Rota-Baxter operators on groups and post-groups

Yunhe Sheng
(Joint work with Chengming Bai, Li Guo, Honglei Lang and Rong Tang)

Department of Mathematics, Jilin University, China

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Poisson Lie groups and Lie bialgebras

In Lie Theory:

$$\text{Lie groups } G \underset{\text{integration}}{\overset{\text{differentiation}}{\rightleftarrows}} \text{Lie algebras } \mathfrak{g}$$

In Poisson Geometry:

$$\text{Poisson Lie groups } (M,\pi) \underset{\text{integration}}{\overset{\text{differentiation}}{\rightleftarrows}} \text{Lie bialgebras } (\mathfrak{g},\delta)$$

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Integrability of Lie algebroids, Courant algebroids, Leibniz algbras, L_∞-algebras:

M. Crainic and R. L. Fernandes, Integrability of Lie brackets. *Ann. of Math. (2)* **157** (2003), 575-620.

A. Henriques. Integrating L_∞-algebras. *Compos. Math.* 144(4) (2008), 1017-1045.

Y. Sheng and C. Zhu, Higher Extensions of Lie Algebroids, *Commun. Contemp. Math.* 19 (3) (2017), 1650034, 41 pages.

C. Laurent-Gengoux and F. Wagemann, Lie rackoids integrating Courant algebroids, *Ann. Global Anal. Geom.* 57 (2020), no. 2, 225-256.

Definition

Let $\phi : \mathfrak{g} \to \mathrm{Der}(\mathfrak{h})$ be an action of a Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ on a Lie algebra $(\mathfrak{h}, [\cdot,\cdot]_{\mathfrak{h}})$. A linear map $T : \mathfrak{h} \to \mathfrak{g}$ is called a **relative Rota-Baxter operator of weight λ** on $\mathfrak{g}$ with respect to $(\mathfrak{h};\phi)$ if

$$[T(u), T(v)]_{\mathfrak{g}} = T\Big(\phi(T(u))v - \phi(T(v))u + \lambda[u,v]_{\mathfrak{h}}\Big), \quad \forall u, v \in \mathfrak{h}.$$

If $\mathfrak{h} = \mathfrak{g}$ and $\phi = \mathrm{ad}$, then we call B a Rota-Baxter operator of weight λ.

L. Guo, An introduction to Rota-Baxter algebra. Surveys of Modern Mathematics, 4. International Press, Somerville, MA; Higher Education Press, Beijing, 2012. xii+226 pp.

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Operator form of CYBE

triangular Lie bialgebra:

$$r_+(\mathrm{ad}^*_{r_+\xi}\eta - \mathrm{ad}^*_{r_+\eta}\xi) = [r_+(\xi), r_+(\eta)].$$

$r_+ : \mathfrak{g}^* \to \mathfrak{g}$ is a relative Rota-Baxter operator of weight 0 with respect to the coadjoint representation.

quasitriangular Lie bialgebra:

$$r_+(\mathrm{ad}^*_{r_+\xi}\eta - \mathrm{ad}^*_{r_+\eta}\xi + \mathrm{ad}^*_{I(\eta)}\xi) = [r_+(\xi), r_+(\eta)].$$

It turns out that $[\xi, \eta]_I \triangleq \mathrm{ad}^*_{I(\eta)}\xi$ defines a Lie bracket on $\mathfrak{g}^*$, and ad^* is an action of the Lie algebra $\mathfrak{g}$ on $(\mathfrak{g}^*, [\cdot,\cdot]_I)$.

$r_+ : \mathfrak{g}^* \to \mathfrak{g}$ is a relative Rota-Baxter operator of weight 1 on $\mathfrak{g}$ with respect to the action ad^* on $(\mathfrak{g}^*, [\cdot,\cdot]_I)$.

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Questions

What is the integration of Rota-Baxter operators on Lie algebras?

Li Guo, Honglei Lang and Yunhe Sheng, Integration and geometrization of Rota-Baxter Lie algebras, **Adv. Math.** 387 (2021), 107834.

Rota-Baxter operators on Lie groups $\xrightarrow{\text{differentiation}}$ Rota-Baxter operators on Lie algebras

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Rota-Baxter Lie groups

Definition (Guo-Lang-S.)

A Rota-Baxter operator of weight 1 on a Lie group G is a smooth map $\mathfrak{B} : G \to G$ such that

$$\mathfrak{B}(g)\mathfrak{B}(h) = \mathfrak{B}(g\mathrm{Ad}_{\mathfrak{B}(g)}h), \qquad g, h \in G.$$

Theorem (Guo-Lang-S.)

If $(G, \mathfrak{B})$ is a Rota-Baxter Lie group, then $(\mathfrak{g}, B = \mathfrak{B}_{*e})$ is a Rota-Baxter Lie algebra of weight 1.

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Useful formulas

Let G be a Lie group and e its identity. Let $\mathfrak{g} = T_eG$ be the Lie algebra of G and let

$$\exp^{(\cdot)} : \mathfrak{g} \longrightarrow G$$

be the exponential map. Then the relation between the Lie bracket $[\cdot,\cdot]_{\mathfrak{g}}$ and the Lie group multiplication is given by the following important formula:

$$[u,v]_{\mathfrak{g}} = \frac{d^2}{dtds}\Big|_{t,s=0} \exp^{tu}\exp^{sv}\exp^{-tu}, \quad \forall\ u,v \in \mathfrak{g}.$$

Since $B = \mathfrak{B}_{*e}$ is the tangent map of $\mathfrak{B}$ at e, we have the following relation for sufficiently small t:

$$\frac{d}{dt}\Big|_{t=0} \mathfrak{B}(\exp^{tu}) = \frac{d}{dt}\Big|_{t=0} \exp^{tB(u)} = B(u), \quad \forall\ u \in \mathfrak{g}.$$

Proof.

$$\begin{aligned}
& [B(u), B(v)] \\
= & \frac{d^2}{dtds}\Big|_{t,s=0} \exp^{tB(u)} \exp^{sB(v)} \exp^{-tB(u)} \\
= & \frac{d^2}{dtds}\Big|_{t,s=0} \mathfrak{B}(\exp^{tu})\mathfrak{B}(\exp^{sv})\mathfrak{B}(\exp^{-tu}) \\
= & \frac{d^2}{dtds}\Big|_{t,s=0} \mathfrak{B}(\exp^{tu})\mathfrak{B}(\exp^{sv} \mathrm{Ad}_{\mathfrak{B}(\exp^{sv})} \exp^{-tu}) \\
= & \frac{d^2}{dtds}\Big|_{t,s=0} \mathfrak{B}(\exp^{tu}(\mathrm{Ad}_{\mathfrak{B}(\exp^{tu})} \exp^{sv})(\mathrm{Ad}_{\mathfrak{B}(\exp^{tu})\mathfrak{B}(\exp^{sv})} \exp^{-tu})) \\
= & \mathfrak{B}_{*e}\left(\frac{d^2}{dtds}\Big|_{t,s=0} \mathrm{Ad}_{\mathfrak{B}(\exp^{tu})} \exp^{sv} + \frac{d^2}{dtds}\Big|_{t,s=0} \mathrm{Ad}_{\mathfrak{B}(\exp^{sv})} \exp^{-tu} \right. \\
& \left. + \frac{d^2}{dtds}\Big|_{t,s=0} \exp^{tu} \exp^{sv} \exp^{-tu}\right) \\
= & B([B(u), v] - [B(v), u] + [u, v]).
\end{aligned}$$

□

Example

$B : \mathfrak{g} \to \mathfrak{g}, B(u) = -u$ is a Rota-Baxter operator of weight 1 on $\mathfrak{g}$;
$\mathfrak{B} : G \to G, \mathfrak{B}(g) = g^{-1}$ is a Rota-Baxter operator on G.

Example

- Let $\mathfrak{g}$ be a Lie algebra with $\mathfrak{g}_+, \mathfrak{g}_-$ two Lie subalgebras such that $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$. Then the minus of the projections $-P_+, -P_- : \mathfrak{g} \to \mathfrak{g}$ are two RB operators of weight 1 on $\mathfrak{g}$.
- Let G be a Lie group, G_+, G_- two subgroups such that $G = G_+G_-$ and $G_+ \cap G_- = \{e\}$. Then $\mathfrak{B} : G \to G, \mathfrak{B}(g_+g_-) = g_-^{-1}$ is a Rota-Baxter operator on G.

$$SL(n, \mathbb{C}) = SU(n)SB(n, \mathbb{C}), \quad \text{Iwasawa decomposition},$$

where $SB(n, \mathbb{C})$ consists of all upper triangular matrices in $SL(n, \mathbb{C})$ with positive entries on the diagonal.

$$[Bu, Bv] = B([Bu, v]+[u, Bv]+[u, v]), \quad \mathfrak{B}(g)\mathfrak{B}(h) = \mathfrak{B}(g\mathrm{Ad}_{\mathfrak{B}(g)}h).$$

Remark

If $(\mathfrak{g}, B)$ is a Rota-Baxter Lie algebra, then there is s a new Lie algebra structure (called the descendent Lie algebra)

$$[u, v]_B = [Bu, v] + [u, Bv] + [u, v],$$

s.t. $B : (\mathfrak{g}, [\cdot, \cdot]_B) \to \mathfrak{g}$ is a Lie algebra homomorphism.

Proposition

Let $(G, \mathfrak{B})$ be a Rota-Baxter Lie group.

- *The pair $(G, *)$, with the multiplication*

$$g * h := g\mathrm{Ad}_{\mathfrak{B}(g)}h, \qquad \forall\ g, h \in G,$$

  is also a Lie group (called the descendent Lie group), whose Lie algebra is $(\mathfrak{g}, [\cdot,\cdot]_B)$, where $B = \mathfrak{B}_{*e}$.
- *The operator $\mathfrak{B}$ is a Rota-Baxter operator on the Lie group $(G, *)$.*
- *The map $\mathfrak{B} : (G, *) \to G$ is a homomorphism of Rota-Baxter Lie groups from $(G, *, \mathfrak{B})$ to $(G, \mathfrak{B})$.*

Post-Lie algebras

Definition (Vallette)

A **post-Lie algebra** $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \rhd)$ consists of a Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ and a binary product $\rhd : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ such that

$$\begin{aligned} x \rhd [y,z]_{\mathfrak{g}} &= [x \rhd y, z]_{\mathfrak{g}} + [y, x \rhd z]_{\mathfrak{g}}, \\ [x,y]_{\mathfrak{g}} \rhd z &= a_{\rhd}(x,y,z) - a_{\rhd}(y,x,z), \end{aligned}$$

where $a_{\rhd}(x,y,z) = x \rhd (y \rhd z) - (x \rhd y) \rhd z$.

H. Z. Munthe-Kaas and A. Lundervold, On post-Lie algebras, Lie-Butcher series and moving frames, *Found. Comput. Math.* **13** (2013), 583-613.

Y. Bruned, M. Hairer and L. Zambotti, Algebraic renormalisation of regularity structures. Invent. Math. **215** (2019), 1039-1156.

Y. Bruned and F. Katsetsiadis, Post-Lie algebras in regularity structures. *Forum Math. Sigma* **11** (2023), Paper No. e98.

splittings of algebras

Rota-Baxter operators $\rightsquigarrow$ splitting of algebras

Proposition

Let $\mathfrak{B} : \mathfrak{g} \to \mathfrak{g}$ be a Rota-Baxter operator on a Lie algebra $\mathfrak{g}$. Define a multiplication $\rhd_{\mathfrak{B}}$ on $\mathfrak{g}$ by

$$x \rhd_{\mathfrak{B}} y = [\mathfrak{B}(x), y]_{\mathfrak{g}}, \quad \forall x, y \in \mathfrak{g}.$$

Then $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \rhd_{\mathfrak{B}})$ is a post-Lie algebra.

C. Bai, L. Guo and X. Ni, Nonabelian generalized Lax pairs, the classical Yang-Baxter equation and PostLie algebras, *Comm. Math. Phys.* 297 (2010), 553-596.

In a post-Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \rhd)$, if the Lie bracket $[\cdot, \cdot]_{\mathfrak{g}}$ is trivial, then we obtain a pre-Lie algebra, namely a vector space $\mathfrak{g}$ with a multiplication $\rhd$ satisfying

$$a_{\rhd}(x, y, z) - a_{\rhd}(y, x, z) = 0$$

Smoktunowicz proposed the following questions:

Question. Is there a passage from all left nilpotent braces of cardinality p^n, with $n + 1 < p$, to left nilpotent pre-Lie rings?

A. Smoktunowicz, On the passage from finite braces to pre-Lie rings. *Adv. Math.* **409** (2022), 108683.

Questions:

- What is the integration of post-Lie algebras?

$$\begin{array}{ccc}\text{Rota-Baxter Lie group} & \xrightarrow{\text{splitting}} & \text{post-Lie group}\\ \uparrow \text{integration} & & \uparrow \text{integration}\\ \text{Rota-Baxter Lie algebra} & \xrightarrow{\text{splitting}} & \text{post-Lie algebra}\end{array}$$

Chengming Bai, Li Guo, Yunhe Sheng and Rong Tang, Post-groups, (Lie-)Butcher groups and the Yang-Baxter equation, **Math. Ann.** (2023), https://doi.org/10.1007/s00208-023-02592-z

Post-Lie groups $\xrightarrow{\text{differentiation}}$ Post-Lie algebras

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$$\text{Post-Lie groups} \xrightarrow{\text{differentiation}} \text{Post-Lie algebras}$$

Definition (Bai-Guo-S.-Tang)

A **post-group** is a group $(G, \cdot)$ equipped with another binary operation $\rhd$ on G such that

1. for all $a \in G$, the left multiplication

$$L_a^{\rhd} : G \to G, \quad L_a^{\rhd} b = a \rhd b, \quad \forall b \in G,$$

is an automorphism of the group $(G, \cdot)$, that is,

$$a \rhd (b \cdot c) = (a \rhd b) \cdot (a \rhd c), \quad \forall a, b, c \in G;$$

2. the following "weighted" associativity for $\rhd$ holds:

$$a \rhd (b \rhd c) = \big(a \cdot (a \rhd b)\big) \rhd c, \quad \forall a, b, c \in G.$$

Post-groups

Theorem (Bai-Guo-S.-Tang)

Let $(G, \cdot, \rhd)$ be a post-group. Define $\circ : G \times G \to G$ by

$$a \circ b = a \cdot (a \rhd b), \quad \forall a, b \in G.$$

Then $(G, \circ)$ is a group with e being the unit, and the inverse map $\dagger : G \to G$ given by

$$a^{\dagger} := (L^{\rhd}_a)^{-1}(a^{-1}).$$

Moreover, $L^{\rhd} : G \to \mathrm{Aut}(G)$ is an action of the group $(G, \circ)$ on the group $(G, \cdot)$.

The group $G_{\rhd} := (G, \circ)$ is called the **subadjacent group** of the post-group $(G, \cdot, \rhd)$.

Define $\rhd : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ by

$$x \rhd y = L^{\rhd}_{*e}(x)(y) = \frac{d}{dt}\Big|_{t=0} L^{\rhd}_{\exp(tx)} y = \frac{d}{dt}\Big|_{t=0} \frac{d}{ds}\Big|_{s=0} L^{\rhd}_{\exp(tx)} \exp(sy).$$

Theorem (Bai-Guo-S.-Tang)

Let $(G, \cdot, \rhd)$ be a post-Lie group. Then $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \rhd)$ is a post-Lie algebra.

From Rota-Baxter operators to post-groups

Theorem (Bai-Guo-S.-Tang)

Let $\mathfrak{B}: G \longrightarrow G$ be a Rota-Baxter operator on a group $(G, \cdot_G)$. We define a binary product $\triangleright : G \times G \to G$ as following:

$$g \triangleright h = \mathrm{Ad}_{\mathfrak{B}(g)} h, \quad \forall g, h \in G.$$

Then $(G, \cdot_G, \triangleright)$ is a post-group.

From post-groups to Rota-Baxter operators

Proposition (Bai-Guo-S.-Tang)

Let $(G, \cdot, \triangleright)$ be a post-group. Then the identity map $\mathrm{Id} : G \to G$ is a relative Rota-Baxter operator on the subadjacent group $(G, \circ)$ with respect to the action $L^{\triangleright}$ on the group $(G, \cdot)$.

Skew-left braces

Definition (Rump)

A skew-left brace $(G, \circ, \cdot)$ consists of a group $(G, \cdot)$ and a group $(G, \circ)$ such that

$$a \circ (b \cdot c) = (a \circ b) \cdot a^{\cdot -1} \cdot (a \circ c), \quad \forall a, b, c \in G.$$

W. Rump, A decomposition theorem for square-free unitary solutions of the quantum Yang-Baxter equation. **Adv. Math.** 193 (2005), 40-55.

T. Gateva-Ivanova, Set-theoretic solutions of the Yang-Baxter equation, braces and symmetric groups. **Adv. Math.** 338 (2018), 649-701.

F. Cedó, A. Smoktunowicz and L. Vendramin, Skew left braces of nilpotent type. **Proc. Lond. Math. Soc.** 118 (2019), 1367-1392.

Proposition (Bai-Guo-S.-Tang)

Let $(G, \circ, \cdot)$ be a skew-left brace. Define a binary product $\triangleright : G \times G \to G$ by

$$a \triangleright b = a^{-1} \cdot (a \circ b), \quad \forall a, b \in G,$$

here a^{-1} is the inverse of a in $(G, \cdot)$. Then $(G, \cdot, \triangleright)$ is a post-group.

Skew-left braces

Proposition (Bai-Guo-S.-Tang)

Let $(G, \cdot, \triangleright)$ be a post-group. Then $(G, \circ, \cdot)$ is a skew-left brace.

Theorem (Bai-Guo-S.-Tang)

The category of post-groups is isomorphic to the category of skew-left braces.

The Yang-Baxter equation

We show that a post-group gives rise to a braiding group, and thus lead to a solution of the Yang-Baxter equation.

Definition (Yang-Baxter)

Let X be a set. A set-theoretical solution to the **Yang-Baxter equation** on X is a bijective map $R : X \times X \to X \times X$ satisfying:

$$R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23}.$$

Yang-Baxter equations

Let $(G, \cdot, \rhd)$ be a post-group. Define $R_G : G \times G \to G \times G$ by

$$R_G(x, y) = (x \rhd y, (x \rhd y)^{\dagger} \circ x \circ y), \ \forall x, y \in G,$$

where $\circ$ is the subadjacent group structure.

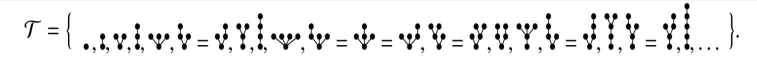
Theorem (Bai-Guo-Sheng-Tang)

Let $(G, \cdot, \rhd)$ be a post-group. Then $\big((G, \circ), R_G\big)$ is a braiding group, and R_G is a solution of the Yang-Baxter equation on the set G.

J. Lu, M. Yan and Y. Zhu, On the set-theoretical Yang-Baxter equation. *Duke Math. J.* **104** (2000), 1-18.

Butcher groups

Let $\mathcal{T}$ be the set of isomorphism classes of rooted trees:



We set $\mathcal{T}^+ = \mathcal{T} \cup \{\emptyset\}$ and denote by

$$\mathcal{B}_{\mathbb{R}} = \{a : \mathcal{T}^+ \to \mathbb{R} | a(\emptyset) = 1\}.$$

Theorem (Hairer-Wanner)

$(\mathcal{B}_{\mathbb{R}}, \circ)$ *is a group, which is called* **Butcher group**, *where*

$$(a \circ b)(\tau) \;=\; a(\tau) + \sum_{c \in AC(\tau)} a(P^c(\tau))b(R^c(\tau)).$$

E. Hairer and G. Wanner, On the Butcher group and general multi-value methods, *Computing* 13 (1974), 1-15.

Butcher group

We define an abelian group structure on $\mathcal{B}_{\mathbb{R}}$ by

$$(a\cdot b)(\emptyset)=1,\ (a\cdot b)(\omega)=a(\omega)+b(\omega),\ \forall\omega\in\mathcal{T},$$

Define the binary product $\triangleright:\mathcal{B}_{\mathbb{R}}\times\mathcal{B}_{\mathbb{R}}\to\mathcal{B}_{\mathbb{R}}$ by

$$\begin{aligned}(a\triangleright b)(\emptyset)&=1,\\ (a\triangleright b)(\tau)&=\sum_{c\in AC(\tau)}a(P^c(\tau))b(R^c(\tau)),\forall\omega\in\mathcal{T}.\end{aligned}$$

Theorem (Bai-Guo-S.-Tang)

With the above notations, $(\mathcal{B}_{\mathbb{R}},\cdot,\triangleright)$ is a post-group, whose subadjacent group is exactly the Butcher group $(\mathcal{B}_{\mathbb{R}},\circ)$.

$\mathcal{P}$-groups

Let $\mathcal{P}$ be a operad. Define $G(\mathcal{P})$ by

$$G(\mathcal{P}) = \{\mathrm{Id}_{\mathcal{P}}\} \times \prod_{n=2}^{+\infty} \mathcal{P}(n)_{\mathbb{S}_n}.$$

Denote an element of $G(\mathcal{P})$ by $\bar{a} = (\mathrm{Id}_{\mathcal{P}}, \overline{a_2}, \cdots, \overline{a_n}, \cdots)$. For all $\bar{a}, \bar{b} \in G(\mathcal{P})$, define $\circ : G(\mathcal{P}) \times G(\mathcal{P}) \to G(\mathcal{P})$ by

$$(\bar{a} \circ \bar{b})_n = \sum_{k=1}^{n} \sum_{t_1+\cdots+t_k=n} \overline{\gamma(b_k; a_{t_1}, \cdots, a_{t_k})}.$$

Theorem (Chapoton-Livernet-van der Laan)

$(G(\mathcal{P}), \circ)$ *is a group, which is called the* $\mathcal{P}$*-group.*

$\mathcal{P}$-groups

We define an abelian group structure on $G(\mathcal{P})$ by

$$(\bar{a}\cdot\bar{b})_1 = \mathrm{Id}_{\mathcal{P}},\ (\bar{a}\cdot\bar{b})_n = \overline{a_n + b_n}, \quad \forall n = 2, 3, \cdots ,$$

Define the binary product $\triangleright : G(\mathcal{P}) \times G(\mathcal{P}) \to G(\mathcal{P})$ by

$$\begin{aligned}(\bar{a} \triangleright \bar{b})_1 &= \mathrm{Id}_{\mathcal{P}},\\ (\bar{a} \triangleright \bar{b})_n &= \sum_{k=2}^{n} \sum_{t_1+\cdots+t_k=n} \overline{\gamma(b_k; a_{t_1}, \cdots, a_{t_k})}.\end{aligned}$$

Theorem (Bai-Guo-Sheng-Tang)

With the above notations, $(G(\mathcal{P}), \cdot, \triangleright)$ is a post-group, whose subadjacent group is exactly the $\mathcal{P}$-group $(G(\mathcal{P}), \circ)$.

Recent developments

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- F. Catino, M. Mazzotta and P. Stefanelli, Rota-Baxter operators on Clifford semigroups and the Yang-Baxter equation. *J. Algebra* (2023).
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- Caranti-Stefanello, Skew braces from RB operators: A cohomological characterisation, and some examples. *Ann. Mat. Pura Appl.* (2023).
- Zhonghua Li, Shukun Wang, Rota-Baxter systems and skew trusses, *J. Algebra* (2023).

Recent developments

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Thanks for your attention!