Infinite-dimensional Lie bialgebras via affinization of Novikov bialgebras

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Outline

- Definitions and motivations
- Novikov bialgebras and infinite-dimensional Lie bialgebras
- Equivalent characterizations of Novikov bialgebras and Novikov Yang-Baxter equation
- Further works on multi-Novikov algebras

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Definitions of Novikov algebras and right Novikov algebras

A **Novikov algebra** is a vector space A with a binary operation $\circ$ satisfying

$$\begin{aligned}(a\circ b)\circ c-a\circ(b\circ c) &= (b\circ a)\circ c-b\circ(a\circ c),\\ (a\circ b)\circ c &= (a\circ c)\circ b,\quad a,b,c\in A.\end{aligned}$$

A **right Novikov algebra** is a vector space A with a binary operation $\diamond$ satisfying

$$\begin{aligned}&(a\diamond b)\diamond c-a\diamond(b\diamond c)=(a\diamond c)\diamond b-a\diamond(c\diamond b),\\ &a\diamond(b\diamond c)=b\diamond(a\diamond c),\quad a,b,c\in A.\end{aligned}$$

- $(A, \circ)$ is a Novikov algebra if and only if its opposite $(A, \diamond)$ defined by $a \diamond b := b \circ a$ for all $a, b \in A$ is a right Novikov algebra.
- Novikov algebras were introduced in connection with Hamiltonian operators in the formal variational calculus I. Gel'fand, I. Dorfman, 1979 and Poisson brackets of hydrodynamic type A. Balinsky, S. Novikov, 1985.
- There is also a correspondence between Novikov algebras and a class of Lie conformal algebras X. Xu, 2000, which was introduced by V. Kac to give an axiomatic description of the singular part of operator product expansion (or rather its Fourier transform) of chiral fields in conformal field theory.

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A natural construction of Novikov algebras S. Gel'fand, 1979

Let $(A, \cdot)$ be a commutative associative algebra and D be a derivation. Then the binary operation

$$a \circ b := a \cdot D(b), \quad a, b \in A, \tag{1}$$

defines a Novikov algebra $(A, \circ)$. The binary operation

$$a \diamond b := D(a) \cdot b, \quad a, b \in A$$

defines a right Novikov algebra $(A, \diamond)$.

A special example

Equip the Laurent polynomial algebra $\mathbf{k}[t, t^{-1}]$ with the natural derivation $D := \frac{d}{dt}$. Then $(B = \mathbf{k}[t, t^{-1}], \diamond)$ is a right Novikov algebra with $\diamond$ given by

$$t^i \diamond t^j := D(t^i)t^j = it^{i+j-1}, \quad i, j \in \mathbb{Z}. \tag{2}$$

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Novikov algebra affinization A. Balinsky and S. Novikov, 1985

Let A be a vector space with a binary operation $\circ$. Define a binary operation on $A[t, t^{-1}] := A \otimes \mathbf{k}[\mathbf{t}, \mathbf{t}^{-1}]$ by

$$[at^i, bt^j] = i(a \circ b)t^{i+j-1} - j(b \circ a)t^{i+j-1}, \quad a, b \in A, i, j \in \mathbb{Z},$$

where $at^i := a \otimes t^i$. Then $(A[t, t^{-1}], [\cdot, \cdot])$ is a Lie algebra if and only if $(A, \circ)$ is a Novikov algebra.

Construction of Lie algebras from Novikov algebras and right Novikov algebras V. Ginzburg and M. Kapranov, 1994

Let $(A, \circ)$ be a Novikov algebra and $(B, \diamond)$ be a $\mathbb{Z}$-graded right Novikov algebra. Define a binary operation on $A \otimes B$ by

$$[a_1 \otimes b_1, a_2 \otimes b_2] = a_1 \circ a_2 \otimes b_1 \diamond b_2 - a_2 \circ a_1 \otimes b_2 \diamond b_1,$$

for all $a_1, a_2 \in A$ and $b_1, b_2 \in B$. Then $(A \otimes B, [\cdot, \cdot])$ is a $\mathbb{Z}$-graded Lie algebra, called **the induced Lie algebra** (from $(A, \circ)$ and $(B, \diamond)$). Further, if $(B, \diamond) = (\mathbf{k}[t, t^{-1}], \diamond)$ is the $\mathbb{Z}$-graded right Novikov algebra defined by Eq. (2) , then $(A \otimes B, [\cdot, \cdot])$ is a $\mathbb{Z}$-graded Lie algebra if and only if $(A, \circ)$ is a Novikov algebra.

Definition of Lie bialgebras

A **Lie bialgebra** is a triple $(L, [\cdot, \cdot], \delta)$ such that $(L, [\cdot, \cdot])$ is a Lie algebra, (L, δ) is a Lie coalgebra, and the following compatibility condition holds.

$$\delta([a, b]) = a.\delta(b) - b.\delta(a), \quad a, b \in L.$$

where $a.\delta(b) := \sum_{(b)}([a, b_{(1)}] \otimes b_{(2)} + b_{(1)} \otimes [a, b_{(2)}])$, if $\delta(b) = \sum_{(b)} b_{(1)} \otimes b_{(2)}$.

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Some facts on (finite-dimensional) Lie bialgebras

- A finite-dimensional Lie bialgebra $(L, [\cdot,\cdot], \delta)$ is equivalent to a Manin triple $(L \oplus L^*, L, L^*)$ of Lie algebras V. Chari, A. Pressley, 1994.
- Let $(L, [\cdot,\cdot])$ be a Lie algebra, and $r = \sum_\alpha x_\alpha \otimes y_\alpha \in L \otimes L$. Then the following equation

$$\begin{aligned}&[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}]\\ = & \sum_{\alpha,\beta}([x_\alpha, x_\beta] \otimes y_\alpha \otimes y_\beta + x_\alpha \otimes [y_\alpha, x_\beta] \otimes y_\beta + x_\alpha \otimes x_\beta \otimes [y_\alpha, y_\beta])\\ = & 0,\end{aligned}$$

  is called the **classical Yang-Baxter equation (CYBE)** in L.
  Note that a skewsymmetric solution r of CYBE in L can produce a Lie bialgebra $(L, [\cdot,\cdot], \delta)$ where $\delta(a) = a.r$ for all $a \in L$ V. Chari, A. Pressley, 1994.

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- Let $(L, [\cdot,\cdot])$ be a Lie algebra and (V, ρ) be a representation of $(L, [\cdot,\cdot])$. A linear map $T : V \to L$ is called an $\mathcal{O}$**-operator** or a **relative Rota-Baxter operator** on $(L, [\cdot,\cdot])$ associated to (V, ρ) if T satisfies

$$[T(u), T(v)] = T(\rho(T(u))v - \rho(T(v))u), \quad u, v \in V.$$

  Let L be finite-dimensional and $T : V \to L$ be a linear map which is identified with $r_T \in L \otimes V^* \subseteq (L \ltimes_{\rho^*} V^*) \otimes (L \ltimes_{\rho^*} V^*)$ through $\mathrm{Hom}_{\mathbf{k}}(V, L) \cong L \otimes V^*$. Then $r = r_T - \tau r_T$ is a solution of the CYBE in the semi-direct product Lie algebra $(L \ltimes_{\rho^*} V^*, [\cdot,\cdot])$ if and only if T is an $\mathcal{O}$-operator on $(L, [\cdot,\cdot])$ associated to (V, ρ) C. Bai, 2007.

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Some facts on (finite-dimensional) Lie bialgebras

- Let $(A, \circ)$ be a finite-dimensional pre-Lie algebra. Then (A, L_A) is a representation of the sub-adjacent Lie algebra $\mathfrak{g}(A)$. Moreover, the identity map is an $\mathcal{O}$-operator on $\mathfrak{g}(A)$ associated to (A, L_A) C. Bai, 2007.
- In summary, in the finite-dimensional case, we have the following diagram.

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Natural questions

- Note that the theory of Lie bialgebras has been well developed. Can we use the definition of Lie bialgebras and Novikov algebra affinization to obtain a definition of Novikov bialgebras? If so, does there exist a similar theory for Novikov bialgebras as Lie bialgebra theory?
- Can we lift the general construction above to the context of bialgebras?

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- In this talk, we always assume that **k** is a field of characteristic 0, $(A, \circ)$ is finite-dimensional and each B_i in $(B = \oplus_{i \in \mathbb{Z}} B_i, \diamond)$ is finite-dimensional.
- These results on Novikov bialgebras and infinite-dimensional Lie bialgebras can be found in Yanyong, Hong, Chengming, Bai, Li, Guo, Infinite-dimensional Lie bialgebras via affinization of Novikov bialgebras and Koszul duality. Comm. Math. Phys. 401 (2023), 2011-2049.

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Definition of Novikov coalgebras

A **Novikov coalgebra** is a vector space A with a linear map $\Delta : A \to A \otimes A$ such that

$$\begin{aligned} & (\mathrm{id} \otimes \Delta)\Delta(a) - (\tau \otimes \mathrm{id})(\mathrm{id} \otimes \Delta)\Delta(a) \\ = \; & (\Delta \otimes \mathrm{id})\Delta(a) - (\tau \otimes \mathrm{id})(\Delta \otimes \mathrm{id})\Delta(a), \\ & (\tau \otimes \mathrm{id})(\mathrm{id} \otimes \Delta)\tau\Delta(a) = (\Delta \otimes \mathrm{id})\Delta(a), \quad a \in A. \end{aligned}$$

Let $\Delta : A \to A \otimes A$ be a linear map and $\cdot : A^* \otimes A^* \to A^*$ be the corresponding binary operation on A^*. The pair (A, Δ) is a Novikov coalgebra if and only if $(A^*, \cdot)$ is a Novikov algebra.

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Next step

We need to give a construction of Lie coalgebras from Novikov coalgebras and right Novikov coalgebras, in particular the Novikov coalgebra affinization. Therefore the first step is to find a coalgebra structure on the space of Laurent polynomials whose graded linear dual is the right Novikov algebra of Laurent polynomials.

An observation

Let $C = \mathbf{k}[\mathbf{x}, \mathbf{x}^{-1}]$ be the $\mathbb{Z}$-graded (by degree) space of Laurent polynomials and let $A = \mathbf{k}[\mathbf{t}, \mathbf{t}^{-1}]$ be the graded linear dual. Then the right Novikov algebra product $\diamond$ on A cannot be the induced product from the graded linear dual of any coproduct $\delta : C \to C \otimes C$.

Proof

Suppose that such a coproduct δ exists. Denote $\delta(x^i) = \sum_{p,q \in \mathbb{Z}} c_{p,q} x^p \otimes x^q$. Then the graded linear duality $\langle x^i, t^j \rangle = \delta_{i,j}$ gives the duality $\langle \delta(x^i), t^j \otimes t^k \rangle = \langle x^i, t^j \diamond t^k \rangle$, yielding $c_{j,k} = \delta_{i-j-k+1,0} j$. Thus

$$\delta(x^i) = \sum_{i-p-q+1=0} \delta_{i-p-q+1,0} p x^p \otimes x^q = \sum_{p \in \mathbb{Z}} p x^p \otimes x^{i-p+1}.$$

This is an infinite sum and so cannot be defined by $\delta : C \to C \otimes C$.

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This is an infinite sum and so cannot be defined by $\delta: C\to C\otimes C$.

Let $C=\oplus_{i\in\mathbb{Z}}C_i$ and $D=\oplus_{j\in\mathbb{Z}}D_j$ be $\mathbb{Z}$-graded vector spaces. We call the **completed tensor product** of C and D to be the vector space

$$C\widehat{\otimes}D:=\prod_{i,j\in\mathbb{Z}}C_i\otimes D_j.$$

If C and D are finite-dimensional, then $C\widehat{\otimes}D$ is just the usual tensor product $C\otimes D$.

Definition of completed right Novikov coalgebras

A **completed right Novikov coalgebra** is a pair (C, Δ) where $C = \oplus_{i\in\mathbb{Z}} C_i$ is a $\mathbb{Z}$-graded vector space and $\Delta : C \to C\widehat{\otimes}C$ is a linear map satisfying

$$\begin{aligned}
& (\Delta\widehat{\otimes}\mathrm{id})\Delta(a) - (\mathrm{id}\widehat{\otimes}\widehat{\tau})(\Delta\widehat{\otimes}\mathrm{id})\Delta(a) \\
= \;& (\mathrm{id}\widehat{\otimes}\Delta)\Delta(a) - (\mathrm{id}\widehat{\otimes}\widehat{\tau})(\mathrm{id}\widehat{\otimes}\Delta)\Delta(a), \\
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\end{aligned}$$

Completed right Novikov coalgebras on $\mathbf{k}[t, t^{-1}]$

On the vector space $B := \mathbf{k}[t, t^{-1}]$, define a linear map $\Delta_B : B \to B\widehat{\otimes}B$ by

$$\Delta_B(t^j) = \sum_{i\in\mathbb{Z}} (i+1)t^{-i-2} \otimes t^{j+i}, \quad j \in \mathbb{Z}. \tag{3}$$

One can directly check that $(B = \mathbf{k}[t, t^{-1}], \Delta_B)$ is a completed right Novikov coalgebra.

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One can directly check that $(B = \mathbf{k}[t, t^{-1}], \Delta_B)$ is a completed right Novikov coalgebra.

Definition of completed Lie coalgebras

A **completed Lie coalgebra** is a pair (L, δ), where $L = \oplus_{i\in\mathbb{Z}} L_i$ is a $\mathbb{Z}$-graded vector space and $\delta : L \to L \widehat{\otimes} L$ is a linear map satisfying

$$\delta(a) = -\widehat{\tau}\delta(a), \tag{4}$$

$$(\mathrm{id}\widehat{\otimes}\delta)\delta(a) - (\widehat{\tau}\widehat{\otimes}\mathrm{id})(\mathrm{id}\widehat{\otimes}\delta)\delta(a) = (\delta\widehat{\otimes}\mathrm{id})\delta(a), \quad a \in L. \tag{5}$$

Theorem 1

Let (A, Δ_A) be a Novikov coalgebra, (B, Δ_B) be a completed right Novikov coalgebra and $L := A \otimes B$. Define the linear map $\delta : L \to L \widehat{\otimes} L$ by

$$\delta(a \otimes b) = (\mathrm{id}_{L\widehat{\otimes}L} - \widehat{\tau})\big(\Delta_A(a) \bullet \Delta_B(b)\big), \quad a \in A, b \in B. \tag{6}$$

Here for $\Delta_A(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)}$ in the Sweedler notation and $\Delta_B(b) = \sum_{i,j,\alpha} b_{1i\alpha} \otimes b_{2j\alpha}$, we set

$$\Delta_A(a) \bullet \Delta_B(b) := \sum_{(a)} \sum_{i,j,\alpha} (a_{(1)} \otimes b_{1i\alpha}) \otimes (a_{(2)} \otimes b_{2j\alpha}). \tag{7}$$

Then (L, δ) is a completed Lie coalgebra. Furthermore, if $(B = \mathbf{k}[t, t^{-1}], \Delta_B)$ is the completed right Novikov coalgebra given as above, then $(L = A \otimes B, \delta)$ is a completed Lie coalgebra if and only if (A, Δ_A) is a Novikov coalgebra.

Definition of quadratic $\mathbb{Z}$-graded right Novikov algebras

A bilinear form $(\cdot,\cdot)$ on a $\mathbb{Z}$-graded vector space $B=\oplus_{i\in\mathbb{Z}}B_i$ is called **graded** if there exists some $m\in\mathbb{Z}$ such that

$$(B_i,B_j)=0 \quad \text{for all } i,j\in\mathbb{Z} \text{ satisfying } i+j+m\neq 0.$$

A graded bilinear form on a $\mathbb{Z}$-graded right Novikov algebra $(B=\oplus_{i\in\mathbb{Z}}B_i,\diamond)$ is called **invariant** if it satisfies

$$(a\diamond b,c)=-(a,b\diamond c+c\diamond b),\quad a,b,c\in B. \tag{8}$$

A **quadratic $\mathbb{Z}$-graded right Novikov algebra**, denoted by $(B=\oplus_{i\in\mathbb{Z}}B_i,\diamond,(\cdot,\cdot))$, is a $\mathbb{Z}$-graded right Novikov algebra $(B,\diamond)$ together with a symmetric invariant nondegenerate graded bilinear form $(\cdot,\cdot)$. In particular, when $B=B_0$, it is simply called a **quadratic right Novikov algebra**.

Example

Let $(B = \oplus_{i\in\mathbb{Z}}\mathbf{k}t^i, \diamond)$ be the $\mathbb{Z}$-graded right Novikov algebra. Define a bilinear form $(\cdot,\cdot)$ on B by

$$(t^i, t^j) = \delta_{i+j+1,0}, \quad i, j \in \mathbb{Z}. \tag{9}$$

It is directly checked that $(B = \mathbf{k}[t, t^{-1}], \diamond, (\cdot,\cdot))$ is a quadratic $\mathbb{Z}$-graded right Novikov algebra.

Lemma 1

Let $(B = \oplus_{i\in\mathbb{Z}} B_i, \diamond, (\cdot,\cdot))$ be a quadratic $\mathbb{Z}$-graded right Novikov algebra. Let $\Delta_B : B \to B\widehat{\otimes}B$ be the dual of $\diamond$ under $(\cdot,\cdot)$, so that

$$(\Delta_B(a), b\otimes c) = (a, b\diamond c), \quad a, b, c \in B. \tag{10}$$

Then (B, Δ_B) is a completed right Novikov coalgebra.

Definition of completed Lie bialgebras

A **completed Lie bialgebra** is a triple $(L, [\cdot,\cdot], \delta)$ such that $(L, [\cdot,\cdot])$ is a Lie algebra, (L, δ) is a completed Lie coalgebra, and the following compatibility condition holds.

$$\delta([a,b]) = (\mathrm{ad}_a \widehat{\otimes} \mathrm{id} + \mathrm{id} \widehat{\otimes} \mathrm{ad}_a)\delta(b) - (\mathrm{ad}_b \widehat{\otimes} \mathrm{id} + \mathrm{id} \widehat{\otimes} \mathrm{ad}_b)\delta(a), \quad a, b \in L,$$

where $\mathrm{ad}_a(b) = [a,b]$ for all $a, b \in L$.

Let $(A, \circ)$ be a Novikov algebra. Define another binary operation $\star$ on A by $a \star b = a \circ b + b \circ a$ for all $a, b \in A$. Let L_A, $R_A : A \to \mathrm{End}_{\mathbf{k}}(A)$ be the linear maps defined respectively by

$$L_A(a)(b) := a \circ b, \quad R_A(a)(b) := b \circ a, \quad a, \quad b \in A.$$

Define $L_{A,\star} : A \to \mathrm{End}_{\mathbf{k}}(A)$ by $L_{A,\star} = L_A + R_A$.

Definition of completed Lie bialgebras

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$$\delta([a,b]) = (\mathrm{ad}_a \widehat{\otimes} \mathrm{id} + \mathrm{id} \widehat{\otimes} \mathrm{ad}_a)\delta(b) - (\mathrm{ad}_b \widehat{\otimes} \mathrm{id} + \mathrm{id} \widehat{\otimes} \mathrm{ad}_b)\delta(a), \quad a, b \in L,$$

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Define $L_{A,\star} : A \to \mathrm{End}_{\mathbf{k}}(A)$ by $L_{A,\star} = L_A + R_A$.

Definition of Novikov bialgebras

A **Novikov bialgebra** is a tripe $(A, \circ, \Delta)$ where $(A, \circ)$ is a Novikov algebra and (A, Δ) is a Novikov coalgebra such that, for all $a, b \in A$, the following conditions are satisfied.

$$\begin{aligned}
& \Delta(a \circ b) = (R_A(b) \otimes \mathrm{id})\Delta(a) + (\mathrm{id} \otimes L_{A,\star}(a))(\Delta(b) + \tau\Delta(b)), \\
& (L_{A,\star}(a) \otimes \mathrm{id})\Delta(b) - (\mathrm{id} \otimes L_{A,\star}(a))\tau\Delta(b) \\
= \ & (L_{A,\star}(b) \otimes \mathrm{id})\Delta(a) - (\mathrm{id} \otimes L_{A,\star}(b))\tau\Delta(a), \\
& (\mathrm{id} \otimes R_A(a) - R_A(a) \otimes \mathrm{id})(\Delta(b) + \tau\Delta(b)) \\
= \ & (\mathrm{id} \otimes R_A(b) - R_A(b) \otimes \mathrm{id})(\Delta(a) + \tau\Delta(a)).
\end{aligned}$$

Theorem 2

Let $(A, \circ, \Delta_A)$ be a Novikov bialgebra and $(B = \oplus_{i\in\mathbb{Z}} B_i, \diamond, (\cdot,\cdot))$ be a quadratic $\mathbb{Z}$-graded right Novikov algebra. Let $L = A \otimes B$ be the induced Lie algebra from $(A, \circ)$ and $(B, \diamond)$, $\Delta_B : B \to B \hat{\otimes} B$ be the linear map defined by Eq. (10), and $\delta : L \to L \hat{\otimes} L$ be the linear map defined by:

$$\delta(a \otimes b) = (\mathrm{id}_{L\hat{\otimes}L} - \widehat{\tau})\big(\Delta_A(a) \bullet \Delta_B(b)\big), \quad a \in A, b \in B.$$

Then $(L, [\cdot,\cdot], \delta)$ is a completed Lie bialgebra. Further, if $(B, \diamond, (\cdot,\cdot)) = (\mathbf{k}[t, t^{-1}], \diamond, (\cdot,\cdot))$ is the quadratic $\mathbb{Z}$-graded right Novikov algebra, then the converse also holds.

Example

Let $(A, \circ)$ be the 2-dimensional Novikov algebra with a basis $\{e_1, e_2\}$ whose multiplication is given by

$$e_1 \circ e_1 = e_1, \quad e_2 \circ e_1 = e_2, \quad e_1 \circ e_2 = e_2 \circ e_2 = 0.$$

Define $\Delta_A : A \to A \otimes A$ by

$$\Delta_A(e_1) = \lambda e_2 \otimes e_2, \quad \Delta_A(e_2) = 0,$$

for a fixed $\lambda \in \mathbf{k}$. Then it is direct to verify that $(A, \circ, \Delta_A)$ is a Novikov bialgebra. Then by Theorem 2, there is a completed Lie bialgebra $(L = A \otimes B, [\cdot, \cdot], \delta)$ given by

$$[e_1 t^i, e_1 t^j] = (i - j) e_1 t^{i+j-1}, \; [e_1 t^i, e_2 t^j] = -j e_2 t^{i+j-1},$$
$$[e_2 t^i, e_2 t^j] = 0, \quad i, j \in \mathbb{Z},$$
$$\delta(e_1 t^j) = \sum_{i \in \mathbb{Z}} \lambda (j + 2i + 2) e_2 t^{-i-2} \otimes e_2 t^{j+i}, \; \delta(e_2 t^j) = 0, \quad j \in \mathbb{Z}.$$

Definition of representations of a Novikov algebra J. Osborn, 1995

A **representation** of a Novikov algebra $(A, \circ)$ is a triple (V, l_A, r_A), where V is a vector space and $l_A, r_A : A \to \mathrm{End}_{\mathbf{k}}(V)$ are linear maps satisfying

$$\begin{aligned}
&l_A(a \circ b - b \circ a)v = l_A(a)l_A(b)v - l_A(b)l_A(a)v,\\
&l_A(a)r_A(b)v - r_A(b)l_A(a)v = r_A(a \circ b)v - r_A(b)r_A(a)v,\\
&l_A(a \circ b)v = r_A(b)l_A(a)v,\\
&r_A(a)r_A(b)v = r_A(b)r_A(a)v, \quad a, b \in A, v \in V.
\end{aligned}$$

Note that (A, L_A, R_A) is a representation of $(A, \circ)$, called the **adjoint representation** of $(A, \circ)$.

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\begin{aligned}
&l_A(a \circ b - b \circ a)v = l_A(a)l_A(b)v - l_A(b)l_A(a)v, \\
&l_A(a)r_A(b)v - r_A(b)l_A(a)v = r_A(a \circ b)v - r_A(b)r_A(a)v, \\
&l_A(a \circ b)v = r_A(b)l_A(a)v, \\
&r_A(a)r_A(b)v = r_A(b)r_A(a)v, \quad a, b \in A, v \in V.
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Note that (A, L_A, R_A) is a representation of $(A, \circ)$, called the **adjoint representation** of $(A, \circ)$.

Proposition 1

Let $(A, \circ)$ be a Novikov algebra. Let V be a vector space and $l_A, r_A : A \to \mathrm{End}_{\mathbf{k}}(V)$ be linear maps. Define a binary operation $\bullet$ on $A \oplus V$ by

$$(a+u) \bullet (b+v) := a \circ b + l_A(a)v + r_A(b)u, \quad a, b \in A, u, v \in V.$$

Then (V, l_A, r_A) is a representation of $(A, \circ)$ if and only if $(A \oplus V, \bullet)$ is a Novikov algebra, called the **semi-direct product** of A by V and denoted by $A \ltimes_{l_A, r_A} V$ or simply $A \ltimes V$.

Let $(A, \circ)$ be a Novikov algebra and V be a vector space. For a linear map $\varphi : A \to \mathrm{End}_{\mathbf{k}}(V)$, define a linear map $\varphi^* : A \to \mathrm{End}_{\mathbf{k}}(V^*)$ by

$$\langle \varphi^*(a)f, v \rangle = -\langle f, \varphi(a)v \rangle, \quad a \in A, f \in V^*, v \in V,$$

where $\langle \cdot, \cdot \rangle$ is the usual pairing between V and V^*.

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Proposition 2

Let $(A, \circ)$ be a Novikov algebra and (V, l_A, r_A) be a representation of $(A, \circ)$. Then $(V^*, l_A^* + r_A^*, -r_A^*)$ is a representation of $(A, \circ)$.

The adjoint representation of a Novikov algebra $(A, \circ)$ gives the representation $(A^*, L_A^* + R_A^*, -R_A^*)$.

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Definition of quadratic Novikov algebras

Let $(A, \circ)$ be a Novikov algebra. A bilinear form $\mathcal{B}(\cdot, \cdot)$ on A is called **invariant** if it satisfies

$$\mathcal{B}(a \circ b, c) = -\mathcal{B}(b, a \star c), \tag{11}$$

A **quadratic Novikov algebra**, denoted by $(A, \circ, \mathcal{B}(\cdot, \cdot))$, is a Novikov algebra $(A, \circ)$ together with a nondegenerate symmetric invariant bilinear form $\mathcal{B}(\cdot, \cdot)$.

Definition of Manin triples of Novikov algebras

A **(standard) Manin triple of Novikov algebras** is a triple of Novikov algebras $(A = A_1 \oplus A_1^*, (A_1, \circ), (A_1^*, \cdot))$ for which

1. as a vector space, A is the direct sum of A_1 and A_1^*;
2. $(A_1, \circ)$ and $(A_1^*, \cdot)$ are Novikov subalgebras of A;
3. the bilinear form on $A = A_1 \oplus A_1^*$ defined by

$$\mathcal{B}(a+f, b+g) := \langle f, b\rangle + \langle g, a\rangle, \quad a, \quad b \in A_1, \quad f, \quad g \in A_1^*, \tag{12}$$

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is invariant.

Theorem 3

Let $(A, \circ)$ be a Novikov algebra and (A, Δ) a Novikov coalgebra. Let $\cdot$ denote the multiplication on the dual space A^* induced by Δ. Then the following conditions are equivalent.

1. There is a Manin triple $(A \oplus A^*, (A, \circ), (A^*, \cdot))$ of Novikov algebras;
2. $(A, \circ, \Delta)$ is a Novikov bialgebra.

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1. There is a Manin triple $(A \oplus A^*, (A, \circ), (A^*, \cdot))$ of Novikov algebras;
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Definition of Novikov Yang-Baxter equation

Let $(A, \circ)$ be a Novikov algebra and $r = \sum_\alpha x_\alpha \otimes y_\alpha \in A \otimes A$. The equation

$$\begin{aligned} & r \diamond r := r_{13} \circ r_{23} + r_{12} \star r_{23} + r_{13} \circ r_{12} \\ = & \sum_{\alpha,\beta} (x_\alpha \otimes x_\beta \otimes y_\alpha \circ y_\beta + x_\alpha \otimes y_\alpha \star x_\beta \otimes y_\beta + x_\alpha \circ x_\beta \otimes y_\beta \otimes y_\alpha) \\ = & \ 0 \end{aligned}$$

is called the **Novikov Yang-Baxter equation (NYBE)** in A.

Theorem 4

Let $(A, \circ)$ be a Novikov algebra and $r \in A \otimes A$ be a skewsymmetric solution of the NYBE in A. Define $\Delta_r : A \to A \otimes A$ by

$$\Delta_r(a) := (L_A(a) \otimes \mathrm{id} + \mathrm{id} \otimes L_{A,\star}(a))r, \quad a \in A.$$

Then $(A, \circ, \Delta_r)$ is a Novikov bialgebra.

For a finite-dimensional vector space A, the isomorphism

$$A \otimes A \cong \mathrm{Hom}_{\mathbf{k}}(A^*, \mathbf{k}) \otimes A \cong \mathrm{Hom}_{\mathbf{k}}(A^*, A)$$

identifies an $r \in A \otimes A$ with a map from A^* to A which we denote by T^r.

Theorem 5

Let $(A, \circ)$ be a Novikov algebra and $r \in A \otimes A$ be skewsymmetric. Then r is a solution of the NYBE in $(A, \circ)$ if and only if T^r satisfies

$$T^r(f) \circ T^r(g) = T^r(L^*_{A,\star}(T^r(f))g) - T^r(R^*_A(T^r(g))f), \quad f, \ g \in A^*.$$

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Definition of $\mathcal{O}$-operators

Let $(A, \circ)$ be a Novikov algebra and (V, l_A, r_A) be a representation. A linear map $T : V \to A$ is called an $\mathcal{O}$**-operator** on $(A, \circ)$ associated to (V, l_A, r_A) if T satisfies

$$T(u) \circ T(v) = T(l_A(T(u))v) + T(r_A(T(v))u), \quad u, \ v \in V.$$

Theorem 6

Let $(A, \circ)$ be a Novikov algebra and (V, l_A, r_A) be a representation. Let $T : V \to A$ be a linear map which is identified with $r_T \in A \otimes V^* \subseteq (A \ltimes_{l_A^*+r_A^*, -r_A^*} V^*) \otimes (A \ltimes_{l_A^*+r_A^*, -r_A^*} V^*)$ through $\mathrm{Hom}_{\mathbf{k}}(V, A) \cong A \otimes V^*$. Then $r = r_T - \tau r_T$ is a solution of the NYBE in the Novikov algebra $(A \ltimes_{l_A^*+r_A^*, -r_A^*} V^*, \bullet)$ if and only if T is an $\mathcal{O}$-operator on $(A, \circ)$ associated to (V, l_A, r_A).

Definition of $\mathcal{O}$-operators

Let $(A, \circ)$ be a Novikov algebra and (V, l_A, r_A) be a representation. A linear map $T : V \to A$ is called an **$\mathcal{O}$-operator** on $(A, \circ)$ associated to (V, l_A, r_A) if T satisfies

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Definition of pre-Novikov algebras

A **pre-Novikov algebra** is a triple $(A, \lhd, \rhd)$, where A is a vector space, and $\lhd$ and $\rhd$ are binary operations such that

$$\begin{aligned}
x \rhd (y \rhd z) &= (x \rhd y + x \lhd y) \rhd z + y \rhd (x \rhd z) \\
&\quad -(y \rhd x + y \lhd x) \rhd z, \\
x \rhd (y \lhd z) &= (x \rhd y) \lhd z + y \lhd (x \lhd z + x \rhd z) - (y \lhd x) \lhd z, \\
(x \lhd y + x \rhd y) \rhd z &= (x \rhd z) \lhd y, \\
(x \lhd y) \lhd z &= (x \lhd z) \lhd y, \quad x, y, z \in A.
\end{aligned}$$

A construction of pre-Novikov algebras

Recall that a **Zinbiel algebra** $(A, \cdot)$ is a vector space A with a binary operation $\cdot : A \otimes A \to A$ satisfying

$$a \cdot (b \cdot c) = (b \cdot a) \cdot c + (a \cdot b) \cdot c, \quad a, b, c \in A.$$

For a derivation D on a Zinbiel algebra $(A, \cdot)$, define binary operations $\lhd$ and $\rhd : A \otimes A \to A$ by

$$a \lhd b := D(b) \cdot a, \quad a \rhd b := a \cdot D(b), \quad a, b \in A.$$

A direct check shows that $(A, \lhd, \rhd)$ is a pre-Novikov algebra.

For a pre-Novikov algebra $(A, \lhd, \rhd)$, define linear maps $L_{\rhd}, R_{\lhd} : A \to \mathrm{End}_{\mathbf{k}}(A)$ by

$$L_{\rhd}(a)(b) := a \rhd b, \quad R_{\lhd}(a)(b) := b \lhd a, \ \ a, b \in A.$$

Proposition 3

Let $(A, \lhd, \rhd)$ be a pre-Novikov algebra. The binary operation

$$\circ : A \otimes A \to A, \quad x \circ y := x \lhd y + x \rhd y, \ \ x, y \in A, \tag{13}$$

defines a Novikov algebra, which is called the **associated Novikov algebra** of $(A, \lhd, \rhd)$. Moreover, $(A, L_{\rhd}, R_{\lhd})$ is a representation of $(A, \circ)$ and the identity map is an $\mathcal{O}$-operator on $(A, \circ)$ associated to the representation $(A, L_{\rhd}, R_{\lhd})$..

For a pre-Novikov algebra $(A, \lhd, \rhd)$, define linear maps $L_{\rhd}, R_{\lhd} : A \to \mathrm{End}_{\mathbf{k}}(A)$ by

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Theorem 6

Let $(A, \lhd, \rhd)$ be a pre-Novikov algebra and $(A, \circ)$ be the associated Novikov algebra. Then

$$r := \sum_{\alpha=1}^{n} (e_\alpha \otimes e_\alpha^* - e_\alpha^* \otimes e_\alpha), \tag{14}$$

is a skewsymmetric solution of the NYBE in the Novikov algebra $A \ltimes_{L_\rhd^* + R_\lhd^*, -R_\lhd^*} A^*$, where $\{e_1, \ldots, e_n\}$ is a linear basis of A and $\{e_1^*, \ldots, e_n^*\}$ is the dual basis of A^*.

There is the following diagram.

pre-Novikov algebras $\longrightarrow$ $\mathcal{O}$-operators on Novikov algebras $\longrightarrow$ solutions of NYBE $\longrightarrow$ Novikov bialgebras

Multi-Novikov algebras

Let r be a positive integer and $I = \{1, 2, \cdots, r\}$. A **multi-Novikov algebra** is a vector space A equipped with the binary operations $\{\circ_i \mid i \in I\}$ satisfying

$$(a \circ_i b) \circ_j c - a \circ_i (b \circ_j c) = (b \circ_i a) \circ_j c - b \circ_i (a \circ_j c),$$
$$(a \circ_i b) \circ_j c - a \circ_i (b \circ_j c) = (a \circ_j b) \circ_i c - a \circ_j (b \circ_i c),$$
$$(a \circ_i b) \circ_j c = (a \circ_j c) \circ_i b, \quad a, b, c \in A, \ i, j \in I.$$

Such structure was introduced by Y. Bruned and V. Dotsenko, 2023 in the study of multi-indices in the context of singular stochastic partial differential equations.

Multi-Novikov algebras

Let r be a positive integer and $I = \{1, 2, \cdots, r\}$. A **multi-Novikov algebra** is a vector space A equipped with the binary operations $\{\circ_i \mid i \in I\}$ satisfying

$$(a \circ_i b) \circ_j c - a \circ_i (b \circ_j c) = (b \circ_i a) \circ_j c - b \circ_i (a \circ_j c),$$
$$(a \circ_i b) \circ_j c - a \circ_i (b \circ_j c) = (a \circ_j b) \circ_i c - a \circ_j (b \circ_i c),$$
$$(a \circ_i b) \circ_j c = (a \circ_j c) \circ_i b, \quad a, b, c \in A, \ i, j \in I.$$

Such structure was introduced by Y. Bruned and V. Dotsenko, 2023 in the study of multi-indices in the context of singular stochastic partial differential equations.

Definition of generalized multi-Novikov algebras

A **generalized multi-Novikov algebra** is a vector space A equipped with binary operations $\{\circ_i \mid i \in I\}$ satisfying

$$a \circ_i (b \circ_j c) - (a \circ_i b) \circ_j c = b \circ_j (a \circ_i c) - (b \circ_j a) \circ_i c, \quad (15)$$

$$(a \circ_i b) \circ_j c + (a \circ_j b) \circ_i c = (a \circ_i c) \circ_j b + (a \circ_j c) \circ_i b, \quad (16)$$

for all $a, b, c \in A$ and $i, j \in I$. We denote it by $(A, \{\circ_i | \; i \in I\})$.

Such structure appeared in the study of Lie conformal algebras in higher dimensions Y. Hong, 2016.

We also point out that a multi-Novikov algebra is a special generalized multi-Novikov algebra.

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Definition of multi-right Novikov algebras

A **multi-right Novikov algebra** is a vector space B equipped with binary operations $\{\diamond_i \mid i \in I\}$ satisfying

$$x \diamond_i (y \diamond_j z) - (x \diamond_i y) \diamond_j z = x \diamond_i (z \diamond_j y) - (x \diamond_i z) \diamond_j y, \quad (17)$$

$$x \diamond_i (y \diamond_j z) - (x \diamond_i y) \diamond_j z = x \diamond_j (y \diamond_i z) - (x \diamond_j y) \diamond_i z, \quad (18)$$

$$x \diamond_i (y \diamond_j z) = y \diamond_j (x \diamond_i z), \quad (19)$$

for all $x, y, z \in B$ and $i, j \in I$. We denote it by $(B, \{\diamond_i \mid i \in I\})$.

Note that $(A, \{\circ_i \mid i \in I\})$ is a multi-Novikov algebra if and only if $(A, \{\diamond_i \mid i \in I\})$ is a multi-right Novikov algebra where $a \diamond_i b := b \circ_i a$ for each i and all $a, b \in A$.

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Construction of (generalized) multi-Novikov algebras Y. Bruned and V. Dotsenko, 2023

Let $(A, \cdot)$ be a commutative associative algebra and D_i $(i \in I)$ are pairwise commuting derivations on A. Define

$$a \circ_i b := a \cdot D_i(b), \quad a, b \in A, i \in I. \tag{20}$$

Then $(A, \{\circ_i \mid i \in I\})$ is a multi-Novikov algebra.

A special example of multi-right Novikov algebras

Let $B = \mathbf{k}[t_1, t_1^{-1}, \cdots, t_r, t_r^{-1}]$ be the Laurent polynomial algebra with r variables. Let $D_i = \frac{\partial}{\partial t_i}$ be the derivations of B. Then $(B = \mathbf{k}[t_1, t_1^{-1}, \cdots, t_r, t_r^{-1}], \{\diamond_i \mid i \in I\})$ is a multi-right Novikov algebra where $\diamond_i$ is defined by

$$\begin{aligned} t_1^{m_1} t_2^{m_2} \cdots t_r^{m_r} \diamond_i t_1^{n_1} t_2^{n_2} \cdots t_r^{n_r} &= \frac{\partial}{\partial t_i}(t_1^{m_1} t_2^{m_2} \cdots t_r^{m_r}) \cdot t_1^{n_1} t_2^{n_2} \cdots t_r^{n_r} \\ &= m_i t_1^{m_1+n_1} \cdots t_i^{m_i+n_i-1} \cdots t_r^{m_r+n_r}. \end{aligned}$$

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Construction of Lie algebras from generalized multi-Novikov algebras and multi-right Novikov algebras Y. Hong, C. Bai, L. Guo, 2024

Let $(A, \{\circ_i \mid i \in I\})$ be a generalized multi-Novikov algebra and $(B, \{\diamond_i \mid i \in I\})$ be a multi-right Novikov algebra. Define a binary operation on the vector space $L = A \otimes B$ by

$$[a \otimes x, b \otimes y] = \sum_{i=1}^{r} (a \circ_i b \otimes x \diamond_i y - b \circ_i a \otimes y \diamond_i x), \quad a, b \in A, x, y \in B$$

Then $(L, [\cdot, \cdot])$ is a Lie algebra, called **the induced Lie algebra** from $(A, \{\circ_i \mid i \in I\})$ and $(B, \{\diamond_i \mid i \in I\})$. In particular, if $(B, \{\diamond_i \mid i \in I\}) = (\mathbf{k}[t_1, t_1^{-1}, \cdots, t_r, t_r^{-1}], \{\diamond_i \mid i \in I\})$ be the multi-right Novikov algebra given as before, then $(L, [\cdot, \cdot])$ is a Lie algebra if and only if $(A, \{\circ_i \mid i \in I\})$ is a generalized multi-Novikov algebra.

Natural questions

- Can we develop a bialgebra theory of generalized multi-Novikov algebras (or multi-Novikov algebras)?
- If so, what is the relationship between generalized multi-Novikov bialgebras (or multi-Novikov bialgebras) and infinite-dimensional Lie bialgebras?

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Thank you!

Proposition 4

Let $(A, \circ)$ be a Novikov algebra and $\big(B = \oplus_{i\in\mathbb{Z}} B_i, \diamond, (\cdot,\cdot)\big)$ be a quadratic $\mathbb{Z}$-graded right Novikov algebra. Let $L = A \otimes B$ be the induced Lie algebra. Suppose that $r = \sum_\alpha x_\alpha \otimes y_\alpha \in A \otimes A$ is a skewsymmetric solution of the NYBE in A. Then for a basis $\{e_p\}_{p\in\Pi}$ consisting of homogeneous elements of B and its homogeneous dual basis $\{f_p\}_{p\in\Pi}$ associated with the bilinear form $(\cdot,\cdot)$, the tensor element

$$r_L := \sum_{p\in\Pi}\sum_{\alpha}(x_\alpha \otimes e_p) \otimes (y_\alpha \otimes f_p) \in L\hat{\otimes}L \tag{22}$$

is a skewsymmetric completed solution of the CYBE in L. Furthermore, if the quadratic $\mathbb{Z}$-graded right Novikov algebra is $(B, \diamond, (\cdot,\cdot)) = (\mathbf{k}[t, t^{-1}], \diamond, (\cdot,\cdot))$, then

$$r_L := \sum_{i\in\mathbb{Z}}\sum_{\alpha} x_\alpha t^i \otimes y_\alpha t^{-i-1} \in L\hat{\otimes}L \tag{23}$$

is a skewsymmetric completed solution of the CYBE in L if and only if r is a skewsymmetric solution of the NYBE in A.