\sigma} \mathrm{J}) \;\mapsto\; (\Sigma[\mathrm{I}] \xrightarrow{\Sigma[\sigma]} \Sigma[\mathrm{J}])$.

The construction Σ is an example of a *vector species*.

Species

Species

André Joyal, Alain Connes, Olivia Caramello and Laurent Lafforgue, IHES (2015)

The theory of *combinatorial species* was introduced by André Joyal in 1980. Species can be seen as a *categorification* of generating functions. It provides a categorical foundation for enumerative combinatorics.

Species

A vector species is a functor

$$\mathsf{p} : \mathsf{FinSet} \longrightarrow \mathsf{Vect}.$$

Species

A vector species is a functor

$$p : \mathsf{FinSet} \longrightarrow \mathsf{Vect}.$$

By functoriality,

$$I \xrightarrow{\alpha} J \xrightarrow{\beta} K \Longrightarrow p[\beta \circ \alpha] = p[\beta] \circ p[\alpha],$$

$$I \xrightarrow{\mathrm{id}_I} I \Longrightarrow p[\mathrm{id}_I] = \mathrm{id}_{p[I]}.$$

In particular, $p[\sigma]^{-1} = p[\sigma^{-1}]$ for every $I \xrightarrow{\sigma} J$.

For every $n \in \mathbb{N}$, $\mathfrak{S}_n$ acts on $p[n]$ via $\sigma \cdot x := p[\sigma](x)$. Therefore,

$$\text{species } p \longleftrightarrow V = (V_n)_{n \geq 0}, V_n \text{ is a } \mathfrak{S}_n\text{-module}.$$

Examples of species

- Species E of **sets**:
$$E[I] := \mathbb{K}\{\mathtt{H}_I\}.$$
- Species E_n of n-**sets**:
$$E_n[I] := \begin{cases} \mathbb{K}\{\mathtt{H}_I\}, & \text{if } |I| = n; \\ (0), & \text{if } |I| \neq n. \end{cases}$$
- Species $X := E_1$ of sets of one element.
- Species $1 := E_0$.
- Species G of **graphs**:
$$G[I] := \mathbb{K}\{\, \mathtt{H}_G \;:\; G \text{ is a finite graphs with vertices in } I \,\}.$$

Examples of species

- Species Π of **partitions**.
- Species L of **linear orders**.
- Species Σ of **set compositions**.
- Species B of **binary trees**.
- Species $\mathfrak{S}$ of **permutations**.
- Species Braid of **braid hyperplane arrangements**.

$$\vdots$$

What is a species, really?

nLab: “A (combinatorial) species is a presheaf (or a higher categorical presheaf) on the groupoid core(FinSet)” (the “permutation groupoid”).

Many variants are obtained by modifying the input category FinSet (replace by total orders, posets, permutations, ...) and/or the output category Vec (replace by sets, modules, algebras, species, ...).

Species is a categorical tool to understand generating functions and their interactions, combinatorially.

Generating functions: *ordinary*, *exponential*, *Dirichlet*, *Lambert*, ...

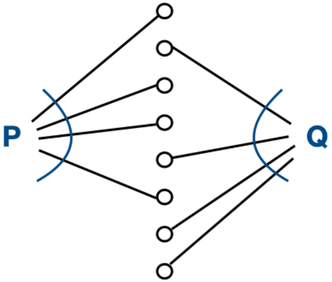
Operations on species

- **Sum of species**

$$(p+q)[I] := p[I] \oplus q[I].$$

- **Product of species** (Cauchy product)

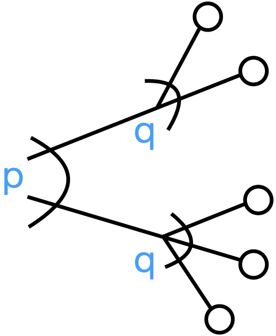
$$(p \cdot q)[I] := \bigoplus_{I = S \sqcup T} p[S] \otimes q[T].$$



Operations on species

- **Composition of species**

$$(p \circ q)[I] := \bigoplus_{\pi \in \Pi[I]} p[\pi] \otimes \bigotimes_{B \in \pi} q[B].$$



Generating function of a species

To every species p it is associated its **exponential generating function**:

$$\mathsf{p}(x) := \sum_{n \geq 0} \dim_{\mathbb{K}} \mathsf{p}[n] \frac{x^n}{n!}.$$

We have:

$$(\mathsf{p} + \mathsf{q})(x) = \mathsf{p}(x) + \mathsf{q}(x),$$
$$(\mathsf{p} \cdot \mathsf{q})(x) = \mathsf{p}(x) \cdot \mathsf{q}(x),$$
$$(\mathsf{p} \circ \mathsf{q})(x) = \mathsf{p}(x) \circ \mathsf{q}(x).$$

For the last identity, $\mathsf{q}[\emptyset] := (0)$.

Enumerative application

A *plane* labelled binary tree is:

- empty;
- a couple of labelled complete binary trees, and the labelled root.

This translates as,

$$\mathsf{B} = \mathsf{1} + \mathsf{X} \cdot \mathsf{B}^2,$$

which implies:

$$\mathsf{B}(x) = 1 + x\mathsf{B}(x)^2.$$

Therefore,

$$\mathsf{B}(x) = \frac{1 - \sqrt{1-4x}}{2x} = \sum_{n \geq 0} n! \left(\frac{1}{n+1} \binom{2n}{n} \right) \frac{x^n}{n!}.$$

Recall that the **Cauchy product** of two species p and q is given by

$$(\mathsf{p} \cdot \mathsf{q})[I] = \bigoplus_{I = S \sqcup T} \mathsf{p}[S] \otimes \mathsf{q}[T].$$

Recall that the **Cauchy product** of two species p and q is given by

$$(\mathsf{p} \cdot \mathsf{q})[I] = \bigoplus_{I = S \sqcup T} \mathsf{p}[S] \otimes \mathsf{q}[T].$$

Endowed with this operation, the *category of species* Sp is symmetric monoidal: we can speak of monoids ($\mu : \mathsf{p} \cdot \mathsf{p} \to \mathsf{p}$), comonoids ($\Delta : \mathsf{p} \to \mathsf{p} \cdot \mathsf{p}$), ..., in species.

$$\mathsf{p}[S] \otimes \mathsf{p}[T] \xrightarrow{\mu_{S,T}} \mathsf{p}[I] \qquad \mathsf{p}[I] \xrightarrow{\Delta_{S,T}} \mathsf{p}[S] \otimes \mathsf{p}[T].$$

Hopf monoids

Hopf monoids refine Hopf algebras. There are a bit more abstract but better suited for many combinatorial purposes.

There is a Fock functor

$$\text{Hopf monoids} \longrightarrow \text{Hopf algebras}$$

Many (combinatorial) phenomena in combinatorial Hopf algebras comes from phenomena in Hopf monoids.

Hopf monoids

A bimonoid (h, μ, Δ) consist of:

- for each finite set I, a vector space h[I];
- for each partition $I = S \sqcup T$, maps

$$\begin{array}{lrl} \text{product} & \mu_{S,T} : h[S] \otimes h[T] \to h[I] \\ \text{coproduct} & \Delta_{S,T} : h[I] \to h[S] \otimes h[T] \\ & x \mapsto x|_S \otimes x/_S \end{array}$$

satisfying associativity, coassociativity, unitality, counitality, compatibility and naturality.

(Bimonoidal object in the braided monoidal category of *vector species*)

Hopf monoids

A Hopf monoid (h, μ, Δ) consist of:

- for each finite set I, a vector space h[I];
- for each partition $I = S \sqcup T$, maps

$$\text{product} \qquad \mu_{S,T} : h[S] \otimes h[T] \to h[I]$$
$$\text{coproduct} \qquad \Delta_{S,T} : h[I] \to h[S] \otimes h[T]$$

- for each finite set I and for each $x \in h[I]$, there exists a formal linear combination

$$s_I(x) \in h[I]$$

$$\sum_{S \sqcup T = I} s_S(x|_S) \cdot x/_S = \sum_{S \sqcup T = I} x|_S \cdot s_T(x/_S) = \begin{cases} 1 & \text{if } I = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Hopf monoids

A Hopf monoid $(\mathsf{h}, \mu, \Delta)$ consist of:

- for each finite set I, a vector space h[I];
- for each partition $I = S \sqcup T$, maps

$$\text{product} \qquad \mu_{S,T} : \mathsf{h}[S] \otimes \mathsf{h}[T] \longrightarrow \mathsf{h}[I]$$
$$\text{coproduct} \qquad \Delta_{S,T} : \mathsf{h}[I] \longrightarrow \mathsf{h}[S] \otimes \mathsf{h}[T]$$

- **Takeuchi's formula**: if h is connected, then

$$s_I = \sum_{F \vDash I} (-1)^{\ell(F)} \mu_F \Delta_F,$$

for any non-empty set I.

Example: Hopf monoid of "ripping and sewing" of graphs

G[I]:=K$\{\mathtt{H}_g$: g is a graph (with half edges) on vertex set $I\}$

Product: $\mathtt{H}_{g_1} \cdot \mathtt{H}_{g_2} = \mathtt{H}_{g_1} \sqcup \mathtt{H}_{g_2}$ (disjoint union)

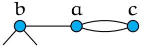
Coproduct: $\Delta_{S,T}(\mathtt{H}_g) = \mathtt{H}_{g|_S} \otimes \mathtt{H}_{g/_S}$, where

$$g|_S = \text{keep everything incident to } S$$

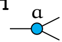
$$g/_S = \text{remove everything incident to } S$$

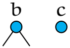
Example: Hopf monoid of graphs

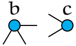
$$\mathrm{G}[I] := \mathbb{K}\{H_G \,:\, G \text{ is a graph (with half edges) on vertex set } I\}$$

$g =$ 

$\in \mathrm{G}[\{a, b, c\}].$

$\Delta_{a,bc}(\mathtt{H}_g) = \mathtt{H}$ 

$\otimes\ \mathtt{H}$ 

$\Delta_{bc,a}(\mathtt{H}_g) = \mathtt{H}$ 

$\otimes\ \mathtt{H}$ 

There are many other examples...

- graphs G
- posets P
- trees T
- matroids M
- linear orders L
- set partitions Π
- set compositions Σ
- paths A
- simplicial complexes SC
- hypergraphs HG
- building sets BS

Character group of a Hopf monoid

Let I a finite set and $\mathrm{I} = \mathrm{S} \sqcup \mathrm{T}$.

A **character** ζ of a Hopf monoid h is a species map

$$\zeta : \mathsf{h} \to \mathbf{E}, \quad \zeta = (\zeta_{\mathrm{I}}), \quad \zeta_{\mathrm{I}} : \mathsf{h}[\mathrm{I}] \to \mathbb{K}$$

such that

$$\zeta_{\mathrm{I}}(x \cdot y) = \zeta_{\mathrm{S}}(x) \cdot \zeta_{\mathrm{T}}(y) \qquad , \qquad \zeta_{\emptyset}(\epsilon) = 1.$$

Character group of a Hopf monoid

The characters of h form a group $\mathbb{X}(h)$ under convolution product:

$$(\zeta\xi)_I(x) = \sum_{I=S\sqcup T} \zeta_S(x|_S)\xi(x/_S).$$

The unit of $\mathbb{X}(h)$ is $u_I(x) := \delta_{x,\epsilon}$; the inverse of $\zeta \in \mathbb{X}(h)$ is

$$\zeta_I^{-1} = \zeta_I \circ s_I.$$

Example: $\mathbb{X}(\mathbf{L}) \cong (\mathbb{K}, +)$.

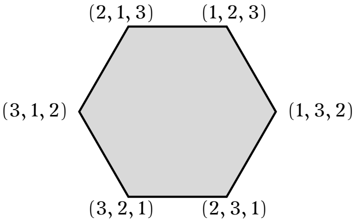
Standard Permutahedron

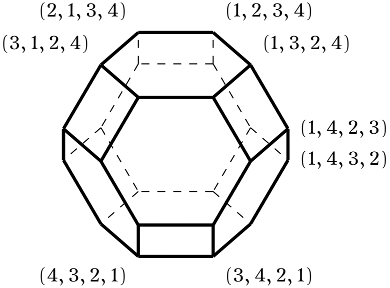
Let I be a non-empty finite set.

Let $\mathbb{R}^I$ be the real vector space of functions $x : I \to \mathbb{R}$. A vector x has coordinates $\{x_i\}_{i \in I}$, where x_i is the value of x at i.

Let $n = |I|$. The *standard permutahedron* π_I is the polytope in $\mathbb{R}^I$ whose vertices consist of all the permutations of the point $(1, 2, \ldots, n)$.

Standard Permutahedron





For example, $\pi_{\{a,b,c\}}$ is a regular hexagon lying on the plane $x_a + x_b + x_c = 6$, and $\pi_{\{a,b,c,d\}}$ lies on the hyperplane $x_a + x_b + x_c + x_d = 10$. We have $\dim \pi_I = n - 1$.

Images from "Hopf monoids and polynomial invariants of combinatorial structures", ECCO course - Marcelo Aguiar and Jose Bastidas.

Standard Permutahedron

For $i \in I$, let $e_i \in \mathbb{R}^I$ denote the *standard vector*

$$e_i = (0, \ldots, 1, \ldots, 0),$$

with j-th coordinate equal to $\delta(i, j)$. The set $\{e_i\}_{i \in I}$ is the *standard basis* of $\mathbb{R}^I$. For any nonempty subset $S \subseteq I$, let

$$e_S = \sum_{i \in S} e_i.$$

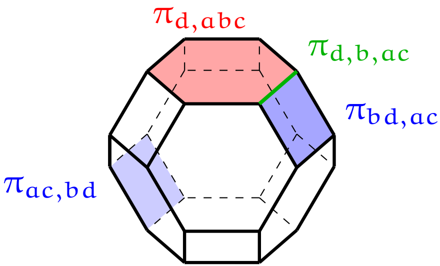
Given two vectors v and w, let $[v, w]$ denote the line segment joining their endpoints:

$$[v, w] = \{\lambda v + (1 - \lambda) w \mid 0 \leq \lambda \leq 1\}.$$

The standard permutahedron coincides with the following Minkowski sum:

$$\pi_I = e_I + \sum_{\{i,j\} \in \binom{I}{2}} [e_i, e_j].$$

The set of $(n-k)$-dimensional faces of π_I is in bijection with the set of compositions of I into k parts.



The closed normal cone of the face π_F is

$$\mathcal{N}_{\pi_I}(\pi_F) = \{\alpha_1 e_{S_1} + \cdots + \alpha_k e_{S_k} \in \mathbb{R}^I \mid \alpha_1 \geq \cdots \geq \alpha_k\}.$$

A fan $\mathcal{F}$ refines a fan $\mathcal{G}$ if every cone of $\mathcal{G}$ is a union of cones of $\mathcal{F}$.

Image from "Hopf monoids and polynomial invariants of combinatorial structures", ECCO course - Marcelo Aguiar and Jose Bastidas.

Generalized Permutahedra

A *generalized permutahedron* $\mathfrak{p} \subseteq \mathbb{R}^I$ is a polytope such that its normal fan is coarser than that of the standard permutahedron.

$$\mathrm{GP}[I] := \{\text{ generalized permutahedra } \mathfrak{p} \subseteq \mathbb{R}^I\}.$$

There is a (commutative) Hopf monoid structure on GP:

- Product: cartesian product;
- Coproduct: if $\mathfrak{p}_S$ denote the face of $\mathfrak{p}$ **maximized** by the functional $x \mapsto \sum_{i \in S} x_i$, then
$$\mathfrak{p}_S = \mathfrak{p}|_S \times \mathfrak{p}/_S.$$

Antipode (Aguiar, Ardila):

$$s_I(\mathfrak{p}) = (-1)^{|I|} \sum_{\mathfrak{q} \leq \mathfrak{p}} (-1)^{\dim(\mathfrak{q})} \mathfrak{q}.$$

Generalized permutahedra: Posets, graphs, matroids

There is a long tradition of modeling combinatorics geometrically.

Stanley (73): graphic zonotope $Z(G)$ associated to a graph G

$$Z(G) = \sum_{ij \in G} (e_i - e_j)$$

Geissinger (81): poset cone C_G associated to a poset P

$$C_P = \text{cone}\{e_i - e_j : i <_P j\}$$

Edmonds (70): matroid polytope P_M associated to a matroid P

$$C_P = \text{conv}\{e_{i_1} + \cdots e_{i_k} : \{i_1, \ldots, i_k\} \text{ is a basis of } M\}$$

Aguiar-Ardila: all the examples of Hopf monoids can be realized as sub-Hopf monoids of GP.

Geometrical application

Consider formal power series

$$A(t) = t + \sum_{n\geq 2} a_{n-1}t^n \qquad \text{and} \qquad B(t) = t + \sum_{n\geq 2} b_{n-1}t^n,$$

such that $A(B(t)) = t$. Then

$$\begin{aligned}
b_1 &= -a_1 \\
b_2 &= -a_2 + 2a_1^2 \\
b_3 &= -a_3 + 5a_2a_1 - 5a_1^3 \\
b_4 &= -a_4 + 6a_3a_1 + 3a_2^2 - 21a_2a_1^2 + 14a_1^4.
\end{aligned}$$

Geometrical application

Consider formal power series

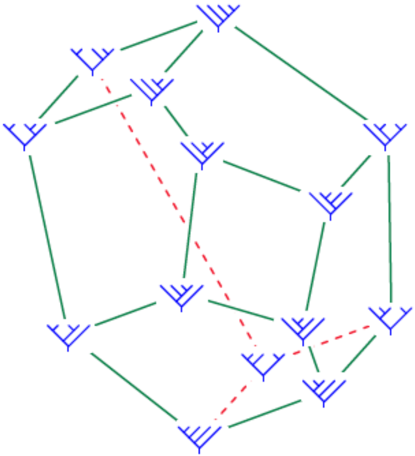
$$A(t) = t + \sum_{n \geq 2} a_{n-1} t^n \qquad \text{and} \qquad B(t) = t + \sum_{n \geq 2} b_{n-1} t^n,$$

such that $A(B(t)) = t$. Then

$$\begin{aligned} b_1 &= -a_1 \\ b_2 &= -a_2 + 2a_1^2 \\ b_3 &= -a_3 + 5a_2a_1 - 5a_1^3 \\ b_4 &= -a_4 + 6a_3a_1 + 3a_2^2 - 21a_2a_1^2 + 14a_1^4. \end{aligned}$$

What do these numbers count? Look at the associahedra!

Counting with the associahedra



Face structure of associahedra

$$b_4 = -a_4 + 6a_3a_1 + 3a_2^2 - 21a_2a_1^2 + 14a_1^4$$

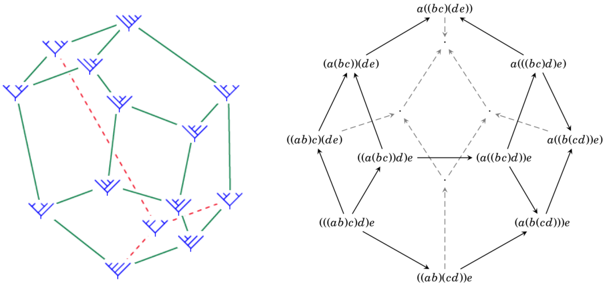
1 three-dimensional associahedron

6 pentagons and 3 squares

21 segments

14 points

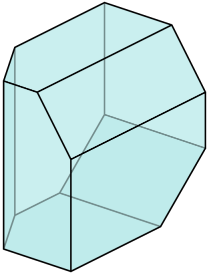
A "mythical polytope"

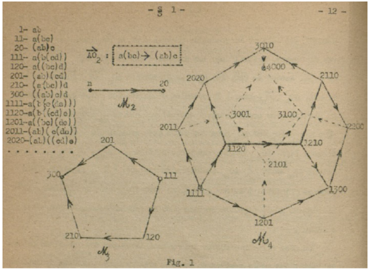


Some realizations of the associahedron

- Tamari (1951): defines the associahedron combinatorially. (lattice).
- Stasheff (1963): realizes the associahedron as a cell complex (polytope).
- Loday, Ronco (2001): relates the associahedron to a Hopf algebra of binary trees.

Associahedron

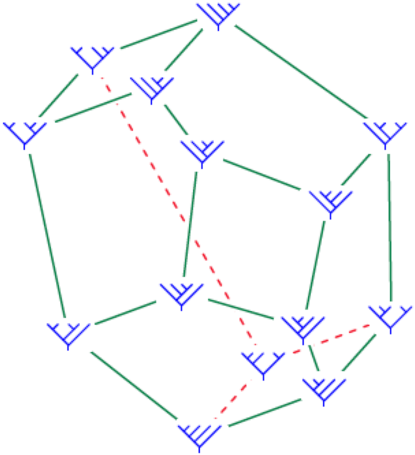




Some realizations of the associahedron

- Tamari (1951): defines the associahedron combinatorially. (lattice).
- Stasheff (1963): realizes the associahedron as a cell complex (polytope).
- Loday, Ronco (2001): relates the associahedron to a **Hopf algebra of binary trees**.

Tamari order



Tamari order

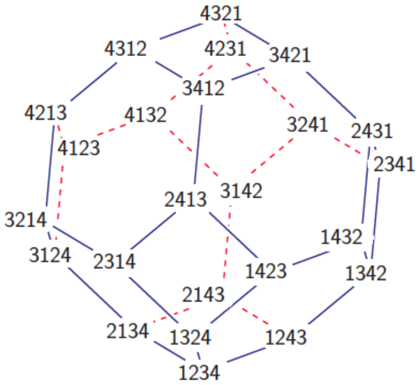
$\mathcal{Y}_n :=$ planar binary trees with $n+1$ leaves.

Covering relation:

$$s \lessdot t,$$

if t is obtained after moving one child node of s, from left to right, across their parent.

(Weak) Bruhat order



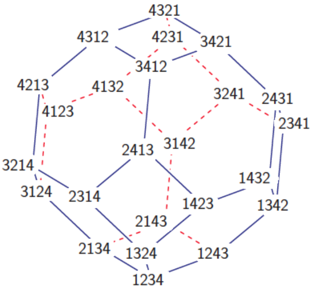
(Weak) Bruhat order on $\mathfrak{S}_n$

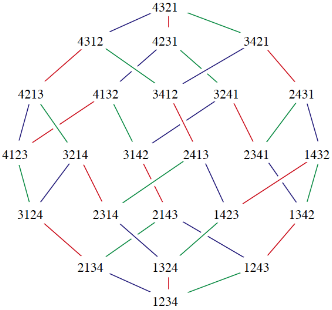
Covering relation:

$$u \lessdot (i\ i+1)u,$$

if the letter i appears before $i+1$ inside u.

Permutohedron

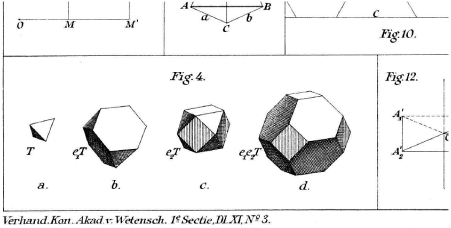


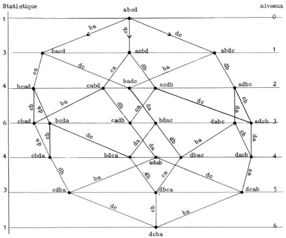


Some realizations of the permutohedron

- Schoute (1911): studies on regular polytopes.
- Guilbaud, Rosenstiehl (1963): derives the permutohedron as a lattice,
- Loday, Ronco (2001): relates the permutohedron to $\mathfrak{S}$Sym.

Permutohedron

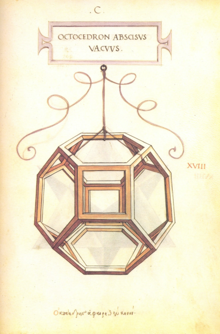


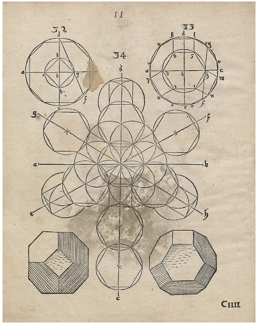


Some realizations of the permutohedron

- Schoute (1911): studies on regular polytopes.
- Guilbaud, Rosenstiehl (1963): derive the permutohedron as a lattice,
- Loday, Ronco (2001): relate the permutohedron to a Hopf algebra of permutations (Malvenuto-Reutenauer).

Permutohedron (3D), in arts



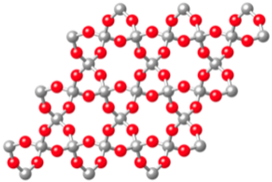


Quelle: Deutsche Fotothek

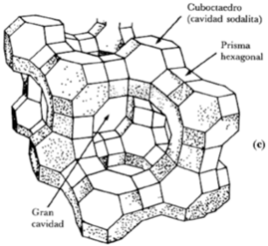
Some realizations of the permutohedron

- Da Vinci (1509): illustration from Luca Pacioli's 1509 book "The Divine Proportion".
- Hirschvogel (1543): Hirschvogel's book "Geometria".

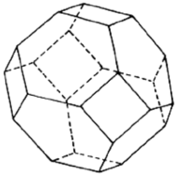
Permutohedron (3D), in nature



Tectosilicates: internal structure based on a three dimensional framework of silicate tetrahedra



(a)



(b)

Permutohedron, in mathematics

Consider formal power series

$$A(t) = 1 + \sum_{n \geq 1} a_n \frac{t^n}{n!} \quad \text{and} \quad B(t) = 1 + \sum_{n \geq 1} b_n \frac{t^n}{n!},$$

such that $A(t)B(t) = 1$. Then

$$\begin{aligned}
b_1 &= -a_1 \\
b_2 &= -a_2 + 2a_1^2 \\
b_3 &= -a_3 + 6a_2a_1 - 6a_1^3 \\
b_4 &= -a_4 + 8a_3a_1 + 6a_2^2 - 36a_2a_1^2 + 25a_1^4.
\end{aligned}$$

Permutohedron, in mathematics

Consider formal power series

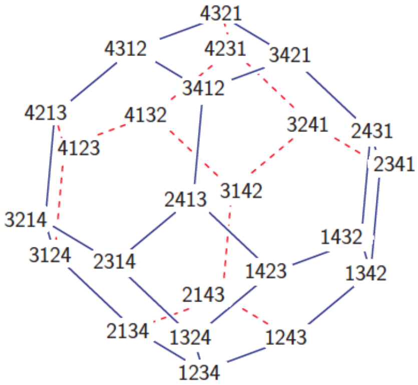
$$A(t) = 1 + \sum_{n \geq 1} a_n \frac{t^n}{n!} \qquad \text{and} \qquad B(t) = 1 + \sum_{n \geq 1} b_n \frac{t^n}{n!},$$

such that $A(t)B(t) = 1$. Then

$$\begin{aligned}
b_1 &= -a_1 \\
b_2 &= -a_2 + 2a_1^2 \\
b_3 &= -a_3 + 6a_2a_1 - 6a_1^3 \\
b_4 &= -a_4 + 8a_3a_1 + 6a_2^2 - 36a_2a_1^2 + 25a_1^4.
\end{aligned}$$

What do these numbers count? Look at the permutohedra!

Counting with the permutohedra



Face structure of permutohedra

$$b_4 = -a_4 + 8a_3a_1 + 6a_2^2 - 36a_2a_1^2 + 25a_1^4$$

1 three-dimensional permutohedron

8 hexagons and 6 squares

36 segments

24 points

- Associahedra “know” how to compute multiplicative inverses.

- Permutohedra “know” how to compute compositional inverses.

This is one of the many consequences of a Hopf monoid structure on *generalized permutahedra* (Aguiar-Ardila).

Generalized Permutahedra: (standard) Permutahedra

If $n := |I|$, let $\mathfrak{p}_I := \text{conv}\{(a_i)_{i \in I} \in \mathbb{R}I : \{a_i\}_{i \in I} = [n]\}$ and $\mathfrak{p}_n := \mathfrak{p}_{[n]}$.
The value $a_n = \zeta(\mathfrak{p}_n)$ determine the character $\zeta \in \mathbb{X}(\overline{P})$.
In the group of characters

$$\mathbb{X}(\overline{P}) \cong \text{ group of exponential formal power series, under multiplication}$$

of the (standard) **Permutahedra** $\overline{P}$, the multiplicative inverse of $1 + a_1 x + a_2 \frac{x^2}{2!} + a_3 \frac{x^3}{3!} + \cdots$ is $1 + b_1 x + b_2 \frac{x^2}{2!} + b_3 \frac{x^3}{3!} + \cdots$, where

$$b_n = (-1)^n \sum_{F \leq \mathfrak{p}_n} (-1)^{\dim F} a_F,$$

with $a_F = a_{f_1} a_{f_2} \cdots a_{f_k}$, for each face $F \cong \mathfrak{p}_{f_1} \times \mathfrak{p}_{f_2} \times \cdots \mathfrak{p}_{f_k}$ of the permutahedron.

Generalized Permutahedra: (Loday's) Associahedra

Let $\mathfrak{a}_n := \sum_{i,j \in [n]} \Delta_{[i,j]}$.

In the group of characters

$$\mathbb{X}(\overline{A}) \cong \text{ group of ordinary formal power series, under composition}$$

of the (Loday's) **Associahedra** $\overline{P}$, the compositional inverse of $x + a_1 x^2 + a_2 x^3 + a_3 x^4 \cdots$ is $x + b_1 x^2 + b_2 x^3 + b_3 x^4 + \cdots$, where

$$b_n = (-1)^n \sum_{F \leq \mathfrak{p}_n} (-1)^{\dim F} a_F,$$

with $a_F = a_{f_1} a_{f_2} \cdots a_{f_k}$, for each face $F \cong \mathfrak{a}_{f_1} \times \mathfrak{a}_{f_2} \times \cdots \mathfrak{a}_{f_k}$ of the associahedron.

Non-commutative probability

- The field of *Free Probability* was created by Dan-Virgil Voiculescu in the 1980s.
- Philosophy: investigate the notion of “freeness" in analogy to the concept of “independence” from (classical) probability theory.
- A combinatorial theory of freeness was developed by Nica and Speicher in the 1990s.
- Voiculescu discovered freeness also asymptotically for many kinds of random matrices (1991).

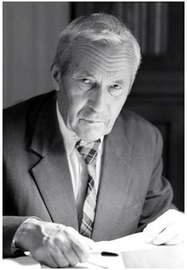
Dan Voiculescu , 2015

Commutative vs non-commutative

Voiculescu: "'Free probability is a probability theory adapted to dealing with variables which have the highest degree of noncommutativity. *Failure of commutativity may occur in many ways*."

- Quantum mechanics' commutation relation: $XY - YX = I$.
- Free product of groups.
- Independent random matrices tend to be asymptotically freely independent, under certain conditions.

Classical probability space



Andrey Kolmogorov

A **probability space** (Kolmogorov, 1930's) is given by the following data:

- a set Ω (sample space),
- a collection $\mathcal{F}$ (event space),
- $\mathbb{P} : \mathcal{F} \to [0, 1]$ (probability function),

satisfying several axioms.

Expectation: for every bounded random variable $X \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$, let

$$\mathbb{E}[X] := \int_\Omega X(\omega)\, d\mathbb{P}(\omega).$$

Intuition: replace $(L^\infty(\Omega, \mathcal{F}, \mathbb{P}), \mathbb{E})$ by a more general pair $(\mathcal{A}, \varphi)$.

Non-commutative probability space

A **non-commutative probability space** is a pair $(\mathcal{A}, \varphi)$ such that

- $\mathcal{A}$ is a unital associative algebra over $\mathbb{C}$;
- $\varphi : \mathcal{A} \to \mathbb{C}$ is a linear functional such that $\varphi(1_{\mathcal{A}}) = 1$.

Non-commutative probability space

A **non-commutative probability space** is a pair $(\mathcal{A}, \varphi)$ such that

- $\mathcal{A}$ is a unital associative algebra over $\mathbb{C}$;
- $\varphi : \mathcal{A} \to \mathbb{C}$ is a linear functional such that $\varphi(1_{\mathcal{A}}) = 1$.

Examples: $(L^\infty(\Omega, \mathcal{F}, \mathbb{P}), \mathbb{E})$, $\left(\mathrm{Mat}_n(\mathbb{C}), \frac{1}{n}\mathrm{Tr}\right)$, $(\mathrm{Mat}_n(\Omega), \varphi)$.

Non-commutative probability space

A **non-commutative probability space** is a pair $(\mathcal{A}, \varphi)$ such that

- $\mathcal{A}$ is a unital associative algebra over $\mathbb{C}$;
- $\varphi : \mathcal{A} \to \mathbb{C}$ is a linear functional such that $\varphi(1_{\mathcal{A}}) = 1$.

Examples: $(L^{\infty}(\Omega, \mathcal{F}, \mathbb{P}), \mathbb{E})$, $\left(\mathrm{Mat}_n(\mathbb{C}), \frac{1}{n}\mathrm{Tr}\right)$, $(\mathrm{Mat}_n(\Omega), \varphi)$.

$$\varphi(a) := \int_{\Omega} \mathrm{tr}(a(\omega))\, d\mathbb{P}(\omega)$$

Non-commutative probability space

Random variable: $a \in \mathcal{A}$

Moments: $(\varphi(a), \varphi(a^2), \varphi(a^3), \dots) \longleftrightarrow \mu : \mathbb{C}[x] \to \mathbb{C}, \mu(t^i) := \varphi(a^i)$

Join distribution of $(a_1, \dots, a_k)$: if $1 \le i_1, \dots, i_n \le k$,

$$\mu : \mathbb{C}\langle t_1, \dots, t_k \rangle \to \mathbb{C} \quad , \quad \mu(t_{i_1} \cdots t_{i_n}) := \varphi(a_{i_1} \cdots a_{i_n})$$

Non-commutative probability space

A **non-commutative probability space** is a pair $(\mathcal{A}, \varphi)$ such that

- $\mathcal{A}$ is a unital associative algebra over $\mathbb{C}$;
- $\varphi : \mathcal{A} \to \mathbb{C}$ is a linear functional such that $\varphi(1_{\mathcal{A}}) = 1$.

In a (classical) probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the notion of independence between two random variables $X, Y : \Omega \to \mathbb{C}$ implies

$$\mathbb{E}(X^m Y^n) = \mathbb{E}(X^m)\mathbb{E}(Y^n).$$

Free independence

Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space.

Free independence

Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space.

Consider $\{\mathcal{A}_i\}_{i \in I}$ unital subalgebras of $\mathcal{A}$.

Free independence

Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space.

Consider $\{\mathcal{A}_i\}_{i \in I}$ unital subalgebras of $\mathcal{A}$.

The family $\{\mathcal{A}_i\}_{i \in I}$ of algebras is **freely independent** if for every $n \in \mathbb{N}$ and for every choice of $(i_1, \ldots, i_n)$ of "different neighbouring indices" (i.e., $i_{j-1} \neq i_j \neq i_{j+1}$), we have

$$\varphi(a_1 \cdots a_n) = 0,$$

whenever $a_j \in \mathcal{A}_{i_j}$ and $\varphi(a_j) = 0$, for every $1 \leq j \leq n$.

Free independence

Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space.

Consider $\{\mathcal{A}_i\}_{i\in I}$ unital subalgebras of $\mathcal{A}$.

The family $\{\mathcal{A}_i\}_{i\in I}$ of algebras is **freely independent** if for every $n \in \mathbb{N}$ and for every choice of $(i_1, \ldots, i_n)$ of "different neighbouring indices" (i.e., $i_{j-1} \neq i_j \neq i_{j+1}$), we have

$$\varphi(a_1 \cdots a_n) = 0,$$

whenever $a_j \in \mathcal{A}_{i_j}$ and $\varphi(a_j) = 0$, for every $1 \leq j \leq n$.

A family $(a_i)_{i\in I}$ of non-commutative random variables is called **free** if the family of subalgebras $(\langle 1_{\mathcal{A}}, a_i\rangle)_{i\in I}$ is freely independent.

Free independence

Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space.

Consider $\{\mathcal{A}_i\}_{i\in I}$ unital subalgebras of $\mathcal{A}$.

The family $\{\mathcal{A}_i\}_{i\in I}$ of algebras is **freely independent** if for every $n \in \mathbb{N}$ and for every choice of $(i_1, \dots, i_n)$ of "different neighbouring indices" (i.e., $i_{j-1} \neq i_j \neq i_{j+1}$), we have

$$\varphi(a_1 \cdots a_n) = 0,$$

whenever $a_j \in \mathcal{A}_{i_j}$ and $\varphi(a_j) = 0$, for every $1 \leq j \leq n$.

A family $(a_i)_{i\in I}$ of non-commutative random variables is called **free** if the family of subalgebras $(\langle 1_{\mathcal{A}}, a_i \rangle)_{i\in I}$ is freely independent.

Sets of variables in $(\mathcal{A}, \varphi)$ are free if the algebras they generate are free.

It looks artificial...

Free independence

Free independence provides a rule to compute mixed moments.

Free independence

Free independence provides a rule to compute mixed moments.

Let $(\mathcal{A}, \varphi)$ be a n.c.p.s. and let $a, b \in \mathcal{A}$ free n.c.r.v.

Free independence

Free independence provides a rule to compute mixed moments.

Let $(\mathcal{A}, \varphi)$ be a n.c.p.s. and let $a, b \in \mathcal{A}$ free n.c.r.v.

What is $\varphi(ab)$?

Free independence

Free independence provides a rule to compute mixed moments.

Let $(\mathcal{A}, \varphi)$ be a n.c.p.s. and let $a, b \in \mathcal{A}$ free n.c.r.v.

What is $\varphi(ab)$? $\varphi((a - \varphi(a)1_{\mathcal{A}})(b - \varphi(b)1_{\mathcal{A}})) = 0$, so

Free independence

Free independence provides a rule to compute mixed moments.

Let $(\mathcal{A}, \varphi)$ be a n.c.p.s. and let $a, b \in \mathcal{A}$ free n.c.r.v.

What is $\varphi(ab)$? $\varphi((a - \varphi(a)1_{\mathcal{A}})(b - \varphi(b)1_{\mathcal{A}})) = 0$, so

$$\begin{aligned} 0 &= \varphi((a - \varphi(a) \cdot 1_{\mathcal{A}})(b - \varphi(b) \cdot 1_{\mathcal{A}})) \\ &= \varphi(ab) - \varphi(a \cdot 1_{\mathcal{A}})\varphi(b) - \varphi(a)\varphi(1_{\mathcal{A}} \cdot b) + \varphi(a)\varphi(b)\varphi(1_{\mathcal{A}}) \\ &= \varphi(ab) - \varphi(a)\varphi(b) - \varphi(a)\varphi(b) + \varphi(a)\varphi(b) \end{aligned}$$

Free independence

Free independence provides a rule to compute mixed moments.

Let $(\mathcal{A}, \varphi)$ be a n.c.p.s. and let $a, b \in \mathcal{A}$ free n.c.r.v.

What is $\varphi(ab)$? $\varphi((a - \varphi(a)1_{\mathcal{A}})(b - \varphi(b)1_{\mathcal{A}})) = 0$, so

$$\begin{aligned} 0 &= \varphi((a - \varphi(a) \cdot 1_{\mathcal{A}})(b - \varphi(b) \cdot 1_{\mathcal{A}})) \\ &= \varphi(ab) - \varphi(a \cdot 1_{\mathcal{A}})\varphi(b) - \varphi(a)\varphi(1_{\mathcal{A}} \cdot b) + \varphi(a)\varphi(b)\varphi(1_{\mathcal{A}}) \\ &= \varphi(ab) - \varphi(a)\varphi(b) - \varphi(a)\varphi(b) + \varphi(a)\varphi(b) \end{aligned}$$

Therefore, $\varphi(ab) = \varphi(a)\varphi(b)$.

Free independence

Free independence provides a rule to compute mixed moments.

Let $(\mathcal{A}, \varphi)$ be a n.c.p.s. and let $\{a_1, a_2\}, \{b\} \subseteq \mathcal{A}$ free n.c.r.v.

Free independence

Free independence provides a rule to compute mixed moments.

Let $(\mathcal{A}, \varphi)$ be a n.c.p.s. and let $\{a_1, a_2\}, \{b\} \subseteq \mathcal{A}$ free n.c.r.v.

What is $\varphi(a_1 b a_2)$?

Free independence

Free independence provides a rule to compute mixed moments.

Let $(\mathcal{A}, \varphi)$ be a n.c.p.s. and let $\{a_1, a_2\}, \{b\} \subseteq \mathcal{A}$ free n.c.r.v.

What is $\varphi(a_1 b a_2)$? From

$$\varphi\Big((a_1 - \varphi(a_1) \cdot 1_{\mathcal{A}})(b - \varphi(b) \cdot 1_{\mathcal{A}})(a_2 - \varphi(a_2) \cdot 1_{\mathcal{A}})\Big) = 0,$$

we obtains

$$\varphi(a_1 b a_2) = \varphi(a_1 a_2)\varphi(b).$$

Free independence

Free independence provides a rule to compute mixed moments.

Let $(\mathcal{A}, \varphi)$ be a n.c.p.s. and let $\{a_1, a_2\}, \{b\} \subseteq \mathcal{A}$ free n.c.r.v.

What is $\varphi(a_1 b a_2)$? From

$$\varphi\Big((a_1 - \varphi(a_1)\cdot 1_{\mathcal{A}})(b - \varphi(b)\cdot 1_{\mathcal{A}})(a_2 - \varphi(a_2)\cdot 1_{\mathcal{A}})\Big) = 0,$$

we obtains

$$\varphi(a_1 b a_2) = \varphi(a_1 a_2)\varphi(b).$$

If $\{a_1, a_2\}, \{b_1, b_2\} \subseteq \mathcal{A}$ free n.c.r.v,

$$\begin{aligned}\varphi(a_1 b_1 a_2 b_2) =& \varphi(a_1 a_2)\varphi(b_1)\varphi(b_2) + \varphi(a_1)\varphi(a_2)\varphi(b_1 b_2)\\ &- \varphi(a_1)\varphi(a_2)\varphi(b_1)\varphi(b_2).\end{aligned}$$

Free independence

Free independence provides a rule to compute mixed moments.

Let $(\mathcal{A}, \varphi)$ be a n.c.p.s. and let $\{a_1, a_2\}, \{b\} \subseteq \mathcal{A}$ free n.c.r.v.

What is $\varphi(a_1 b a_2)$? From

$$\varphi\Big((a_1 - \varphi(a_1) \cdot 1_{\mathcal{A}})(b - \varphi(b) \cdot 1_{\mathcal{A}})(a_2 - \varphi(a_2) \cdot 1_{\mathcal{A}})\Big) = 0,$$

we obtains

$$\varphi(a_1 b a_2) = \varphi(a_1 a_2)\varphi(b).$$

If $\{a_1, a_2\}, \{b_1, b_2\} \subseteq \mathcal{A}$ free n.c.r.v,

$$\begin{aligned}\varphi(a_1 b_1 a_2 b_2) =& \varphi(a_1 a_2)\varphi(b_1)\varphi(b_2) + \varphi(a_1)\varphi(a_2)\varphi(b_1 b_2) \\ & - \varphi(a_1)\varphi(a_2)\varphi(b_1)\varphi(b_2).\end{aligned}$$

$$\Rightarrow \varphi(abab) = \varphi(a^2)\varphi(b)^2 + \varphi(a)^2\varphi(b^2) - \varphi(a)^2\varphi(b)^2.$$

Freeness from the free product

Voiculescu gave the definition of freeness in the context of von Neumann algebras of free products of groups.

$$F(G) := \{\alpha : G \to \mathbb{C} : |\{g \in G \mid \alpha(g) \neq 0\}| < \infty\},$$
$$(\alpha * \beta)(g) := \sum_{h \in G} \alpha(gh^{-1})\beta(h),$$

$$\varphi_G : F(G) \to \mathbb{C} \qquad , \qquad \alpha \mapsto \alpha(e).$$

$F(G)$ is linearly generated by $\{\delta_g : g \in G\}$, where

$$\delta_g(h) = \begin{cases} 1, & h = g \\ 0, & h \neq g \end{cases}$$

Freeness from the free product

Theorem

If $\{G_i\}_{i\in I}$ *subgroups of* G *are algebraically free, then* $\{F(G_i)\}_{i\in I} \subseteq F(G)$ *are freely independent in* $(F(G), \varphi_G)$.

Sketch of the proof:

Consider $(i_1, \ldots, i_n) \in I^n$ such that $i_1 \neq i_2 \neq \cdots \neq i_n$, and $\alpha_k \in F(G_{i_k})$ such that $\alpha_k(e) = 0$, for $1 \leq k \leq n$.

Freeness from the free product

Theorem

If $\{G_i\}_{i\in I}$ *subgroups of* G *are algebraically free, then* $\{F(G_i)\}_{i\in I} \subseteq F(G)$ *are freely independent in* $(F(G), \varphi_G)$.

Sketch of the proof:
Consider $(i_1, \dots, i_n) \in I^n$ such that $i_1 \neq i_2 \neq \cdots \neq i_n$, and $\alpha_k \in F(G_{i_k})$ such that $\alpha_k(e) = 0$, for $1 \leq k \leq n$.

$$\begin{aligned}\varphi(\alpha_1 * \cdots * \alpha_n) &= (\alpha_1 * \cdots * \alpha_n)(e) \\ &= \sum_{\substack{g_1,\dots,g_n \in G \\ g_1\cdots g_n = e}} \alpha_1(g_1)\cdots\alpha_n(g_n).\end{aligned}$$

Since $G_{i_1}, \dots, G_{i_n}$ are algebraically free. there exists k such that $g_k = e$, leading to $\varphi(\alpha_1 * \cdots * \alpha_n)$.

Non-commutative independence

Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space. Consider $\{\mathcal{A}_i\}_{i \in I}$ unital subalgebras of $\mathcal{A}$. Let $a_1 \in \mathcal{A}_{i_1}, \ldots, a_n \in \mathcal{A}_{i_n}$ such that $i_j \neq i_{j+1}$.

The family $\{\mathcal{A}_i\}_{i \in I}$ is

- **freely independent** if

$$\varphi(a_1 \cdots a_n) = 0,$$

  when $\varphi(a_j) = 0$, for all $1 \leq j \leq n$;

- **boolean independent** if

$$\varphi(a_1 \cdots a_n) = \varphi(a_1) \cdots \varphi(a_n);$$

Other notions: *monotone independence*, *conditional monotone*, ...

Back to the examples

$$\begin{aligned}
\varphi(ab) &= \varphi(a)\varphi(b) \\
\varphi(a_1 b a_2) &= \varphi(a_1 a_2)\varphi(b) \\
\varphi(a_1 b_1 a_2 b_2) &= \varphi(a_1 a_2)\varphi(b_1)\varphi(b_2) + \varphi(a_1)\varphi(a_2)\varphi(b_1 b_2) \\
&\quad - \varphi(a_1)\varphi(a_2)\varphi(b_1)\varphi(b_2) \\
\varphi(a_1 b_1 c b_2 a_2 d a_3) &= \varphi(a_1 \color{blue}{b_1 c b_2} \color{red}{a_2 d a_3}) \\
&= \varphi(a_1 \color{red}{a_2 d a_3})\varphi(\color{blue}{b_1 c b_2}) \\
&= \varphi(a_1 a_2 d a_3)\varphi(b_1 c b_2) \\
&= \varphi(a_1 a_2 a_3)\varphi(b_1 b_2)\varphi(c)\varphi(d).
\end{aligned}$$

"Non-crossing moments" factorize; "crossing moments" do not.

Back to $(\mathcal{A}, \varphi)$

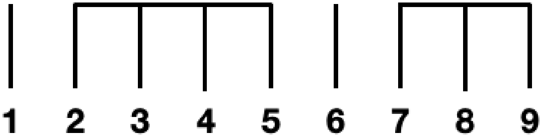
Let $n \in \mathbb{N}$ and $a_1, a_2, \ldots, a_n \in \mathcal{A}$.

Consider $\{f_n : \mathcal{A}^n \to \mathbb{C} \,|\, n \geq 0\}$ a family of multilinear functionals.

Let $\pi = \{B_1, \ldots, B_k\} \in NC(n)$. We define

$$f_\pi(a_1, \ldots, a_n) := \prod_{\substack{B \in \pi \\ B = \{b_1 < b_2 < \cdots < b_r\}}} f_{|B|}(b_1, b_2, \ldots, b_r).$$

Back to $(\mathcal{A}, \varphi)$



If $\pi = \{\{1\}, \{2,3,4,5\}, \{6\}, \{7,8,9\}\}$, then

$$f_\pi(a_1, \ldots, a_9) = f_1(a_1)\, f_4(a_2, a_3, a_4, a_5)\, f_1(a_6)\, f_3(a_7, a_8, a_9).$$

Moment to cumulant relations in $(\mathcal{A}, \varphi)$

Consider the multilinear functionals

$\{r_n : \mathcal{A}^n \to \mathbb{C}\}_{n\geq 1}$ (Free cumulants) , $\{b_n : \mathcal{A}^n \to \mathbb{C}\}_{n\geq 1}$ (Boolean cumulants) , $\{h_n : \mathcal{A}^n \to \mathbb{C}\}_{n\geq 1}$ (Monotone cumulants)

defined by

$$\varphi(a_1 \cdots a_n) = \sum_{\pi \in NC(n)} r_\pi(a_1, \ldots, a_n),$$

$$\varphi(a_1 \cdots a_n) = \sum_{\pi \in NC_{\mathrm{Int}}(n)} b_\pi(a_1, \ldots, a_n),$$

$$\varphi(a_1 \cdots a_n) = \sum_{\pi \in NC(n)} \frac{1}{\tau(\pi)!} h_\pi(a_1, \ldots, a_n).$$

Double tensor Hopf algebra

Double tensor Hopf algebra $T(T_+(V))$: non-commutative and non-cocommutative Hopf algebra, with graduation

$$T(T_+(V))_n := \bigoplus_{n_1+\cdots+n_k=n} V^{\otimes n_1} \otimes \cdots \otimes V^{\otimes n_k}.$$

Elements in $T(T_+(V))_n$ are written as (linear combinations of) words with bars

$$w_1|\cdots|w_k,$$

where $w_i \in V^{\otimes n_i}$ for some $n_1 + \cdots + n_k = n$. We call this elements **words on (non-empty) words**.

Double tensor Hopf algebra

Let V be a $\mathbb{K}$-vector space.

If $k \geq 0$, we write elementary tensors from $V^{\otimes k}$ as **words**, $u_1 u_2 \cdots u_k$, with $u_i \in V$. We called the $\mathbb{K}$-vector spaces

$$\mathsf{T}(V) := \bigoplus_{k \geq 0} V^{\otimes k} \quad , \quad \mathsf{T}_+(V) := \bigoplus_{k \geq 1} V^{\otimes k}$$

the **tensor module** and **reduced tensor module**, respectively, generated by V.

- *Product rule*: if $u \in T(T_+(V))_n$ and $v \in_m$, then

$$u|v := u_1|\cdots|u_r|v_1|\cdots|v_s \in T(T_+(V))_{n+m}.$$

- *Coproduct rule*: given a word $u = u_1 \cdots u_n \in V^{\otimes n}$ and $A = \{a_1, \ldots, a_k\} \subset \mathbb{N}$, we write $u_A := u_{a_1} \cdots u_{a_k}$.
Consider the map $\Delta : T_+(V) \to T(V) \otimes T(T_+(V))$ given by

$$\begin{aligned} \Delta(u) : &= \sum_{A \subseteq [n]} u_A \otimes u_{K(A,[n])} \\ &= \sum_{A \subseteq [n]} u_A \otimes u_{K_1}|\cdots|u_{K_r}. \end{aligned}$$

Finally, we extend the map Δ multiplicatively to all of $T(T_+(V))$, by setting

$$\Delta(w_1|\cdots|w_k) := \Delta(w_1)\cdots\Delta(w_k).$$

For example, we have

$$\Delta(ab) = 1 \otimes ab + a \otimes b + b \otimes a + ab \otimes 1;$$

$$\Delta(a|b) = 1 \otimes a|b + a \otimes b + b \otimes a + a|b \otimes 1;$$

$$\Delta(abc) = 1\otimes abc + a\otimes bc + b \otimes a|c + c\otimes ab + ab\otimes c + ac\otimes b + bc\otimes a + 1\otimes abc;$$

$$\begin{aligned}\Delta(a|bc) =& 1 \otimes a|bc + a \otimes bc + b \otimes a|c + c \otimes a|b \\ & + a|b \otimes c + a|c \otimes b + bc \otimes a + 1 \otimes a|bc;\end{aligned}$$

Algebraic approach to cumulants (Ebrahimi-Fard, Patras)

- $(\mathcal{A}, \varphi)$ non-commutative probability space.
- $H = T(T_+(\mathcal{A}))$ *words on non-empty words on* $\mathcal{A}$.
- The coproduct Δ in H is *codendriform*: $\Delta = \Delta_< + \Delta_>$.
- The space $(\mathrm{Hom}_{\mathrm{lin}}(H, \mathbb{K}), <, >)$ is a dendriform algebra, with $* = < + >$.
- The linear form φ is extended to $T_+(\mathcal{A})$ by defining to all words $u = a_1 \cdots a_n \in \mathcal{A}^{\otimes n}$

$$\varphi(a_1 a_2 \cdots a_n) := \varphi(a_1 \cdot_{\mathcal{A}} a_2 \cdot_{\mathcal{A}} \cdots \cdot_{\mathcal{A}} a_n).$$

This is the **multivariate moment** of u.

Algebraic approach to cumulants (Ebrahimi-Fard, Patras)

- $(\mathcal{A}, \varphi)$ non-commutative probability space.
- $H = T(T_+(\mathcal{A}))$ *words on non-empty words on* $\mathcal{A}$.
- The coproduct Δ in H is *codendriform*: $\Delta = \Delta_< + \Delta_>$.
- The space $(\mathrm{Hom}_{\mathrm{lin}}(H, \mathbb{K}), <, >)$ is a dendriform algebra, with $* = < + >$.
- The linear form φ is extended to $T_+(\mathcal{A})$ by defining to all words $u = a_1 \cdots a_n \in \mathcal{A}^{\otimes n}$

$$\varphi(a_1 a_2 \cdots a_n) := \varphi(a_1 \cdot_{\mathcal{A}} a_2 \cdot_{\mathcal{A}} \cdots \cdot_{\mathcal{A}} a_n).$$

  This is the **multivariate moment** of u.
  The map φ is then extended multiplicatively to a map $\Phi : T(T_+(\mathcal{A})) \to \mathbb{K}$ with $\Phi(\mathbf{1}) := 1$ and

$$\Phi(u_1 | \cdots | u_k) := \varphi(u_1) \cdots \varphi(u_k).$$

Cumulants as infinitesimal characters

Proposition (Ebrahimi-Fard, Patras -2015)

Let $\rho, \kappa, \beta \in \mathfrak{g}(\mathcal{A})$ *the infinitesimal characters solving*

$$\Phi = \exp_*(\rho),$$

$$\Phi = \epsilon + \kappa \prec \Phi$$

and

$$\Phi = \epsilon + \Phi \succ \beta.$$

Then, ρ, κ, β *correspond to the* ***monotone cumulants, free cumulants and boolean cumulants****, respectively.*

For any word $u = a_1 \cdots a_n \in \mathcal{A}^{\otimes n}$, we have

$$h_n(a_1, \ldots, a_n) = \rho(u), r_n(a_1, \ldots, a_n) = \kappa(u), b_n(a_1, \ldots, a_n) = \beta(u).$$

Series on species

From species to vector spaces I

There are functors

$$\mathcal{K}, \overline{\mathcal{K}}, \mathcal{K}^{\vee}, \overline{\mathcal{K}} : \text{Hopf monoids in species} \longrightarrow \mathbb{N}\text{-graded Hopf algebras.}$$

$$\mathcal{K}(\mathsf{h}) = \mathcal{K}^{\vee}(\mathsf{h}) := \bigoplus_{n \geq 0} \mathsf{h}[n]$$

$$\overline{\mathcal{K}}(\mathsf{h}) := \bigoplus_{n \geq 0} \mathsf{h}[n]_{\mathfrak{S}_n} \quad , \quad \overline{\mathcal{K}}^{\vee}(\mathsf{h}) := \bigoplus_{n \geq 0} \mathsf{h}[n]^{\mathfrak{S}_n}$$

From species to vector spaces I

There are functors

$$\mathcal{K}, \overline{\mathcal{K}}, \mathcal{K}^{\vee}, \overline{\mathcal{K}} : \text{Hopf monoids in species} \longrightarrow \mathbb{N}\text{-graded Hopf algebras.}$$

$$\mathcal{K}(\mathrm{h}) = \mathcal{K}^{\vee}(\mathrm{h}) := \bigoplus_{n \geq 0} \mathrm{h}[n]$$

$$\overline{\mathcal{K}}(\mathrm{h}) := \bigoplus_{n \geq 0} \mathrm{h}[n]_{\mathfrak{S}_n} \quad , \quad \overline{\mathcal{K}}^{\vee}(\mathrm{h}) := \bigoplus_{n \geq 0} \mathrm{h}[n]^{\mathfrak{S}_n}$$

Patras-Schocker-Reutenauer:

$$\mathcal{K}(\mathrm{h}) : \text{ cosymmetrized bialgebra}$$

$$\mathcal{K}^{\vee}(\mathrm{h}) : \text{ symmetrized bialgebra}$$

From species to vector spaces I

There are functors

$$\mathcal{K}, \overline{\mathcal{K}}, \mathcal{K}^{\vee}, \overline{\mathcal{K}} : \text{Hopf monoids in species} \longrightarrow \mathbb{N}\text{-graded Hopf algebras.}$$

$$\mathcal{K}(\mathrm{h}) = \mathcal{K}^{\vee}(\mathrm{h}) := \bigoplus_{n \geq 0} \mathrm{h}[n]$$

$$\overline{\mathcal{K}}(\mathrm{h}) := \bigoplus_{n \geq 0} \mathrm{h}[n]_{\mathfrak{S}_n} \quad , \quad \overline{\mathcal{K}}^{\vee}(\mathrm{h}) := \bigoplus_{n \geq 0} \mathrm{h}[n]^{\mathfrak{S}_n}$$

- $\mathcal{K}(\mathrm{h}) \cong \overline{\mathcal{K}}(\mathrm{L} \times \mathrm{h})$.
- If h is finite-dimensional, then $\overline{\mathcal{K}}(\mathrm{h}^*) \cong \overline{\mathcal{K}}(\mathrm{h})^*$.
- If h is cocommutative, then so are $\mathcal{K}(\mathrm{h})$ and $\overline{\mathcal{K}}(\mathrm{h})$.
- If h is commutative, so is $\overline{\mathcal{K}}(\mathrm{h})$.

From species to vector spaces II

Let p be a species.

From species to vector spaces II

Let p be a species.
A **series** s of p is a collection of elements

$$s_I \in \mathrm{p}[I],$$

one for each finite set I, such that

$$\mathrm{p}[\sigma](s_I) = s_J,$$

for each bijection $\sigma : I \to J$.

From species to vector spaces II

Let p be a species.
A **series** s of p is a collection of elements

$$s_I \in \mathsf{p}[I],$$

one for each finite set I, such that

$$\mathsf{p}[\sigma](s_I) = s_J,$$

for each bijection $\sigma : I \to J$.

The space $\mathscr{S}(\mathsf{p})$ of all series of p is a vector space:

$$(s + t)_I = s_I + t_I \quad , \quad (\lambda \cdot s)_I := \lambda\, s_I,$$

for $s, t \in \mathscr{S}(\mathsf{p})$ and $\lambda \in \mathbb{K}$.

From species to vector spaces II

Let p be a species.
A **series** s of p is a collection of elements

$$s_I \in \mathrm{p}[I],$$

one for each finite set I, such that

$$\mathrm{p}[\sigma](s_I) = s_J,$$

for each bijection $\sigma : I \to J$.

From species to vector spaces II

Let p be a species.
A **series** s of p is a collection of elements

$$s_I \in \mathsf{p}[I],$$

one for each finite set I, such that

$$\mathsf{p}[\sigma](s_I) = s_J,$$

for each bijection $\sigma : I \to J$.

Let E be the exponential map. A series s of p corresponds to the morphism of species

$$\mathsf{E} \to \mathsf{p}$$

$$*_I \mapsto s_I,$$

so $\mathscr{S}(\mathsf{p}) \cong \mathrm{Hom}_{\mathsf{Sp}}(\mathsf{E}, \mathsf{p})$.

From species to vector spaces II

Let p be a species.
A **series** s of p is a collection of elements

$$s_I \in \mathrm{p}[I],$$

one for each finite set I, such that

$$\mathrm{p}[\sigma](s_I) = s_J, \tag{1}$$

for each bijection $\sigma : I \to J$.

From species to vector spaces II

Let p be a species.
A **series** s of p is a collection of elements

$$s_I \in \mathsf{p}[I],$$

one for each finite set I, such that

$$\mathsf{p}[\sigma](s_I) = s_J, \tag{1}$$

for each bijection $\sigma : I \to J$.

Property (2) implies that each $s_{[n]}$ is an $\mathfrak{S}_n$-invariant element of $\mathsf{p}[n]$. In fact,

$$\mathscr{S}(\mathsf{p}) \cong \prod_{n \geq 0} \mathsf{p}[n]^{\mathfrak{S}_n}$$

$$s \mapsto (s_{[n]})_{n \geq 0}.$$

From species to vector spaces II

Let p be a species.
A **series** s of p is a collection of elements

$$s_I \in \mathrm{p}[I],$$

one for each finite set I, such that

$$\mathrm{p}[\sigma](s_I) = s_J, \tag{2}$$

for each bijection $\sigma : I \to J$.

From species to vector spaces II

Let p be a species.
A **series** s of p is a collection of elements

$$s_{\mathrm{I}} \in \mathrm{p}[\mathrm{I}],$$

one for each finite set I, such that

$$\mathrm{p}[\sigma](s_{\mathrm{I}}) = s_{\mathrm{J}}, \tag{2}$$

for each bijection $\sigma : \mathrm{I} \to \mathrm{J}$.

There is a functor

$$\mathscr{S} : \mathsf{Sp} \to \mathsf{Vec}.$$

The functor $\mathscr{S}$ is *braided lax monoidal*: it preserves monoids, commutative monoids, Lie monoids . . .

Decorated series

Let V be a vector space.

Decorated series

Let V be a vector space. Recall that a series of p corresponds to a morphism of species $\mathsf{E} \longrightarrow \mathsf{p}$.

Decorated series

Let V be a vector space. Recall that a series of p corresponds to a morphism of species $\mathsf{E} \longrightarrow \mathsf{p}$.

A V-**decorated series**, or **decorated series**, is a morphism of species

$$\mathsf{E}_V \longrightarrow \mathsf{p},$$

where E_V is the **exponential decorated exponential** given by

$$\mathsf{E}_V[I] := \mathbb{K}\{f : I \to V\}.$$

Let $\mathscr{S}_V(\mathsf{p})$ be the space of decorated series.

Decorated series

A series s in $\mathscr{S}_V(\mathsf{p})$ is a collection of elements

$$s_{I,f} \in \mathsf{p}[I],$$

one for each finite set I and for each map $f : I \to V$, such that

$$\mathsf{p}[\sigma](s_{I,f}) = s_{J, f\circ\sigma^{-1}},$$

for each bijection $\sigma : I \to J$.

Cumulants from decorated series

Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space.

Cumulants from decorated series

Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space.

Consider the *ripping and sewing* Hopf monoid P.

Cumulants from decorated series

Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space.

Consider the *ripping and sewing* Hopf monoid P. As a species, $\mathrm{P} = \mathrm{L} \circ \mathrm{L}_+$.

Cumulants from decorated series

Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space.

Consider the *ripping and sewing* Hopf monoid P. As a species, $\mathsf{P} = \mathsf{L} \circ \mathsf{L}_+$.

Define $\Phi \in \mathscr{S}_{\mathcal{A}}(\mathsf{P}^*)$ as follows: if I is a finite set and $f : I \to \mathcal{A}$, let

$$\Phi_{I,f} \in \mathsf{P}^*[I]$$

given by

$$\Phi_{I,f}(w_1 w_2 \cdots w_n) := \varphi(w_1) \cdots \varphi(w_n),$$

where for each $w_k = x_1^k \cdots x_r^k \in \mathsf{L}_+[I_k]$,

$$\varphi(w) := (\varphi \circ f)(x_1^k) \cdots (\varphi \circ f)(x_r^k).$$

Cumulants from decorated series

Proposition (V. - 2024)

Let $(\mathcal{A}, \varphi)$ *be a non-commutative probability space. For every species p,*

consider the space $\mathrm{C}_{\mathcal{A}}(p) := \mathscr{S}_{\mathcal{A}}((L \circ p_+)^*)$.

- *Classical cumulants are obtained from* $p = X$
- *Non-commutative cumulants are obtained from* $p = L$

Problem : structure on p giving a more general ripping and sewing coproduct on the *free monoid* $\mathrm{L} \circ \mathrm{p}_+$?

(In progress: structure of *hereditary species* on p)

Series of the internal Hom species

Given two species p and q, let $\mathcal{H}(\mathsf{p},\mathsf{q})$ be the species defined by

$$\mathcal{H}(\mathsf{p},\mathsf{q})[I] := \mathsf{Hom}_{\mathbb{K}}(\mathsf{p}[I],\mathsf{q}[I]).$$

Series of the internal Hom species

Given two species p and q, let $\mathcal{H}(p,q)$ be the species defined by

$$\mathcal{H}(p,q)[I] := \mathrm{Hom}_{\mathbb{K}}(p[I], q[I]).$$

If $\sigma : I \to J$ is a bijection and $f \in \mathcal{H}(p,q)[I]$, then

$$\mathcal{H}(p,q)[\sigma](f) \in \mathcal{H}(p,q)[J]$$

is defined as the composition

$$p[J] \xrightarrow{p[\sigma^{-1}]} p[I] \xrightarrow{f} q[I] \xrightarrow{q[\sigma]} q[J].$$

Series of the internal Hom species

Given two species p and q, let $\mathcal{H}(\mathsf{p},\mathsf{q})$ be the species defined by

$$\mathcal{H}(\mathsf{p},\mathsf{q})[I] := \mathsf{Hom}_{\mathbb{K}}(\mathsf{p}[I],\mathsf{q}[I]).$$

Series of the internal Hom species

Given two species p and q, let $\mathcal{H}(\mathsf{p},\mathsf{q})$ be the species defined by

$$\mathcal{H}(\mathsf{p},\mathsf{q})[I] := \mathsf{Hom}_{\mathbb{K}}(\mathsf{p}[I],\mathsf{q}[I]).$$

There is a natural isomorphism

$$\mathrm{Hom}_{\mathsf{Sp}_{\Bbbk}}(\mathsf{p}\times\mathsf{q},\mathsf{r}) \cong \mathrm{Hom}_{\mathsf{Sp}_{\Bbbk}}\big(\mathsf{p},\mathcal{H}(\mathsf{q},\mathsf{r})\big),$$

for species p, q and r. This says that the functor $\mathcal{H}$ is the *internal Hom* in the symmetric monoidal category $(\mathsf{Sp}_{\Bbbk},\times)$ of species under Hadamard product.

System of products in PAQFT

Given a vector space V, the *decorated Fock functor* $\mathcal{K}_{\mathrm{V}}$ is given by

$$\mathcal{K}_{\mathrm{V}}(\mathsf{p}) := \bigoplus_{n \geq 0} \mathsf{p}[n] \otimes \mathrm{V}^{\otimes n}.$$

A *system of products* (Norledge) is a homomorphism of algebras

$$\bigoplus_{n \geq 0} \mathsf{p}[n] \otimes \mathrm{V}^{\otimes n} \to \mathcal{A} \qquad \text{(Wick algebra)}$$

In the language of species, this is precisely a species map

$$\mathsf{h} \times \mathbf{E}_{\mathrm{V}} \to \mathcal{U}_{\mathcal{A}},$$

after applying $\mathcal{K}_{\mathrm{V}}$. Here, $\mathcal{U}_{\mathcal{A}}[\mathrm{I}] := \mathcal{A}$, for every finite set I.

System of products in PAQFT

Given two species p and q, let $\mathcal{H}(\mathsf{p},\mathsf{q})$ be the species defined by

$$\mathcal{H}(\mathsf{p},\mathsf{q})[I] := \mathsf{Hom}_{\mathbb{K}}(\mathsf{p}[I],\mathsf{q}[I]).$$

A system of products

$$\mathsf{h} \times \mathbf{E}_V \to \mathcal{U}_{\mathcal{A}}$$

is equivalent to a map of species

$$\mathsf{h} \to \mathcal{H}(\mathbf{E}_V, \mathcal{U}_{\mathcal{A}}).$$

A system of fully (re)normalized time-ordered products, as defined in PAQFT/causal perturbation theory, is a system of products for the Hopf monoid $\mathsf{h} := \Sigma$.

System of products in PAQFT

More precisely, a map

$$T : \Sigma \times \mathbf{E}_{\mathcal{F}_{\mathsf{loc}}[[\hbar]]} \to \mathcal{U}_{\mathcal{F}_{\mathsf{mc}}((\hbar))}$$

$$a \otimes A_{i_1} \otimes \cdots \otimes A_{i_n} \mapsto T_I(a \otimes A_{i_1} \otimes \cdots \otimes A_{i_n})$$

(satisfying causal factorization, T_I inclusion) is equivalent to having linear maps

$$T(S_1) \cdots T(S_n) : \mathbf{E}_{\mathcal{F}_{\mathsf{loc}}[[\hbar]]}[I] \to \mathcal{U}_{\mathcal{F}_{\mathsf{mc}}((\hbar))}[I],$$

where $I = S_1 \sqcup \cdots \sqcup S_n$.

Given two species p and q, let $\mathcal{H}(\mathsf{p},\mathsf{q})$ be the species defined by

$$\mathcal{H}(\mathsf{p},\mathsf{q})[I] := \mathsf{Hom}_{\mathbb{K}}(\mathsf{p}[I],\mathsf{q}[I]).$$

Given two species p and q, let $\mathcal{H}(\mathsf{p},\mathsf{q})$ be the species defined by

$$\mathcal{H}(\mathsf{p},\mathsf{q})[I] := \mathsf{Hom}_{\mathbb{K}}(\mathsf{p}[I],\mathsf{q}[I]).$$

A series of the species $\mathcal{H}(\mathsf{p},\mathsf{q})$ is a morphism of species from p to q:

$$\mathscr{S}(\mathcal{H}(\mathsf{p},\mathsf{q})) = \mathsf{Hom}_{\mathsf{Sp}}(\mathsf{p},\mathsf{q}).$$

Given two species p and q, let $\mathcal{H}(\mathsf{p},\mathsf{q})$ be the species defined by

$$\mathcal{H}(\mathsf{p},\mathsf{q})[I] := \mathsf{Hom}_{\mathbb{K}}(\mathsf{p}[I],\mathsf{q}[I]).$$

A series of the species $\mathcal{H}(\mathsf{p},\mathsf{q})$ is a morphism of species from p to q:

$$\mathscr{S}(\mathcal{H}(\mathsf{p},\mathsf{q})) = \mathsf{Hom}_{\mathsf{Sp}}(\mathsf{p},\mathsf{q}).$$

In analogy with a non-commutative space $(\mathcal{A},\varphi)$, consider the pair (h,φ) formed by a connected bimonoid and a map $\varphi : \mathsf{h} \to \mathsf{E}$ such that

$$\begin{aligned}\varphi_{\emptyset} : \mathsf{h}[\emptyset] &\to \mathbb{K}\\ 1 &\mapsto 1_{\mathbb{K}}.\end{aligned}$$

Given two species p and q, let $\mathcal{H}(\mathsf{p},\mathsf{q})$ be the species defined by

$$\mathcal{H}(\mathsf{p},\mathsf{q})[I] := \mathsf{Hom}_{\mathbb{K}}(\mathsf{p}[I],\mathsf{q}[I]).$$

A series of the species $\mathcal{H}(\mathsf{p},\mathsf{q})$ is a morphism of species from p to q:

$$\mathscr{S}(\mathcal{H}(\mathsf{p},\mathsf{q})) = \mathsf{Hom}_{\mathsf{Sp}}(\mathsf{p},\mathsf{q}).$$

In analogy with a non-commutative space $(\mathcal{A},\varphi)$, consider the pair (h,φ) formed by a connected bimonoid and a map $\varphi : \mathsf{h} \to \mathsf{E}$ such that

$$\begin{aligned}\varphi_{\emptyset} : \mathsf{h}[\emptyset] &\to \mathbb{K}\\ 1 &\mapsto 1_{\mathbb{K}}.\end{aligned}$$

This leads to consider the space $\mathsf{C}_{\mathsf{h}}(\mathsf{p}) := \mathscr{S}(\mathcal{H}(\mathsf{h},(\mathsf{L}\circ\mathsf{p}_{+})^{*}))$.

Cumulants from decorated series (V. 2024)

$$\mathsf{C}_{\mathsf{h}}(\mathsf{p}) := \mathscr{S}(\mathcal{H}(\mathsf{h}, (\mathsf{L} \circ \mathsf{p}_{+})^{*})).$$

Cumulants from decorated series (V. 2024)

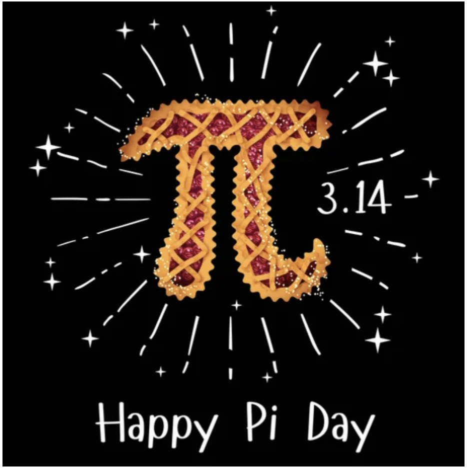
$$C_h(p) := \mathscr{S}(\mathcal{H}(h, (L \circ p_+)^*)).$$

Particular case: $p := X$, (h, φ) a connected bimonoid with

$$\varphi_I(x) := \dim_{\mathbb{K}} h[I],$$

for all $x \in h[I]$.

Thanks for your attention!



References I

Marcelo Aguiar and Swapneel Mahajan.
Hopf monoids in the category of species.
Hopf algebras and tensor categories, 585:17–124, 2013.

Octavio Arizmendi and Adrián Celestino.
Monotone cumulant-moment formula and schróder trees.
arXiv preprint arXiv:2111.02179, 2021.

Kurusch Ebrahimi-Fard and Frédéric Patras.
A group-theoretical approach to conditionally free cumulants.
arXiv preprint arXiv:1806.06287, 2018.

References II

Hillary Einziger.
Incidence Hopf algebras: Antipodes, forest formulas, and noncrossing partitions.
PhD thesis, The George Washington University, 2010.

Takahiro Hasebe and Franz Lehner.
Cumulants, spreadability and the campbell-baker-hausdorff series.
arXiv preprint arXiv:1711.00219, 2017.

Matthieu Josuat-Vergès, Frédéric Menous, Jean-Christophe Novelli, and Jean-Yves Thibon.
Free cumulants, schröder trees, and operads.
Advances in Applied Mathematics, 88:92–119, 2017.

References III

Matthieu Josuat-Vergès, Frédéric Menous, Jean-Christophe Novelli, and Jean-Yves Thibon.
Free cumulants, schröder trees, and operads.
Advances in Applied Mathematics, 88:92–119, 2017.

Franz Lehner, Jean-Christophe Novelli, and Jean-Yves Thibon.
Combinatorial hopf algebras in noncommutative probabilility.
arXiv preprint arXiv:2006.02089, 2020.

Naofumi Muraki.
The five independences as natural products.
Infinite Dimensional Analysis, Quantum Probability and Related Topics, 6(03):337–371, 2003.

References IV

Yannic Vargas.
Free cumulants for bimonoids in species.
In preparation, 2024.

Dan Voiculescu.
Symmetries of some reduced free product c*-algebras.
In *Operator algebras and their connections with topology and ergodic theory*, pages 556–588. Springer, 1985.