Introduction to combinatorial species

Course 1

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Algebraic, analytic and geometric structures emerging from quantum field theory

4-16 March 2024, Chengdu, China

Content

What is a (set) species?

Species

André Joyal, Alain Connes, Olivia Caramello and Laurent Lafforgue, IHES (2015)

The theory of *combinatorial species* was introduced by André Joyal in 1980. Species can be seen as a *categorification* of generating functions. It provides a categorical foundation for enumerative combinatorics.

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It follows that each map $\mathrm{P}[\sigma]$ is invertible, with inverse $\mathrm{P}[\sigma^{-1}]$.

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- $\mathrm{L}[I] := \{$ linear orders $\ell : [n] \to I$ on $I\}$.
- If ℓ is a linear order on I and $\sigma : I \to J$ is a bijection, then $\mathrm{L}[\sigma](\ell)$ is the list obtained by replacing each $i \in I$ for $\sigma(i) \in J$:

$$\mathrm{L}[\sigma] := \sigma \circ \ell.$$

Species of linear orders

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When defining species, we often omit the specification of relabeling maps.

Partitions and compositions

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$$X = \{\{a, c, d\}, \{b, g\}, \{e, f, h\}\} \vdash I \quad , \quad F = \big(\{a, c, d\}, \{b, g\}, \{e, f, h\}\big) \vDash I.$$

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Exercise: Let $\Pi[I]$ be the set of all partitions of I. Show that the "obvious" definition for relabeling partitions induces a species Π.

We call Π the *species of partitions*.

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for every bijection $\sigma : I \to J$. This is a naturality axiom. Therefore, forgetting the order among the blocks of a composition induces a surjective morphism

$$\pi : \Sigma \twoheadrightarrow \Pi.$$

There is also an injective morphism $\mathrm{L} \hookrightarrow \Sigma$ obtained by identifying a linear order wits a composition into singletons.

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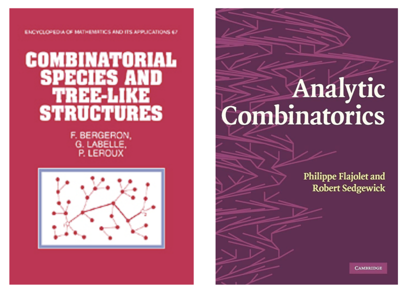
Species and morphisms of species form a category Sp, in which E is terminal.

Generating function associated to a species



"Following a slow (and forced) evolution in the history of mathematics, the modern notion of function (due to Dirichlet, 1837) has been made independent of any actual description format. A similar process has led André Joyal to introduce in combinatorics the notion of 'Species' to make the description of structures independent of any specific format. The theory serves as an elegant 'explanation' for the *surprising power of generating function uses for the solution of structure enumeration*. "
– François Bergeron, Gilbert Labelle, Pierre Leroux.

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For example,

$$\mathrm{E}(z) = \sum_{n \geq 0} 1 \frac{z^n}{n!} = e^z \quad \text{and} \quad \mathrm{L}(z) = \sum_{n \geq 0} n! \frac{z^n}{n!} = \frac{1}{1-z}.$$

Generating function associated to a species

Recall that a morphism $f : \mathrm{P} \to \mathrm{Q}$ between set species P and Q is a collection of maps

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Lemma
Let P and Q be two finite species. Then

$$\mathrm{P} = \mathrm{Q} \Longrightarrow \mathrm{P}(z) = \mathrm{Q}(z).$$

The reciprocal is not true in general. We will see an example in a moment.

Combinatorial operations

Cauchy product

Given set species P and Q, their *Cauchy product* is the set species $\mathrm{P} \cdot \mathrm{Q}$ defined by

$$\big(\mathrm{P} \cdot \mathrm{Q}\big)[I] = \coprod_{S \sqcup T = I} \mathrm{P}[S] \times \mathrm{Q}[T]$$

for each finite set I.

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Given $(x, y) \in \mathrm{P}[S] \times \mathrm{Q}[T]$, its relabeling under a bijection $\sigma : I \to J$ is the $(\mathrm{P} \cdot \mathrm{Q})$-structure

$$(x', y') \in \mathrm{P}[S'] \times \mathrm{Q}[T'],$$

where $S' = \sigma(S)$, $T' = \sigma(T)$, and x' and y' are obtained by relabeling x and y under the restricted bijections $\sigma|_S : S \to S'$ and $\sigma|_T : T \to T'$, respectively.

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It is the unit for the Cauchy product: for any set species P, there are canonical isomorphisms for which

$$1 \cdot \mathrm{P} = \mathrm{P} = \mathrm{P} \cdot 1.$$

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Exercise. □

Recall that

$$\left(\sum_{n\geq 0} a_n \frac{z^n}{n!}\right) \cdot \left(\sum_{n\geq 0} b_n \frac{z^n}{n!}\right) = \sum_{n\geq 0} \left(\sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}\right) \frac{z^n}{n!}.$$

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From the previous proposition,

$$(\mathrm{E} \cdot \mathrm{E})(z) = e^{2z} \quad \text{and} \quad (\mathrm{E}^{\cdot k})(z) = e^{kz}.$$

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$$f : I \to [k] \quad \longleftrightarrow \quad f^{-1}(1) \sqcup \cdots \sqcup f^{-1}(k) = I.$$

From the previous proposition,

$$(\mathrm{E} \cdot \mathrm{E})(z) = e^{2z} \quad \text{and} \quad (\mathrm{E}^{\cdot k})(z) = e^{kz}.$$

Expanding these power series we recover the facts that the number of subsets of $[n]$ is 2^n and the number of functions $[n] \to [k]$ is k^n.

Derangements and bijections

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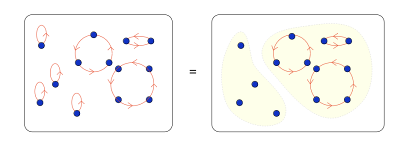

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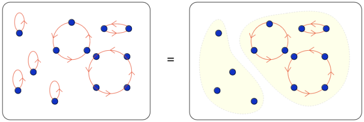

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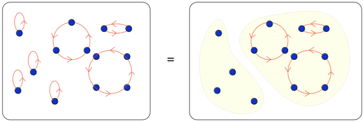

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Notices that $|\mathrm{B}[n]| = n! = |\mathrm{L}[n]|$, for each $n \geq 0$.

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Linear orders and permutations are not transported in the same manner along bijections

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Let Q be a set species with $\mathrm{Q}[\emptyset] = \emptyset$ and P an arbitrary set species. The *substitution* of P into Q is the set species $\mathrm{P} \circ \mathrm{Q}$ defined by

$$\big(\mathrm{P} \circ \mathrm{Q}\big)[I] = \coprod_{X \vdash I} \left(\mathrm{P}[X] \times \prod_{B \in X} \mathrm{Q}[B] \right)$$

for each finite set I.

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- a Q-structure on each of the blocks (which are subsets of I).

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Substitution is associative and unital. The unit is the *singleton species* X defined by

$$\mathrm{X}[I] = \begin{cases} \{*_I\} & \text{if } I \text{ is a singleton,} \\ \emptyset & \text{otherwise.} \end{cases}$$

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Corollary (The exponential formula)

$$(\mathrm{E} \circ \mathrm{Q})(z) = e^{\mathrm{Q}(z)}.$$

A structure of species $\mathrm{E} \circ \mathrm{Q}$ on I consists of a partition of I together with a Q-structure on each block of the partition.

Given a set species Q, its *positive part* is the species Q_+ defined by

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A structure of type $\mathrm{L} \circ \mathrm{E}_+$ on I is simply a partition of I, endowed with a total order on the set of its blocks.Hence,

$$\mathrm{L} \circ \mathrm{E}_+ = \Sigma,$$

which implies

$$\Sigma(z) = \frac{1}{2 - e^z}.$$

Addition

The *addition* of two species P and Q is the new species $\mathrm{P}+\mathrm{Q}$ defined by

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The *empty species* 0, defined by

$$0[I] := \emptyset,$$

for all finite set I, is a neutral element for addition. For any set species P, there are canonical isomorphisms for which

$$0 + \mathrm{P} = \mathrm{P} = \mathrm{P} + 0.$$

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A family $(\mathrm{P}_\lambda)_{\lambda\in\Lambda}$ is said to be *summable* if for any finite set I, $\mathrm{P}_\lambda[I] = \emptyset$, except for a finite number of indices $\lambda \in \Lambda$. The sum of a summable family $(\mathrm{P}_\lambda)_{\lambda\in\Lambda}$ is the species $\sum_{\lambda\in\Lambda} \mathrm{P}_\lambda$ defined by

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Notices that the disjoint union is finite. Each species P gives rise canonically to a summable family $(\mathrm{P}_n)_{n\geq 0}$.

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Given a set species P and a positive integer n, consider the new species P_n defined by

$$\mathrm{P}_n[I] = \begin{cases} \mathrm{P}[I] & \text{if } |I| = n, \\ \emptyset & \text{otherwise.} \end{cases}$$

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The power X^n of the singleton species X is isomorphic to the lists of lentgth n. The family $(\mathrm{X}^n)_{n \geq 0}$ is summable and we have the identity

$$\mathrm{L} = \mathrm{X}^0 + \mathrm{X}^1 + \mathrm{X}^2 + \mathrm{X}^3 + \cdots .$$

Derivative

Let P be a finite species. The derivative of the generating function $\mathrm{P}(z)$ is

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Given a bijection $\sigma : I \to J$, consider the map $\sigma^+ : I^+ \to J^+$ given by

$$\sigma^+(i) := \sigma(i) \text{ for } i \in I \quad \text{and} \quad \sigma^+(*_I) := *_J.$$

Then, we define $\mathrm{P}'[\sigma] : \mathrm{P}'[I] \to \mathrm{P}'[J]$ as

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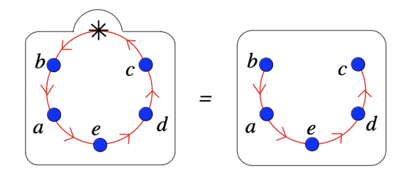
Derivative

Let C be the *species of cycles*. Formally:

$$\mathrm{C}[I] := \{\sigma : I \xrightarrow{\sim} I \,|\, \sigma \text{ is a cycle } \}.$$

Removing an element from a cycle allows to obtain a linear order. Therefore,

$$\mathrm{C}' = \mathrm{L}.$$



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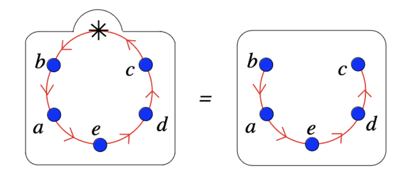

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$$\mathrm{C}(z) = \sum_{n\geq 0} |\mathrm{C}[n]| \frac{z^n}{n!} = \sum_{n\geq 0} (n-1)! \frac{z^n}{n!} = \log \frac{1}{1-z},$$

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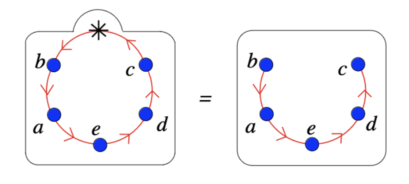

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Proposition
Let P be a species. Then $(\mathrm{P}(z))' = \mathrm{P}'(z)$.

Derivative

Exercise: proof the following "combinatorial rules of calculus".

1. $(\mathrm{P} \cdot \mathrm{Q})' = \mathrm{P}' \cdot \mathrm{Q} + \mathrm{P} \cdot \mathrm{Q}'$.
2. $(\mathrm{P} \circ \mathrm{Q})' = \mathrm{Q}' \cdot (\mathrm{P}' \circ \mathrm{Q})$.
3. $(\mathrm{P} + \mathrm{Q})' = \mathrm{P}' + \mathrm{Q}'$.
4. $1' = 0$.

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We can define A recursively: a rooted tree with node set I consists of the choice of:

- a node $r \in I$ (the root);
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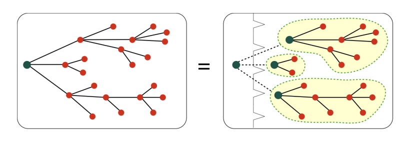

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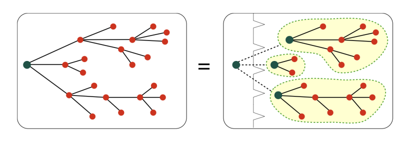

Therefore, $\mathrm{A} = \mathrm{X} \cdot (\mathrm{E} \circ \mathrm{A})$. This is well defined, since $\mathrm{A}[\emptyset] = \emptyset$.

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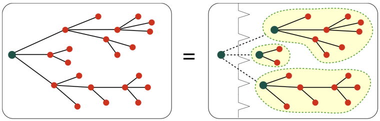
- a node $r \in I$ (the root);
- a partition X of $I \setminus \{r\}$ (whose blocks consist of the nodes in the branches stemming out of r);
- a rooted tree with node set S for each $S \in X$.



Therefore, $\mathrm{A} = \mathrm{X} \cdot (\mathrm{E} \circ \mathrm{A})$. This is well defined, since $\mathrm{A}[\emptyset] = \emptyset$. In particular,

$$\mathrm{A}(z) = ze^{\mathrm{A}(z)} = z + 2\frac{x^2}{2!} + 9\frac{z^3}{3!} + 64\frac{z^4}{4!} + 625\frac{z^5}{5!} + \cdots .$$

Implicit equations

Exercise: in a rooted tree with root r, the children of a node s are those nodes adjacent to s that are not between s and r. A *planar rooted tree* is a rooted tree in which the children of each node are given a linear order. Let $\overrightarrow{\mathrm{A}}$ denote the species of planar rooted trees.

(a) Show that $\overrightarrow{\mathrm{A}} = \mathrm{X} \cdot (\mathrm{L} \circ \overrightarrow{\mathrm{A}})$.

(b) Deduce that

$$\overrightarrow{\mathrm{A}}(x) = \frac{x}{1 - \overrightarrow{\mathrm{A}}(x)} \quad \text{and} \quad \overrightarrow{\mathrm{A}}(x) = \frac{1 - \sqrt{1 - 4x}}{2}.$$

Thus the number of planar rooted trees on n nodes is $n!C_{n-1}$, where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the n-th *Catalan number*.

More tree-like structures

Let a be the species of *non-rooted trees* (a is for "arbre"). An element is $\mathrm{a}[I]$ is just an acyclic graph (with vertex set I), without the choice of a vertex.

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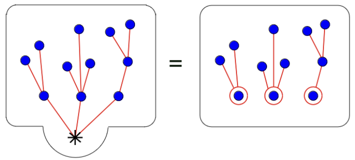
Removing a vertex of a non-rooted tree leads to a set of rooted trees. Hence,

$$\mathrm{a}' = \mathrm{E} \circ \mathrm{A}.$$



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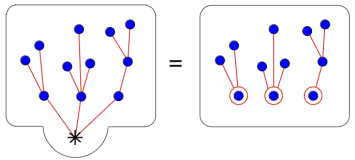

The species $\mathrm{F} := \mathrm{E} \circ \mathrm{A}$ is called the *species of forests*.

Questions?